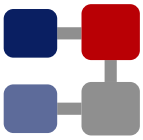
Engineering Mathematics



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Welcome

Chemical engineering occupies a unique position at the interface between molecular sciences and engineering. Intimately linked with the fundamental subjects of chemistry, biology, **mathematics**, and physics — and in close collaboration with fellow engineering disciplines like materials science, computer science, and mechanical, electrical, and civil engineering — chemical engineering offers unparalleled opportunities to do great things

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Figure 1: https://seoultechpse.github.io/EngMath/

References

- D.G. Zill and W.S. Wright, **Advanced Engineering Mathematics**, 7^{th} *ed.*, Jones and Bartlett, 2022.
- E.A. Coddington, **An Introduction to Ordinary Differential Equations**, Prentice-Hall, 1961. (Dover Publications Inc., 1989 - Unabridged, corrected republication).
- G.E. Shilov, **Linear Algebra**, Prentice-Hall, 1971. (Dover Publications Inc., 1977 - Unabridged republication)
- S.J. Farlow, **Partial Differential Equations for Scientists and Engineers**, John Wiley & Sons, 1982. (Dover Publications Inc., 1993 – Unabridged, corrected republication).
- H.F. Weinberger, **A First Course in Partial Differential Equations with Complex Variables and Transform Methods**, Blaisdell Publishing Company, 1965. (Dover Publications Inc., 1995 - New edition).
- R. Johansson, **Numerical Python: Scientific Computing and Data Science Applications with Numpy, Scipy and Matplotlib**, 2^{nd} *ed.*, Apress, 2018.

Part I

First Semester

1 Introduction to Differential Equations

Learning objectives After this chapter you should be able to

- classify a differential equation by type (ODE or PDE), order and linearity
- verify that a function, explicit or implicit, is a solution on an interval, and recognize families of solutions
- state the conditions under which a first-order initial-value problem has a unique solution
- translate a physical situation, such as radioactive decay or a draining tank, into a differential equation

1.1 Definitions and Terminology

- A **differential equation (DE)** is an equation containing the **derivatives** of **one or more dependent variables** with respect to **one or more independent variables**
- Notation
  - Leibniz notation $\frac{d^2y}{dx^2}$
  - Prime notation y''
  - Newton's dot notation $\ddot{y}$
  - Subscript notation $u_{xx} + u_{yy} = 0$

1.1.1 Classification by Type

- **Ordinary differential equation (ODE):** Derivatives are with respect to a single independent variable

$$\frac{dx}{dt} + \frac{dy}{dt} = 3x + 2y$$

- **Partial differential equation (PDE):** Derivatives are with respect to two or more independent variables

$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x}$$

1.1.2 Classification by Order

- The order of an ODE or PDE is the order of the highest derivative in the equation

$$\underbrace{\frac{d^2y}{dx^2}}_{\text{highest order}} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

$$2\underbrace{\frac{\partial^4 u}{\partial x^4}}_{\text{highest order}} + \frac{\partial^2 u}{\partial t^2} = 0$$

1.1.3 Classification by Linearity

- An n-th order ODE, $F\left(x, y, y', \cdots, y^{(n)}\right) = 0$, is **linear** in the variable y if F is linear in $y, y', \cdots, y^{(n)}$
- An ODE is **nonlinear** if
  - The coefficients of $y, y', \cdots, y^{(n)}$ contain the dependent variable y or its derivatives
  - Powers of $y, y', \cdots, y^{(n)}$ appear in the equation *or*
  - Nonlinear functions of the dependent variable or its derivatives (e.g., $\sin y$ or $e^{y'}$) appear in the equation

1.1.4 Solution of an ODE

- Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an ODE reduces the equation to an identity, is **a solution**
- Interval I can be an open interval (a, b), a closed interval $[a, b]$, an infinite interval (a, ∞), etc.
- A solution of a differential equation that is identically zero on an interval I is a **trivial solution**
- The graph of a solution ϕ of an ODE is a **solution curve** and it is continuous on its interval I while the domain of ϕ may differ from the interval I
- An **explicit solution** is one in which the dependent variable is expressed solely in terms of the independent variable and constants
- $G(x, y) = 0$ is an **implicit solution**, if at least one function ϕ exists, that satisfies the relation G and the ODE on I

1.1.5 Families of Solutions

- Similar to integration, we obtain a solution to a first-order differential equation containing an arbitrary constant c
- A solution with a constant c represents a set $G(x, y, c) = 0$ of solutions, called a **one-parameter family of solutions**
- An n-parameter family of solutions

$$G(x, y, c_1, c_2, \cdots, c_n) = 0$$

  solves an n-th order differential equation

1.1.6 Systems of Differential Equations

- Two or more equations involving the derivatives of two or more unknown functions of a single independent variable are called a **system of differential equations**

$$\begin{aligned}\frac{dx}{dt} &= f(t, x, y)\\ \frac{dy}{dt} &= g(t, x, y)\end{aligned}$$

- A solution of a system is a set of functions defined on a common interval I that satisfies each equation of the system on this interval

1.2 Initial-Value Problems - IVP

In an **IVP**, we seek a solution $y(x)$ of a differential equation so that $y(x)$ satisfies **initial conditions** at x_0

- n-th order differential equation and Initial conditions

$$\begin{aligned}&\frac{d^n y}{dx^n} = f\left(x, y, y', \cdots, y^{(n-1)}\right)\\ &y(x_0) = y_0,\ y'(x_0) = y_1, \cdots,\ y^{(n-1)}(x_0) = y_{n-1}\end{aligned}$$

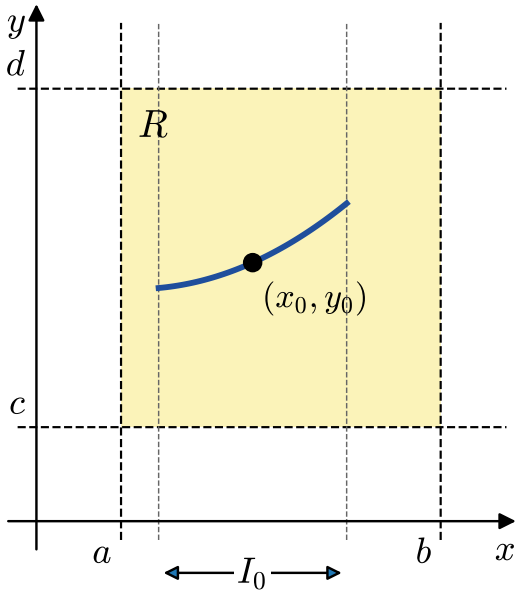
$$\Downarrow$$

$$\frac{d\mathbf{y}}{dx} = \mathbf{f}(x, \mathbf{y}),\quad \mathbf{y}(x_0) = \mathbf{y}_0$$

When solving an IVP, consider whether a solution exists and whether the solution is unique

- **Existence:** Does the differential equation possess solutions and do any of the solution curves pass through the initial point (x_0, y_0)?
- **Uniqueness:** When can we be certain there is precisely one solution curve passing through the initial point (x_0, y_0)?

Theorem 1.1. *gives conditions that are sufficient to guarantee the existence and uniqueness of a solution $y(x)$ to a first-order IVP*



Let R be a rectangular region in the xy-plane defined by $a \le x \le b$, $c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f/\partial y$ are continuous on R, then there exists some interval I_0: $(x_0 - h,\ x_0 + h)$, $h > 0$, contained in $[a, b]$, and a unique function $y(x)$, defined on I_0, that is a solution of the IVP $dy/dx = f(x, y)$, $y(x_0) = y_0$

Example First-order IVP

- $y(x) = ce^x$ is a one-parameter family of solutions of the first-order DE $y' = y$
- Find a solution of the first-order IVP with initial condition $y(0) = 3$

Solution

- From the initial condition, we obtain $3 = ce^0$
- Solving, we find $c = 3$
- The solution of the IVP is $y = 3e^x$

Example Second-order IVP

- $x(t) = c_1 \cos 4t + c_2 \sin 4t$ is a two-parameter family of solutions of the second-order DE $x'' + 16x = 0$
- Find a solution of the second-order IVP with initial conditions $x\left(\frac{\pi}{2}\right) = -2,\ x'\left(\frac{\pi}{2}\right) = 1$

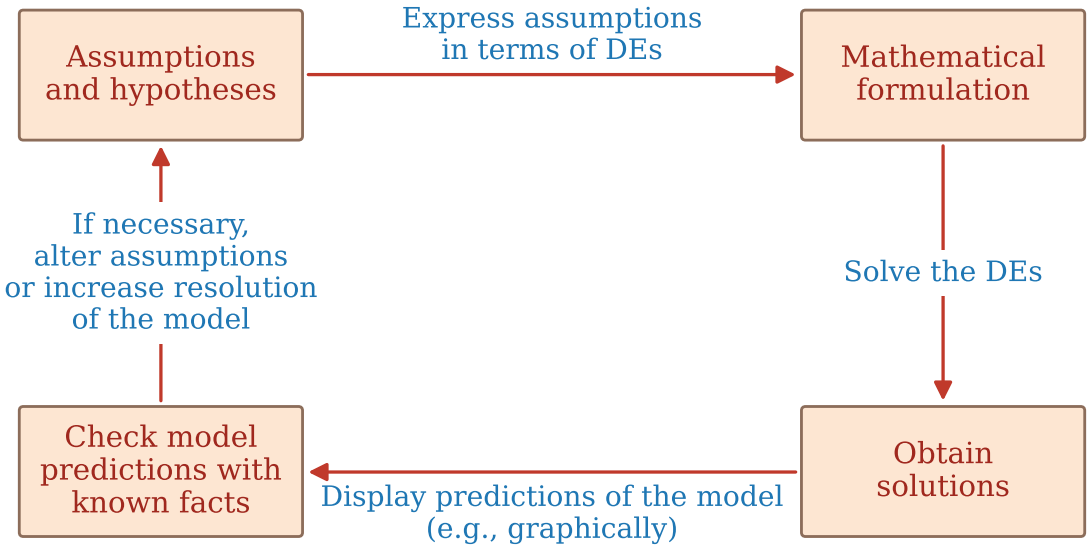
Solution

- Substituting for initial conditions and solving for constants, we find $c_1 = -2$ and $c_2 = 1/4$
- The solution of the IVP is $x(t) = -2\cos 4t + \frac{1}{4}\sin 4t$

Try it in Python: Solve initial-value problems symbolically with `sympy.dsolve(..., ics=...)`: Symbolic solution to ODEs

1.3 Differential Equations as Mathematical Models

- A **mathematical model** is a description of a system or a phenomenon
- Differential equation models are used to describe behavior in various fields
  - Biology
  - Chemistry
  - Physics
- Steps of the modeling process



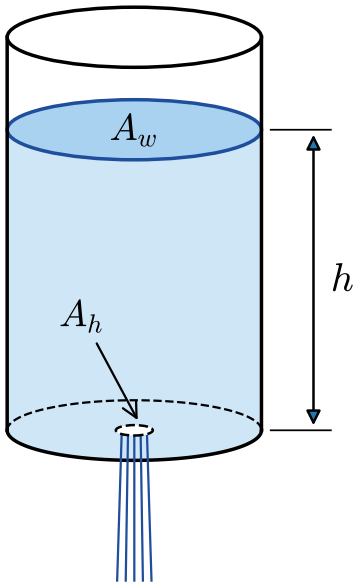
Example Radioactive Decay

- In modeling the phenomenon, it is assumed that the rate $\frac{dA}{dt}$ at which the nuclei of a substance decay is proportional to the amount $A(t)$ remaining at time t given an initial amount of radioactive substance on hand A_0

$$\frac{dA}{dt} = -kA, \; A(0) = A_0$$

- This differential equation also describes a first-order chemical reaction

Example Draining Tank



- Torricelli's law states that the exit speed v of water through a sharp-edged hole at the bottom of a tank filled to a depth h is the same as the speed it would acquire falling from a height h, $v = \sqrt{2gh}$
- The volume V of water leaving the tank per second is the area of the hole A_h times the exit speed

$$\frac{dV}{dt} = -A_h\sqrt{2gh}$$

- The volume in the tank at t is $V(t) = A_w h$, in which A_w is the *constant* area of the upper water surface
- Combining expressions gives the differential equation for the height of water at time t

$$\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2gh}$$

Summary

Concept	Key point
Order	order of the highest derivative in the equation
Linear ODE	$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$: linear in $y, y', \cdots, y^{(n)}$
Solution	a function ϕ on an interval I that reduces the ODE to an identity; explicit $y = \phi(x)$ or implicit $G(x, y) = 0$
n-parameter family	$G(x, y, c_1, \cdots, c_n) = 0$ solves an n-th order ODE
Initial-value problem	ODE with $y(x_0) = y_0,\ y'(x_0) = y_1,\ \cdots,\ y^{(n-1)}(x_0) = y_{n-1}$
Existence and uniqueness	$y' = f(x, y),\ y(x_0) = y_0$: if f and $\partial f/\partial y$ are continuous on a rectangle containing (x_0, y_0), there is exactly one solution near x_0

Concept	Key point
Models	radioactive decay $\dfrac{dA}{dt} = -kA$; draining tank $\dfrac{dh}{dt} = -\dfrac{A_h}{A_w}\sqrt{2gh}$

Worked Exercises

1.2: 1. $y = 1/(1 + c_1 e^{-x})$ is a one-parameter family of solutions of the first-order DE $y' = y - y^2$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition:

$$y(0) = -\frac{1}{3}$$

Solution

$$\begin{aligned} y &= \frac{1}{1 + c_1 e^{-x}} \\ &\Downarrow \text{ at } x{=}0 \\ -\frac{1}{3} &= \frac{1}{1 + c_1} \\ &\Downarrow \\ c_1 &= -4 \\ &\Downarrow \\ y &= \frac{1}{1 - 4e^{-x}} \end{aligned}$$

1.2: 3. $y = 1/(x^2 + c)$ is a one-parameter family of solutions of the first-order DE $y' + 2xy^2 = 0$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition. Give the largest interval I over which the solution is defined

$$y(2) = \frac{1}{3}$$

Solution

$$\begin{aligned} y &= \frac{1}{x^2 + c} \\ &\Downarrow \text{ at } x{=}2 \\ \frac{1}{3} &= \frac{1}{4 + c} \\ &\Downarrow \\ c = -1\,, &\quad I = (1, \infty) \\ &\Downarrow \\ y &= \frac{1}{x^2 - 1} \end{aligned}$$

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-5, 5, 200)
y = 1.0 /(x**2 -1)

threshold = 20
y[y > threshold] = np.inf
y[y <-threshold] = np.inf

plt.plot(x,y)
plt.scatter(2, 1/3, color='red')

plt.title(r'$y=\frac{1}{x^2 -1}$')
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.grid()

plt.show()
```

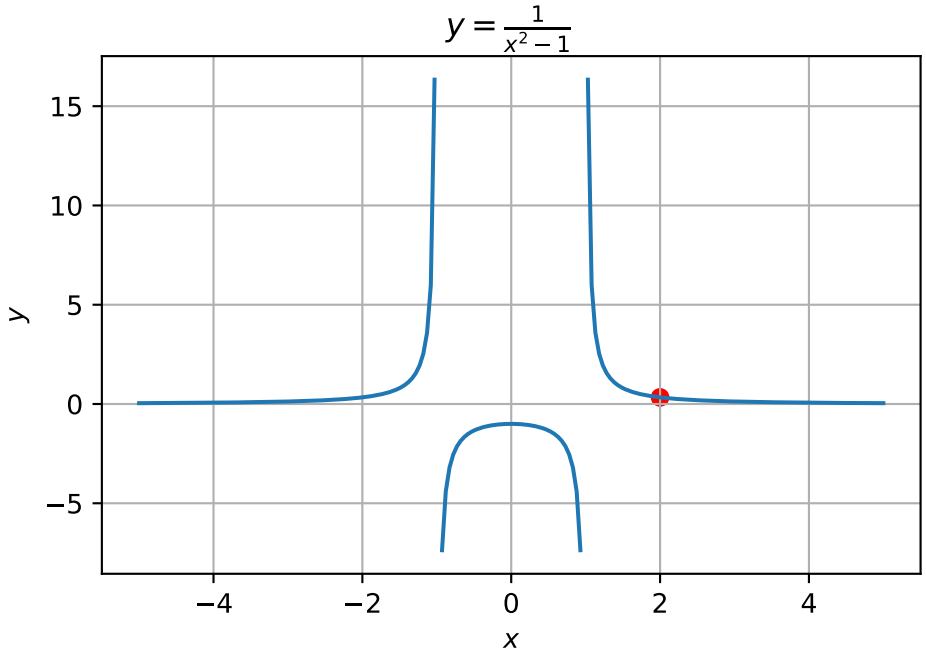


Figure 1.1: Exercises 1.2: 3

1.2: 7. $x(t) = c_1 \cos t + c_2 \sin t$ is a two-parameter family of solutions of the second-order DE $x'' + x = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions

$$x(0) = -1,\ x'(0) = 8$$

Solution

$$
\begin{aligned}
x(t) &= c_1 \cos t + c_2 \sin t \\
&\Downarrow \ {\scriptstyle x(0)=-1,\, x'(0)=8} \\
-1 &= c_1 \\
8 &= c_2 \\
&\Downarrow \\
x(t) &= -\cos t + 8 \sin t
\end{aligned}
$$

1.2: 17. Determine a region of the xy-plane for which the given differential equation would have a unique solution whose graph passes through a point (x_0, y_0) in the region.

$$\frac{dy}{dx} = y^{2/3}$$

Solution

Step 1: Identify the Function and Its Partial Derivative

Let:

$$f(x, y) = y^{2/3}$$

Then the partial derivative with respect to y is:

$$\frac{\partial f}{\partial y} = \frac{2}{3} y^{-1/3}$$

Step 2: Analyze Continuity

- $f(x, y) = y^{2/3}$ is continuous for all real values of x and y
- However, $\frac{\partial f}{\partial y} = \frac{2}{3} y^{-1/3}$ is undefined at $y = 0$, since $y^{-1/3}$ is not defined at $y = 0$

Step 3: Apply the Existence and Uniqueness Theorem

- Both f and $\partial f/\partial y$ are continuous in the half-plane $y > 0$ and in the half-plane $y < 0$
- Hence the theorem guarantees a unique solution through every point (x_0, y_0) with $y_0 \neq 0$: the region can be taken as either half-plane $y > 0$ or $y < 0$
- The theorem gives only a sufficient condition, so it says nothing at $y_0 = 0$. There uniqueness actually fails: both $y = 0$ and $y = (x - x_0)^3/27$ satisfy $y' = y^{2/3}$ and $y(x_0) = 0$

1.2: 29.

1. By inspection, find a one-parameter family of solutions of the differential equation $xy' = y$. Verify that each member of the family is a solution of the initial value problem $xy' = y$, $y(0) = 0$
2. Explain part 1. by determining a region R in the xy-plane for which the differential equation $xy' = y$ would have a unique solution through a point (x_0, y_0) in R

3. Verify that the piecewise-defined function

$$y = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

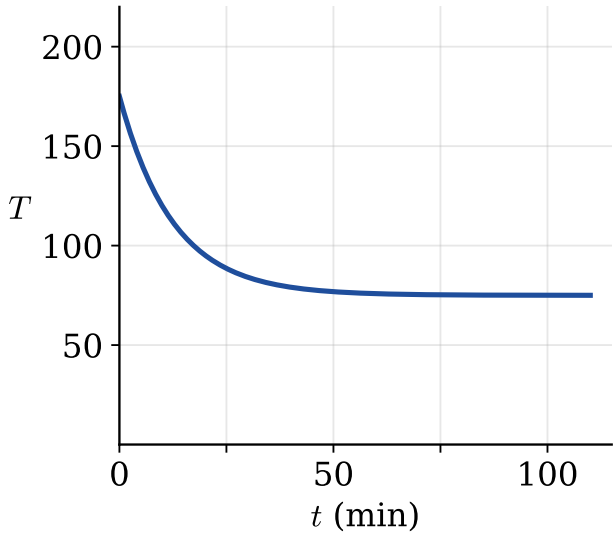
satisfies the condition $y(0) = 0$. Determine whether this function is also a solution of the initial-value problem in part 1

Solution

1. By inspection, $y = cx$ satisfies $xy' = x \cdot c = y$ for every constant c. Every member satisfies $y(0) = c \cdot 0 = 0$, so the IVP $xy' = y$, $y(0) = 0$ has infinitely many solutions
2. In normal form $y' = f(x, y) = y/x$, and f and $\partial f/\partial y = 1/x$ are continuous wherever $x \neq 0$. So a region R is either half-plane $x > 0$ or $x < 0$; through any point (x_0, y_0) with $x_0 \neq 0$ there is a unique solution. The initial point $(0, 0)$ lies in no such region, so the theorem does not apply there, which is consistent with part 1
3. The piecewise function satisfies $y(0) = 0$, and it satisfies $xy' = y$ for $x < 0$ $(0 = 0)$ and for $x > 0$ $(x \cdot 1 = x)$. However, its left derivative at $x = 0$ is 0 and its right derivative is 1, so $y'(0)$ does not exist. Therefore it is **not** a solution of the IVP on any interval containing $x = 0$

1.3: 5. A cup of coffee cools according to Newton's law of cooling. Use data from the graph of the temperature $T(t)$ in figure to estimate the constants T_m, T_0, and k in a model of the form of the first-order initial-value problem

$$\frac{dT}{dt} = k(T - T_m), \quad T(0) = T_0$$



Solution

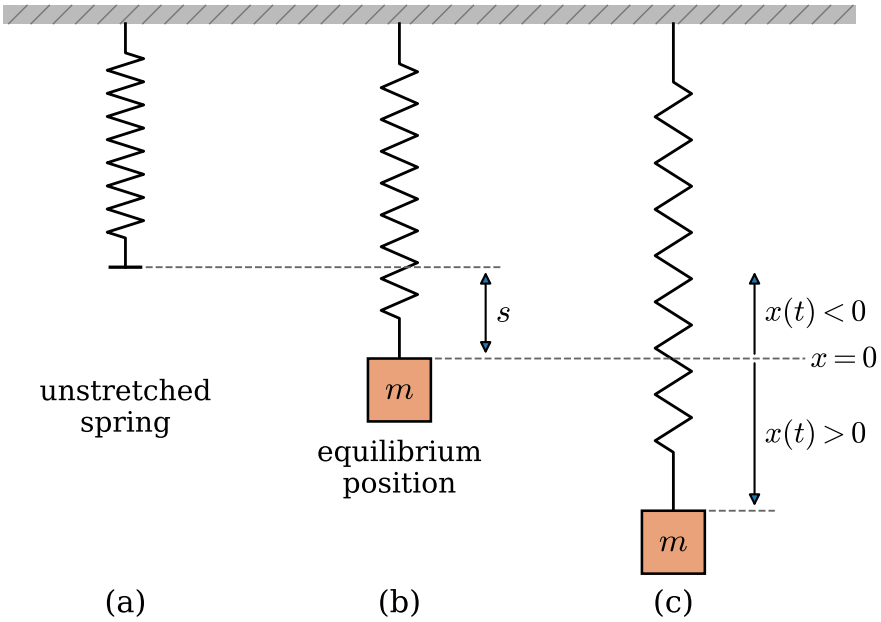
- The solution of the IVP is $T(t) = T_m + (T_0 - T_m)e^{kt}$
- $T_0 = T(0)$ is read at $t = 0$: $T_0 \approx 175$

- $T_m = \lim_{t\to\infty} T(t)$ is the level the curve approaches: $T_m \approx 75$
- k follows from one more point on the curve. The curve reaches $T \approx 100$ at $t \approx 17$ min, so

$$k = \frac{1}{t}\ln\frac{T - T_m}{T_0 - T_m} \approx \frac{1}{17}\ln\frac{100 - 75}{175 - 75} = \frac{\ln 0.25}{17} \approx -0.08 \text{ min}^{-1}$$

(equivalently, k is the slope of $\ln[(T - T_m)/(T_0 - T_m)]$ plotted against t)

1.3: 19. After a mass m is attached to a spring, it stretches s units and then hangs at rest in the equilibrium position as shown in Figure (b). After the spring/mass system has been set in motion, let $x(t)$ denote the directed distance of the mass beyond the equilibrium position. As indicated in Figure (c), assume that the downward direction is positive, that the motion takes place in a vertical straight line through the center of gravity of the mass, and that the only forces acting on the system are the weight of the mass and the restoring force of the stretched spring. Use Hooke's law: The restoring force of a spring is proportional to its total elongation. Determine a differential equation for the displacement $x(t)$ at time t



Solution

- **Equilibrium (Figure (b)):** the spring is stretched by s and the weight is balanced by the spring force (Hooke's law, spring constant k)

$$mg = ks$$

- **In motion (Figure (c)):** the total elongation is $s + x$ (downward positive), so the forces on the mass are the weight mg (down) and the restoring force $-k(s + x)$. Newton's second law gives

$$m\frac{d^2x}{dt^2} = mg - k(s + x) = \underbrace{mg - ks}_{0} - kx$$

- Hence

$$m\frac{d^2x}{dt^2} + kx = 0$$

Measuring x from the equilibrium position removes the weight from the equation

2 First-Order Differential Equations

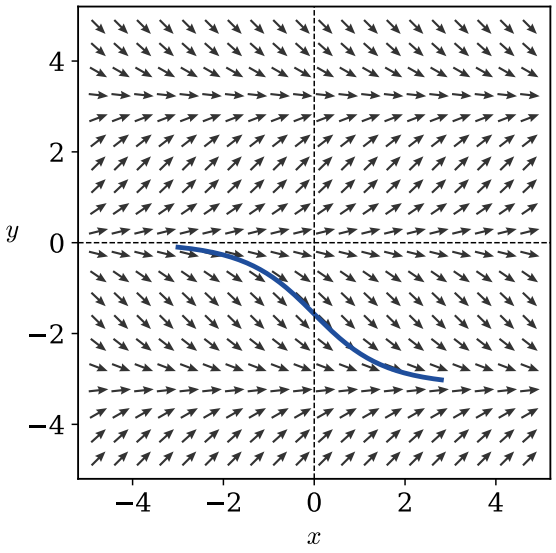
Learning objectives After this chapter you should be able to

- sketch solution curves from a direction field, and find and classify the critical points of an autonomous equation $dy/dx = f(y)$
- recognize separable, linear, exact, homogeneous, Bernoulli and $y' = f(Ax + By + C)$ equations, and solve each by its method
- approximate a solution by Euler's method and see how the step size h affects the error
- set up and solve first-order models: growth and decay, series circuits, the logistic equation, radioactive series and predator-prey systems

2.1 Solution Curves Without a Solution

- DEs can be analyzed qualitatively, allowing us to approximate a solution curve without solving the problem
- Two approaches are:
  - Direction fields
  - Autonomous first-order DEs
- Direction fields:
  - Slope of the **lineal element** at $(x, y(x))$ on a solution curve is the value of $\frac{dy}{dx}$ at this point
  - **Direction/slope fields** of $\frac{dy}{dx} = f(x, y)$ are collections of lineal slope elements that visually suggest the shape of a family of solution curves

For example, $\dfrac{dy}{dx} = \sin y$

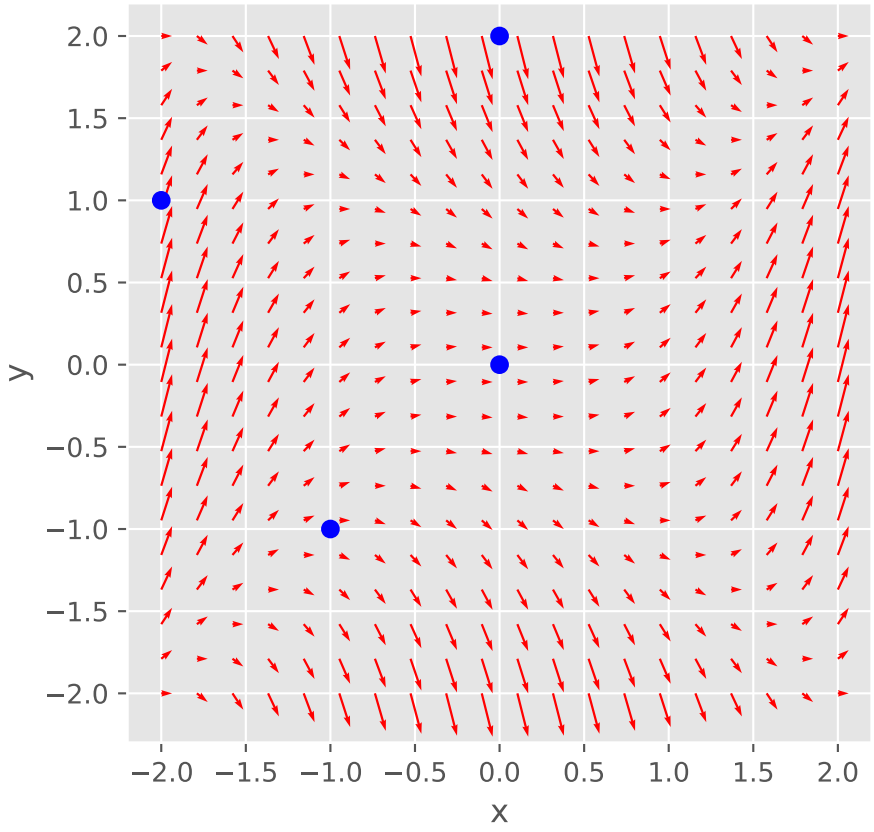


- Using the given computer-generated direction field, sketch, by hand, an approximate solution curve that passes through each of the indicated points:

$$\frac{dy}{dx} = x^2 - y^2$$

(a) $y(-2) = 1$
(b) $y(-1) = -1$
(c) $y(0) = 2$
(d) $y(0) = 0$

```
import numpy as np
import matplotlib.pyplot as plt
plt.style.use('ggplot')

x = np.linspace(-2, 2, 20)
y = np.linspace(-2, 2, 20)
X, Y = np.meshgrid(x, y)

dy = (X*X -Y*Y)
dx = np.ones(dy.shape)

plt.figure(figsize=(5, 5))
plt.quiver(X, Y, dx, dy, color='red')
plt.plot([-2, -1, 0, 0], [1, -1, 2, 0], 'o', color="blue")
plt.xlim([-2.2, 2.2])
plt.ylim([-2.3, 2.2])
plt.xticks(np.arange(-2, 2.5, 0.5))
plt.yticks(np.arange(-2, 2.5, 0.5))
plt.xlabel('x')
plt.ylabel('y')

plt.show()
```



Answer 1 (end of chapter) · Solution

- **Autonomous** first-order DEs, $\dfrac{dy}{dx} = f(y)$

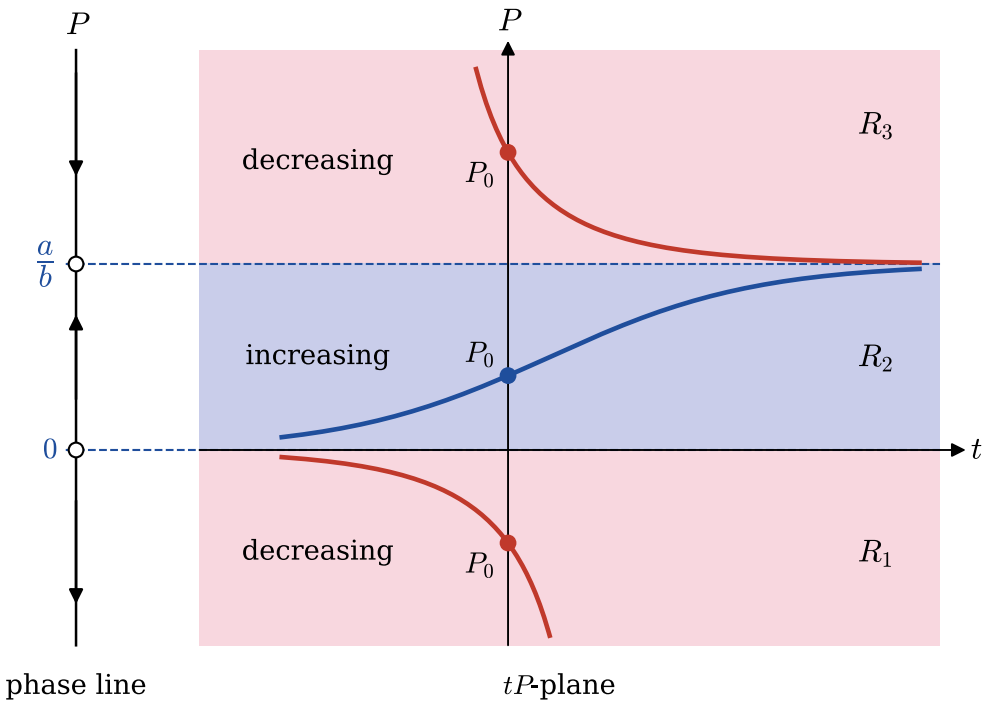
  An ODE in which the independent variable does not appear explicitly

$$\frac{dy}{dx} = 1 + y^2 \quad \text{autonomous}$$
$$\frac{dy}{dx} = 0.2\,xy \quad \text{nonautonomous}$$

- **Critical points**, $f(c) = 0$, are constant (or equilibrium) solutions of autonomous DEs
- A **phase portrait** (phase line) is made by putting the critical points on a vertical line, with arrows pointing up or down depending on the sign of $f(y)$ on the intervals between the points
- The horizontal lines $y = c$ through the critical points divide the xy-plane into sub-regions (horizontal strips) R_i
- Some conclusions can be drawn about nonconstant solution curves to autonomous DEs
  - If a solution $y(x)$ passes through (x_0, y_0) in sub-region R_i, then $y(x)$ remains in R_i
  - By continuity, $f(y)$ cannot change signs in a sub-region R_i
  - Since $f(y)$ is either positive or negative in R_i, a solution is either increasing or decreasing and has no relative extremum

Example Phase portrait and solution curves

$$\frac{dP}{dt} = P(a - bP)$$



Example Consider the autonomous first-order differential equation

$$\frac{dy}{dx} = y^2 - y^4$$

and the initial condition $y(0) = y_0$. Sketch the graph of a typical solution $y(x)$ when y_0 has the given values

$$\begin{aligned} (a) &\quad y_0 < -1 \\ (b) &\quad -1 < y_0 < 0 \\ (c) &\quad 0 < y_0 < 1 \\ (d) &\quad 1 < y_0 \end{aligned}$$

Answer 2 (end of chapter) · Solution

Try it in Python: Draw direction fields for any $dy/dx = f(x, y)$: Direction fields

2.2 Separable Equations

- Consider $\dfrac{dy}{dx} = f(x)$
  - When f does not depend on y, $\dfrac{dy}{dx} = f(x)$, which can be solved by integration
  - The solution $y = \displaystyle\int f(x)dx = F(x) + c$, where $F(x)$ is an antiderivative (indefinite integral)
  - Some functions, termed **nonelementary**, do not possess an antiderivative that is an elementary function
- A first-order DE of the form $\dfrac{dy}{dx} = g(x)h(y)$ is said to be **separable**, or have **separable variables**
  - A separable equation can be rewritten in the form

$$\frac{1}{h(y)}dy = g(x)dx$$

    which is solved by integrating both sides

Example Solve a separable equation $\dfrac{dy}{dx} = y^2 - 9, \;\; y(0) = 0$

- Separating and using partial fractions

$$\frac{dy}{(y-3)(y+3)} = dx$$
$$\Downarrow$$
$$\frac{1}{6}\left[\frac{1}{y-3} - \frac{1}{y+3}\right]dy = dx$$

- Integrating and solving for y yields

$$\frac{1}{6}\ln\left|\frac{y-3}{y+3}\right| = x + c_1$$
$$\Downarrow c = \mp e^{6c_1}$$
$$y = 3\frac{1 - ce^{6x}}{1 + ce^{6x}}$$

- Finally, applying $y(0) = 0$ gives $c = 1$

Example Solve the given differential equation by separation of variables

- $\dfrac{dy}{dx} = \sin 5x$
- $dx + e^{3x}dy = 0$

- $\dfrac{dy}{dx} = x\sqrt{1-y^2}$

Answer 3 (end of chapter) · Solution

```
import sympy
from sympy import pi, dsolve
sympy.init_printing()

x = sympy.Symbol('x')
y = sympy.Function('y')

eq = y(x).diff(x) -x *sympy.sqrt(1 -y(x) *y(x))

dsolve(eq, hint='separable')
```

$$y(x) = \sin\left(C_1 + \frac{x^2}{2}\right)$$

Example Find an implicit and an explicit solution of the given initial-value problem

- $x^2\dfrac{dy}{dx} = y - xy, \quad y(-1) = -1$
- $\dfrac{dx}{dt} = 4(x^2+1), \quad x(\pi/4) = 1$

Answer 4 (end of chapter) · Solution

```
t = sympy.Symbol('t')
x = sympy.Function('x')

eq = x(t).diff(t) -4 *(x(t)**2 +1)

dsolve(eq, ics={x(pi/4): 1}, hint='separable')
```

$$x(t) = \tan\left(4t + \frac{\pi}{4}\right)$$

Try it in Python: Newton's law of cooling and other first-order equations with `sympy.dsolve`, including initial conditions: Symbolic solution to ODEs

2.3 Linear Equations

A first-order DE of the form $a_1(x)\dfrac{dy}{dx} + a_0(x)y = g(x)$ is a **linear equation** in the dependent variable y

- The DE is **homogeneous** when $g(x) = 0$; otherwise, it is **nonhomogeneous**
- The standard form of a linear DE is obtained by dividing both sides by the lead coefficient

$$\frac{dy}{dx} + P(x)y = f(x)$$

The **standard form** equation has the property that its solution y is the sum of the solution of the associated homogeneous equation y_h and the particular solution of the nonhomogeneous equation y_p :

$$y = y_h + y_p$$

- The homogeneous equation $\dfrac{dy_h}{dx} + P(x)y_h = 0$ is separable, allowing us to solve for y_h

$$\frac{dy_h}{y_h} = -P(x)dx$$
$$\Downarrow$$
$$\ln|y_h| = -\int P(x)\,dx + c \;\Rightarrow\; y_h = \bar{c}\exp\left(-\int P(x)\,dx\right)$$

- **Variation of parameters** $y_p = u(x)y_h$ can be used to find a particular solution y_p of the nonhomogeneous equation

$$y_h\frac{du}{dx} + \underbrace{\left(\frac{dy_h}{dx} + P(x)y_h\right)}_{=0}u = f(x)$$
$$du = \frac{f(x)}{y_h}dx \;\Rightarrow\; u(x) = \int\frac{f(x)}{y_h(x)}dx$$
$$\Downarrow$$
$$y_p = y_h\int\frac{f(x)}{y_h(x)}dx$$

Example Find the general solution of the given differential equation:

- $\dfrac{dy}{dx} + 2y = 0$
- $y' + 2xy = x^3$
- $x\dfrac{dy}{dx} + 2y = 3$
- $xy' + (1+x)y = e^{-x}\sin 2x$

Answer 5 (end of chapter) · Solution

Example Solve the given initial-value problem. Give the largest interval I over which the solution is defined

- $y\dfrac{dx}{dy} - x = 2y^2, \quad y(1) = 5$
- $(x+1)\dfrac{dy}{dx} + y = \ln x, \quad y(1) = 10$

Answer 6 (end of chapter) · Solution

```
x = sympy.Symbol('x')
y = sympy.Function('y')

eq = (x + 1) *y(x).diff(x) +y(x) -sympy.log(x)

dsolve(eq, ics={y(1): 10}, hint='1st_linear')
```

$$y(x) = \frac{x\log(x) - x + 21}{x+1}$$

Example The given differential equation is not linear in y. Nevertheless, find a general solution of the equation

- $dx = (x + y^2)dy$
- $ydx + (2x + xy - 3)dy = 0$

Answer 7 (end of chapter) · Solution

Example The sine integral function is defined as

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$$

where the integrand is defined to be 1 at $x = 0$. Express the solution of the initial value problem

$$x^3\frac{dy}{dx} + 2x^2 y = 10\sin x, \quad y(1) = 0$$

in terms of $\mathrm{Si}(x)$

Answer 8 (end of chapter) · Solution

2.4 Exact Equations

- A differential expression $M(x,y)dx + N(x,y)dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x,y)$:

$$df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

- **Criterion for an exact differential:** if M, N and their first partial derivatives are continuous in a rectangular region R, then $M\,dx + N\,dy$ is exact in R if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- $M(x,y)\,dx + N(x,y)\,dy = 0$ is an **exact equation** if the left side is an exact differential

Example Solving an exact DE, $2xy\,dx + (x^2-1)\,dy = 0$

Answer 9 (end of chapter) · Solution

- **Integrating Factor** of the first-order linear DE

$$\begin{gathered}
\frac{dy}{dx} + P(x)y = f(x) \\
\Downarrow \\
(P(x)y - f(x))\,dx + dy = 0 \\
\Downarrow \times I(x): \text{ Integrating Factor} \\
I(x)\,(P(x)y - f(x))\,dx + I(x)dy = 0
\end{gathered}$$

To be an exact equation $\Downarrow$

$$\frac{\partial}{\partial y}\{I(x)\,(P(x)y - f(x))\} = I(x)P(x) = \frac{d}{dx}I(x)$$

$$\Downarrow \quad I(x) = \exp\left(\int P(x)dx\right)$$

Then

$$I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)f(x) \;\Rightarrow\; \frac{d}{dx}\{I(x)y\} = I(x)f(x)$$

$$\Downarrow$$

$$y(x) = I(x)^{-1}y(x_0)I(x_0) + I(x)^{-1}\int_{x_0}^{x} I(t)f(t)\,dt$$

Example Solve $\dfrac{dy}{dx} - 2xy = 2,\ y(0) = 1$

$$\begin{aligned}
&\frac{dy}{dx} - 2xy = 2 \\
&\qquad\qquad \Downarrow \times e^{-x^2} \\
&\frac{d}{dx}[e^{-x^2} y] = 2e^{-x^2} \\
&\qquad y = ce^{x^2} + 2e^{x^2} \int_0^x e^{-t^2}\, dt \\
&\qquad \Downarrow y(0) = 1 \to c = 1 \\
&\qquad y = e^{x^2} \left[1 + \sqrt{\pi} \underbrace{\left(\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \right)}_{\operatorname{erf}(x)} \right] \\
&\qquad = e^{x^2} \left[1 + \sqrt{\pi} \operatorname{erf}(x) \right]
\end{aligned}$$

Example Determine whether the given differential equation is exact. If it is exact, solve it

- $(2x - 1)dx + (3y + 7)dy = 0$
- $(5x + 4y)dx + (4x - 8y^2)dy = 0$
- $(2xy^2 - 3)dx + (2x^2 y + 4)dy = 0$
- $(x^2 - y^2)dx + (x^2 - 2xy)dy = 0$

Answer 10 (end of chapter) · Solution

```
x = sympy.Symbol('x')
y = sympy.Function('y')

eq = (2 *x -1) +(3 *y(x) +7) *y(x).diff(x)

dsolve(eq, hint='1st_exact')
```

$$\left[y(x) = -\frac{\sqrt{C_1 - 6x^2 + 6x}}{3} - \frac{7}{3},\ y(x) = \frac{\sqrt{C_1 - 6x^2 + 6x}}{3} - \frac{7}{3} \right]$$

Example Solve the given initial-value problem

- $(x + y)^2 dx + (2xy + x^2 - 1)dy = 0,\ \ y(1) = 1$
- $(4y + 2t - 5)dt + (6y + 4t - 1)dy = 0,\ \ y(-1) = 2$

Answer 11 (end of chapter) · Solution

Example Solve the given differential equation by finding an appropriate integrating factor

- $y(x + y + 1)dx + (x + 2y)dy = 0$

Answer 12 (end of chapter) · Solution

2.5 Solutions by Substitutions

Substitution is often used to get a DE in a form that a known procedure can be used to find a solution

- Reduction to separation of variables can be facilitated in the DE

$$\frac{dy}{dx} = f(Ax + By + C)$$

by substituting $u = Ax + By + C,\ B \neq 0$

Example Solve the IVP $\frac{dy}{dx} = (-2x + y)^2 - 7,\ y(0) = 0$

Let $u = -2x + y$, then $\frac{du}{dx} = -2 + \frac{dy}{dx}$ giving $\frac{du}{dx} = u^2 - 9$

Answer 13 (end of chapter) · Solution

```
x = sympy.Symbol('x')
y = sympy.Function('y')

eq = y(x).diff(x) -(-2 *x +y(x))**2 +7

dsolve(eq, ics={y(0): 0})
```

$$y(x) = \frac{-2xe^{6x} - 2x + 3e^{6x} - 3}{-e^{6x} - 1}$$

- **Homogeneous** first-order DE

  A first-order ordinary DE in the form:

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

  is of homogeneous type if both functions $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree n:

$$M(\lambda x, \lambda y) = \lambda^n M(x,y), \quad N(\lambda x, \lambda y) = \lambda^n N(x,y)$$

  Thus, we can let $t = 1/x$ to simplify this quotient to a function f of the single variable y/x:

$$\frac{M(x,y)}{N(x,y)} = \frac{M(tx,ty)}{N(tx,ty)} = \frac{M\left(1, \frac{y}{x}\right)}{N\left(1, \frac{y}{x}\right)} = -f\left(\frac{y}{x}\right) = -\frac{dy}{dx}$$

The change of variables $y = ux$ transforms the original differential equation into the separable form:

$$\frac{du}{f(u) - u} = \frac{dx}{x}$$

Example Solve $(x^2 + y^2)dx + (x^2 - xy)dy = 0$

Answer 14 (end of chapter) · Solution

- **Bernoulli** DE:

$$y' + P(x)y = Q(x)y^n \quad \text{where} \quad n \neq 0 \quad \text{and} \quad n \neq 1$$

  - Bernoulli equations are special because they are nonlinear differential equations with known exact solutions
  - The substitution $u = y^{1-n}$ reduces any Bernoulli equation to a linear differential equation

Example Solve $y' - \frac{2}{x}y = -x^2y^2$

Changing variables $u = \frac{1}{y}$, $u' = -\frac{1}{y^2}y'$ gives the equation

$$u' + \frac{2}{x}u = x^2$$

Answer 15 (end of chapter) · Solution

```
x = sympy.Symbol('x')
y = sympy.Function('y')

eq = y(x).diff(x) -2 /x *y(x) +x**2 *y(x)**2

dsolve(eq, hint='Bernoulli')
```

$$y(x) = \frac{5x^2}{C_1 + x^5}$$

- **Riccati** DE

– Riccati equation is any first-order ordinary differential equation that is quadratic in the unknown function. In other words, it is an equation of the form

$$y' = q_0(x) + q_1(x)y + q_2(x)y^2$$

where $q_0(x) \neq 0$ and $q_2(x) \neq 0$. If $q_0(x) = 0$, it is a Bernoulli equation, and if $q_2(x) = 0$, it is a linear equation

– The new variable $v = yq_2$ satisfies an equation

$$v' = v^2 + R(x)v + S(x)$$

where $S(x) = q_2 q_0$ and $R(x) = q_1 + \left(\frac{q_2'}{q_2}\right)$

– Substituting $v = -\frac{u'}{u}$, u satisfies the linear 2^{nd} order ODE

$$u'' - R(x)u' + S(x)u = 0$$

A solution of this equation will lead to a solution $y = -\frac{u'}{q_2 u}$ of the original Riccati equation

Example Solve the given differential equation by using an appropriate substitution

- $(x - y)dx + xdy = 0$
- $xdx + (y - 2x)dy = 0$
- $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$
- $x\dfrac{dy}{dx} = y + \sqrt{x^2 - y^2}, \quad x > 0$

Answer 16 (end of chapter) · Solution

```
x = sympy.Symbol('x')
y = sympy.Function('y')

eq = x *y(x).diff(x) -y(x) -sympy.sqrt(x**2 -y(x)**2)

dsolve(eq, hint='1st_homogeneous_coeff_best')
```

$$y(x) = -x \sin\left(C_1 - \log\left(x\right)\right)$$

Example Solve the given initial-value problem

- $xy^2\dfrac{dy}{dx} = y^3 - x^3, \quad y(1) = 2$
- $(x^2 + 2y^2)\dfrac{dx}{dy} = xy, \quad y(-1) = 1$

Answer 17 (end of chapter) · Solution

Example Solve the given differential equation by using an appropriate substitution

- $x\dfrac{dy}{dx} + y = \dfrac{1}{y^2}$
- $3(1+t^2)\dfrac{dy}{dt} = 2ty(y^3 - 1)$
- $\dfrac{dy}{dx} = \cos(x+y), \quad y(0) = \dfrac{\pi}{4}$

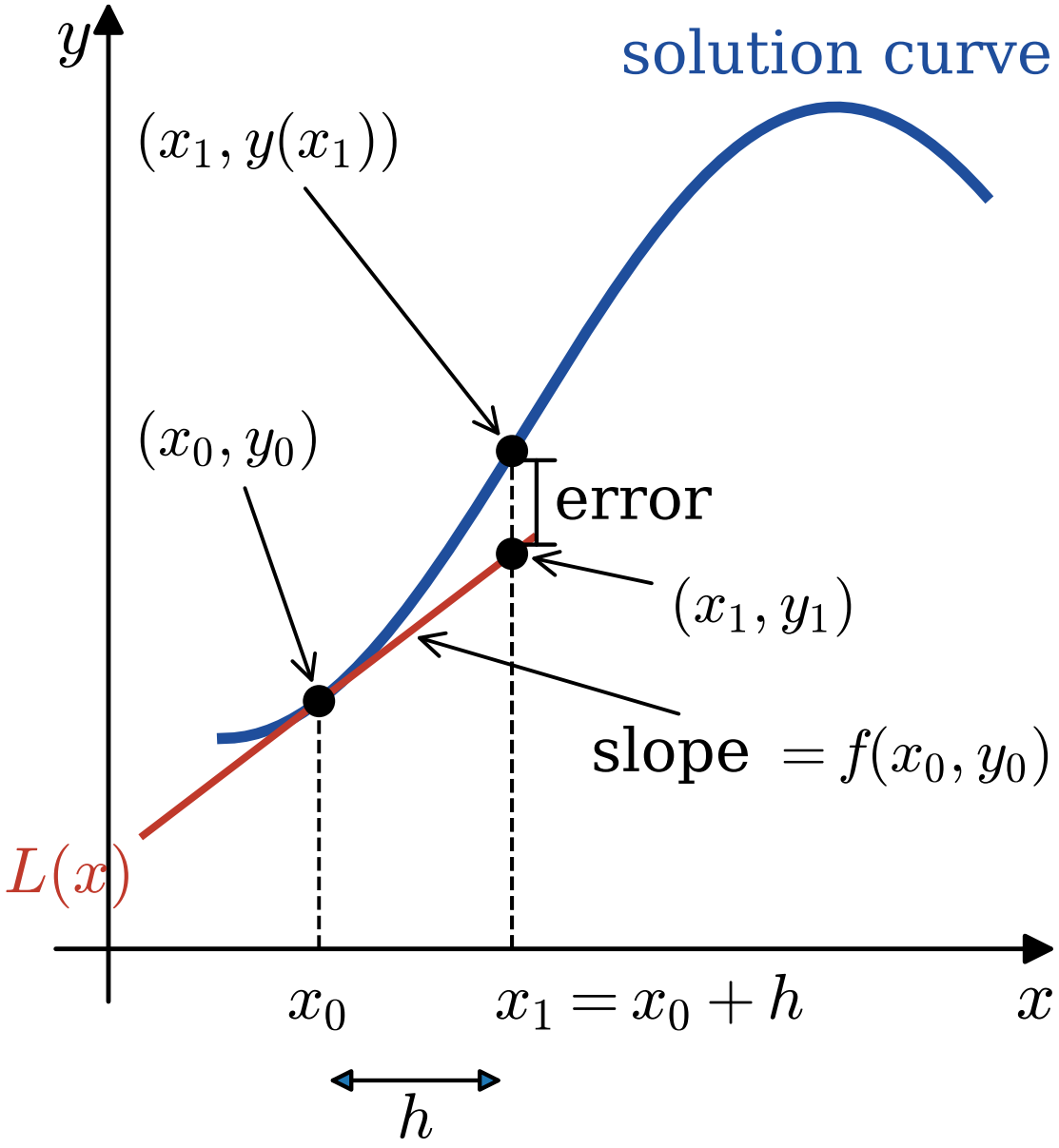
Answer 18 (end of chapter) · Solution

Try it in Python: A Riccati equation reduced in `sympy` to a second-order linear (Airy) equation by the substitution $y = -u'/(q_2 u)$: Symbolic solution to ODEs

2.6 A Numerical Method

Numerical method: an algorithm for approximating the unknown solution to a DE

- **Linearization** approximates solutions within a small area around lineal elements in direction fields
- The procedure of evaluating successive tangent lines is **Euler's method**



Example Consider $y' = 0.1\sqrt{y} + 0.4x^2$, $y(2) = 4$ and approximate $y(2.5)$ using $h = 0.1$

- Substituting into the general formula for Euler's method gives

$$y_{n+1} = y_n + h\left(0.1\sqrt{y_n} + 0.4x_n^2\right)$$

- Considering the initial condition and $n = 0$

$$\begin{aligned} y_1 &= 4 + 0.1\left(0.1\sqrt{4} + 0.4 \cdot 2^2\right) = 4.1800 \\ y_2 &= 4.18 + 0.1\left(0.1\sqrt{4.18} + 0.4 \cdot 2.1^2\right) = 4.3768 \\ &\vdots \\ y_5 &= 5.0768 \end{aligned}$$

```
x0 = 2.0
y0 = 4.0
xf = 2.5

h = 0.1
n = int((xf -x0) /h) + 1

x = np.linspace(x0, xf, n)
y = np.zeros(n)

y[0] = y0
for i in range(1, n):
    y[i] = y[i -1] +h *(0.1 *np.sqrt(y[i -1]) +0.4 *x[i -1]**2)
    print(f'x = {x[i]: 3.1f}, y = {y[i]: 5.4f}')

plt.figure(figsize=(4, 4))
plt.plot(x, y, 'o')
plt.xlabel("$x$")
plt.ylabel("$y$")
plt.title("$h=0.1$")

plt.show()
```

```
x =  2.1, y =  4.1800
x =  2.2, y =  4.3768
x =  2.3, y =  4.5914
x =  2.4, y =  4.8244
x =  2.5, y =  5.0768
```

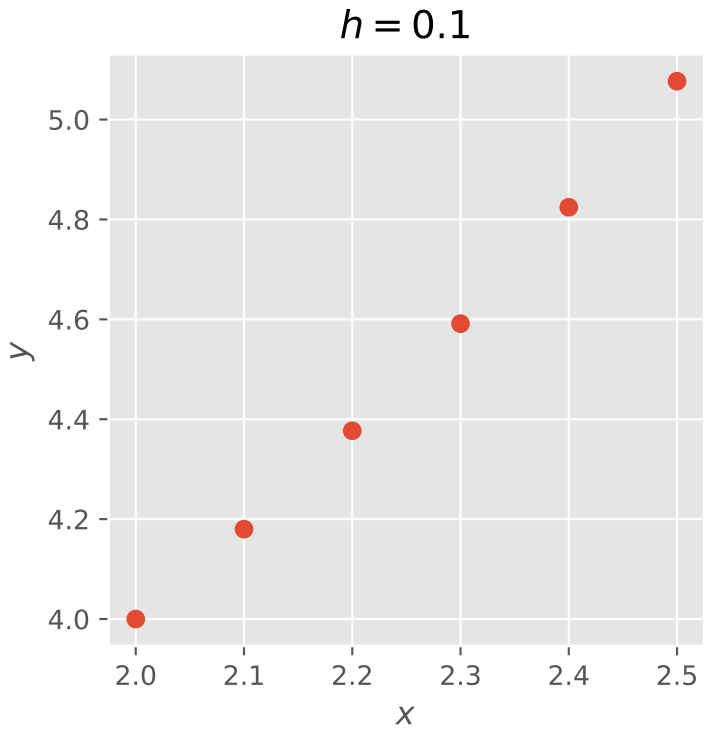


Figure 2.1: Euler's Method

Example Use Euler's method to obtain a four-decimal approximation of the indicated value. Carry out the recursion, first using $h = 0.1$ and then using $h = 0.05$

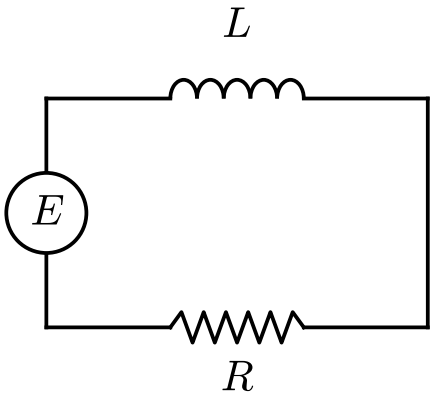
$$y' = 2x - 3y + 1, \quad y(1) = 5; \quad y(1.2)$$

Answer 19 (end of chapter) · Solution

Try it in Python: Euler's method, Runge-Kutta and multistep methods: Numerical methods for solving ODEs; adaptive solvers with `scipy.integrate.solve_ivp`: Numerical integration of ODEs using 'scipy'

2.7 Linear Models

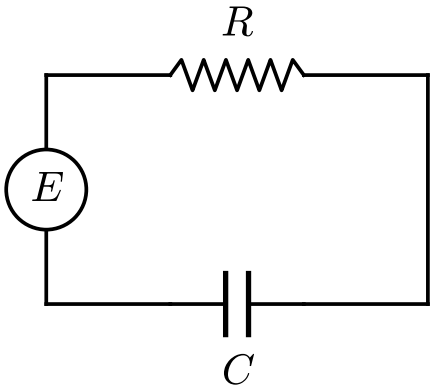
Example Series Circuits - For a series circuit containing a resistor R and an inductor L,



Kirchhoff's second law gives

$$L\frac{di}{dt} + Ri = E(t)$$

For a series circuit containing a resistor and a capacitor,



Kirchhoff's second law gives

$$Ri + \frac{1}{C}q = E(t)$$

where $i = \dfrac{dq}{dt}$. Then $R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$

Example The population of a community is known to increase at a rate proportional to the number of people present at time t. If an initial population P_0 has doubled in 5 years, how long will it take to triple? To quadruple?

Answer 20 (end of chapter) · Solution

2.8 Nonlinear Models

Example The logistic model

- Verhulst proposed a model, called the logistic model, for population growth in 1838. It does not assume unlimited resources. Instead, it assumes there is a carrying capacity K for the population
- This carrying capacity is the stable population level. If the population is above K, then the population will decrease, but if below, then it will increase
- For this model, it is assumed that the rate of change $\frac{dy}{dt}$ of the population y is proportional to the product of the current population y and $K - y$

$$\frac{dy}{dt} = \alpha y(K - y)$$

2.9 Modeling with Systems of First-Order DEs

Example Radioactive Series:

$$X \xrightarrow{k_1} Y \xrightarrow{k_2} Z$$

- The decay of the initial element X is described by

$$\frac{dx}{dt} = -k_1 x$$

- The second element Y is produced by the decay of X and is lost through its own spontaneous decay

$$\frac{dy}{dt} = k_1 x - k_2 y$$

- The stable element Z is generated from the decay of Y

$$\frac{dz}{dt} = k_2 y$$

Example Consider the Lotka-Volterra predator-prey model defined by

$$\frac{dx}{dt} = -0.1x + 0.02xy$$
$$\frac{dy}{dt} = 0.2y - 0.025xy$$

where the populations $x(t)$ (predator) and $y(t)$ (prey) are measured in the thousands and $x(0) = 6$ and $y(0) = 6$

- Use a numerical solver to graph $x(t)$ and $y(t)$

```
# Lotka-Volterra predator-prey model
from scipy import integrate

a, b, c, d = -0.1, 0.02, 0.2, -0.025

def f(t, xy):
    x, y = xy
    return [a *x +b *x *y, c *y +d *x *y]

# Initial condition and Time span
xy0 = [6, 6]
tf = 400
t_eval = np.linspace(0, tf, 5*tf)

# Numerical Solver
sol = integrate.solve_ivp(f, [0, tf], xy0, t_eval=t_eval)

t = sol.t
xy_t = sol.y.T
```

- Use the graphs to approximate the period of each population

```
fig, ax = plt.subplots(1, 1, figsize=(6, 5))

ax.plot(t, xy_t[:, 0], 'r', label="Predator")
ax.plot(t, xy_t[:, 1], 'b', label="Prey")

ax.set_xlabel("Time")
ax.set_ylabel("Number of animals")
ax.set_xlim(0, tf)
ax.set_ylim(0, 12)
ax.legend()

plt.show()
```

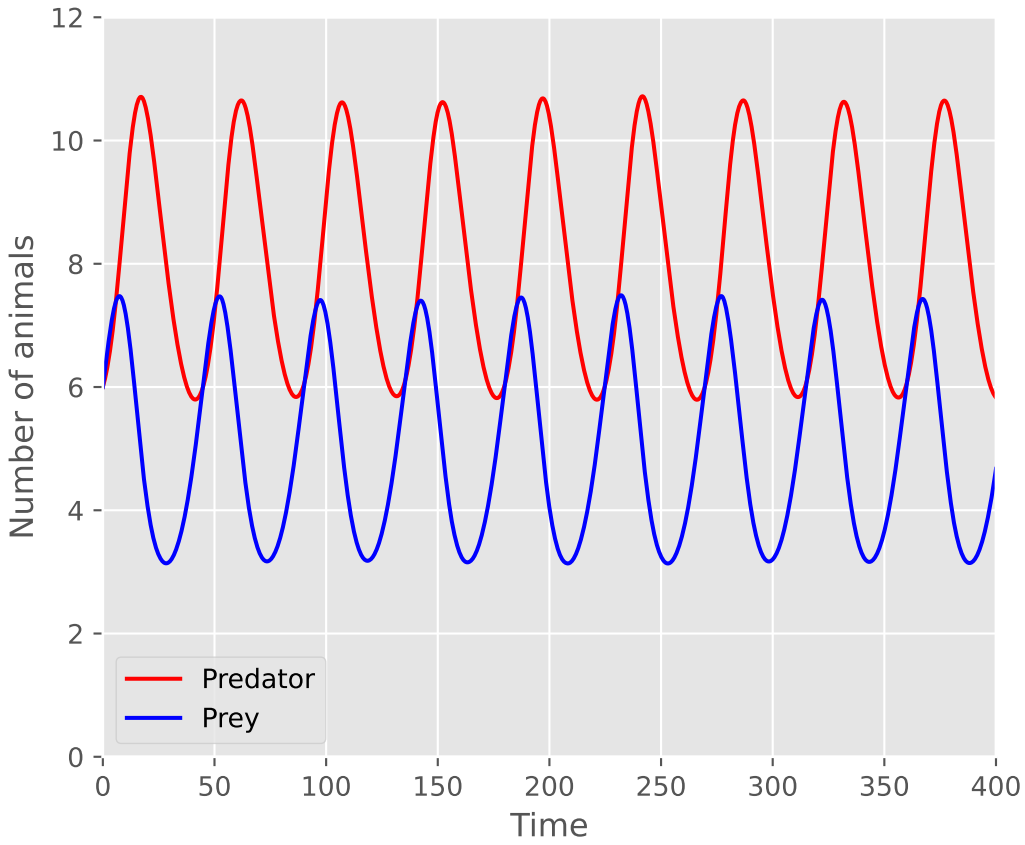


Figure 2.2: Lotka-Volterra predator-prey model

Answer 21 (end of chapter) · Solution

Try it in Python: The Lotka-Volterra and Lorenz systems solved with `solve_ivp`, with phase-plane plots: Numerical integration of ODEs using 'scipy'

Summary

Type	Form	Method
Autonomous	$y' = f(y)$	critical points $f(c) = 0$; the sign of f between them gives stability (phase portrait)
Separable	$y' = g(x)h(y)$	$\int \frac{dy}{h(y)} = \int g(x)\,dx$; check the constant solutions $h(c) = 0$
Linear	$y' + P(x)y = f(x)$	integrating factor $I(x) = e^{\int P\,dx}$: $\frac{d}{dx}[Iy] = If$
Exact	$M\,dx + N\,dy = 0$, $M_y = N_x$	find f with $f_x = M$, $f_y = N$; solution $f(x, y) = c$

Type	Form	Method
Not exact	$M_y \neq N_x$	integrating factor $\mu(x) = e^{\int (M_y - N_x)/N\,dx}$ if $(M_y - N_x)/N$ depends on x only; similarly $\mu(y) = e^{\int (N_x - M_y)/M\,dy}$
Homogeneous	M, N homogeneous of the same degree	$y = ux$ (or $x = vy$) $\rightarrow$ separable
Bernoulli	$y' + P(x)y = f(x)y^n$	$u = y^{1-n}$ $\rightarrow$ linear
$y' = f(Ax + By + C)$		$u = Ax + By + C$ $\rightarrow$ separable
Euler's method	$y' = f(x, y),\ \ y(x_0) = y_0$	$y_{n+1} = y_n + h\, f(x_n, y_n)$

Worked Exercises

Solve the given differential equation by separation of variables

2.2: 1. $\dfrac{dy}{dx} = \sin 5x$

Solution

$$\begin{aligned} \frac{dy}{dx} &= \sin 5x \\ &\Downarrow \\ dy &= \sin 5x\, dx \\ &\Downarrow \\ y &= -\frac{1}{5}\cos 5x + c \end{aligned}$$

2.2: 3. $dx + e^{3x} dy = 0$

Solution

$$\begin{aligned} dy &= -e^{-3x}\, dx \\ &\Downarrow \\ y &= \frac{1}{3}e^{-3x} + c \end{aligned}$$

2.2: 21. $\dfrac{dy}{dx} = x\sqrt{1 - y^2}$

Solution

$$\frac{1}{\sqrt{1-y^2}}dy = x\,dx$$
$$\Downarrow$$
$$\arcsin y = \frac{1}{2}x^2 + c$$
$$\Downarrow$$
$$y = \sin\left(\frac{1}{2}x^2 + c\right)$$

In dividing by $\sqrt{1-y^2}$ we assumed $|y| < 1$; the constant functions $y = 1$ and $y = -1$ are also (singular) solutions

2.2: 25. Find an implicit or an explicit solution of the given initial-value problem

$$x^2\frac{dy}{dx} = y - xy, \quad y(-1) = -1$$

Solution

$$\frac{1}{y}dy = \frac{1-x}{x^2}\,dx$$
$$\Downarrow$$
$$\ln|y| = -\left(\frac{1}{x} + \ln|x|\right) + c$$
$$\Downarrow y(-1) = -1 \;\Rightarrow\; c = -1$$
$$\ln|y| = -\left(1 + \frac{1}{x}\right) - \ln|x|$$
$$\Downarrow$$
$$y = \frac{1}{x}e^{-(1+\frac{1}{x})}$$

2.7: 41. Evaporating Raindrop As a raindrop falls, it evaporates while retaining its spherical shape. If we make the further assumptions that the rate at which the raindrop evaporates is proportional to its surface area and that air resistance is negligible, then a model for the velocity $v(t)$ of the raindrop is

$$\frac{dv}{dt} + \frac{3(k/\rho)}{(k/\rho)t + r_0}v = g$$

Here ρ is the density of water, r_0 is the radius of the raindrop at $t = 0$, $k < 0$ is the constant of proportionality $\dfrac{dm}{dt} = 4\pi r^2 k$, and the downward direction is taken to be the positive direction

(a) Solve for $v(t)$ if the raindrop falls from the rest

(b) Show that the radius of the raindrop at time t is $r(t) = (k/\rho)t + r_0$

(c) If $r_0 = 0.01$ ft and $r = 0.007$ ft at time $t = 10$ sec after the raindrop falls from a cloud, determine the time at which the raindrop has evaporated completely

Solution

(a)

$$\frac{dv}{dt} + \frac{3(k/\rho)}{(k/\rho)t + r_0} v = g$$

$$\Downarrow \text{ multiply by the integrating factor}$$

$$\frac{d}{dt}\left[\left(\frac{k}{\rho}t + r_0\right)^3 v\right] = \left(\frac{k}{\rho}t + r_0\right)^3 g$$

$$\Downarrow$$

$$v(t) = \frac{g\rho}{4k}\left(\frac{k}{\rho}t + r_0\right) + c\left(\frac{k}{\rho}t + r_0\right)^{-3}$$

$$\Downarrow \; v(0) = 0 \; \rightarrow \; c = -\frac{g\rho}{4k} r_0^4$$

$$v(t) = \frac{g\rho}{4k}\left(\frac{k}{\rho}t + r_0\right)\left[1 - \left(\frac{r_0}{\frac{k}{\rho}t + r_0}\right)^4\right]$$

(b)

$$m = \frac{4}{3}\pi r^3 \rho$$

$$\Downarrow$$

$$\frac{dm}{dt} = 4\pi r^2 \rho \frac{dr}{dt}$$

$$\Downarrow \; \frac{dm}{dt} = 4\pi r^2 k$$

$$\frac{dr}{dt} = \frac{k}{\rho}$$

$$\Downarrow \; r = r_0 \text{ at } t = 0$$

$$r(t) = \frac{k}{\rho}t + r_0$$

(c)

$$r(10) = \frac{k}{\rho} \times 10 + 0.01 = 0.007 \; \Rightarrow \; \frac{k}{\rho} = -0.0003$$

$$\Downarrow$$

$$r(t) = -0.0003t + 0.01 = 0 \; \Rightarrow \; t = \frac{0.01}{0.0003} \approx 33.3 \text{ sec}$$

1. Find an explicit solution of the given initial value problem

$$\sqrt{1 - y^2}\, dx - \sqrt{1 - x^2}\, dy = 0, \; y(0) = \sqrt{3}/2$$

Solution

Step 1: Rearrange the Equation

We write the equation in differential form:

$$\sqrt{1 - y^2}\, dx = \sqrt{1 - x^2}\, dy$$

Separate variables:

$$\frac{dx}{\sqrt{1-x^2}} = \frac{dy}{\sqrt{1-y^2}}$$

Step 2: Integrate Both Sides

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dy}{\sqrt{1-y^2}}$$

These are standard integrals:

- $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$
- $\int \frac{dy}{\sqrt{1-y^2}} = \arcsin y + C$

So:

$$\arcsin x = \arcsin y + C$$

Step 3: Solve for the Constant Using the Initial Condition

We're given $y(0) = \frac{\sqrt{3}}{2}$, so plug in $x = 0$, $y = \frac{\sqrt{3}}{2}$:

$$\arcsin(0) = \arcsin\left(\frac{\sqrt{3}}{2}\right) + C$$

$$0 = \frac{\pi}{3} + C \Rightarrow C = -\frac{\pi}{3}$$

Step 4: Write the Explicit Solution

From earlier:

$$\arcsin x = \arcsin y - \frac{\pi}{3}$$

Solve for y:

$$\arcsin y = \arcsin x + \frac{\pi}{3}$$

Now apply sin to both sides:

$$y = \sin\left(\arcsin x + \frac{\pi}{3}\right)$$

or, expanding the sine, $y = \frac{x}{2} + \frac{\sqrt{3}}{2}\sqrt{1-x^2}$

This is a solution only where $\cos\left(\arcsin x + \frac{\pi}{3}\right) \geq 0$, because $\frac{dy}{dx} = \frac{\cos(\arcsin x + \pi/3)}{\sqrt{1-x^2}}$ must equal $\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \geq 0$. Hence $\arcsin x + \frac{\pi}{3} \leq \frac{\pi}{2}$, i.e. the solution is defined for $-1 < x \leq \frac{1}{2}$

2. Find an explicit solution of the given initial value problem

$$\frac{dy}{dx} = -y \ln y,\ y(0) = e$$

Solution

Step 1: Separate Variables

We want to isolate y and x:

$$\frac{dy}{dx} = -y \ln y \quad \Rightarrow \quad \frac{1}{y \ln y}\, dy = -dx$$

Step 2: Integrate Both Sides

Let's handle the left-hand side:

Let $u = \ln y$, then:

$$\frac{du}{dy} = \frac{1}{y} \Rightarrow du = \frac{1}{y}\, dy$$

So:

$$\frac{1}{y \ln y}\, dy = \frac{1}{u} \cdot du = \frac{du}{u}$$

Thus, we have:

$$\int \frac{dy}{y \ln y} = \int \frac{du}{u} = \ln |\ln y|$$

So now integrate both sides:

$$\int \frac{dy}{y \ln y} = \int -dx \quad \Rightarrow \quad \ln |\ln y| = -x + C$$

Step 3: Solve for $y(x)$

Start with:

$$\ln |\ln y| = -x + C$$

Exponentiate both sides:

$$|\ln y| = e^{-x+C} = Ae^{-x}, \quad A > 0$$

So:

$$\ln y = \pm Ae^{-x}$$

We drop the $\pm$ by using a general constant $B = \pm A$, so:

$$\ln y = Be^{-x} \Rightarrow y = e^{Be^{-x}}$$

Step 4: Apply Initial Condition $y(0) = e$

Plug in:

$$y(0) = e^{Be^0} = e^B = e \Rightarrow B = 1$$

Final Answer:

$$y(x) = e^{e^{-x}}$$

This is the explicit solution satisfying the differential equation and the initial condition $y(0) = e$

3. Solve the given initial-value problem

$$\frac{dy}{dx} = \frac{3x + 2y}{3x + 2y + 2},\ y(-1) = -1$$

Solution

Step 1: Make a Substitution

Let us simplify the expression by setting:

$$u = 3x + 2y$$

Differentiate both sides with respect to x:

$$\frac{du}{dx} = 3 + 2\frac{dy}{dx}$$

Now solve for $\frac{dy}{dx}$ from this:

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{du}{dx} - 3\right)$$

But also, from the original equation:

$$\frac{dy}{dx} = \frac{u}{u + 2}$$

So equating both expressions:

$$\frac{1}{2}\left(\frac{du}{dx} - 3\right) = \frac{u}{u + 2}$$

So:

$$\frac{du}{dx} = 3 + \frac{2u}{u+2}$$

Step 2: Simplify the Right-Hand Side

We simplify and so:

$$\frac{du}{dx} = \frac{5u+6}{u+2}$$

Step 3: Separate Variables and Integrate

We separate:

$$\frac{u+2}{5u+6}\,du = dx$$

Let's integrate the left-hand side. Let's simplify:

$$\int \frac{u+2}{5u+6}\,du = \int \left[\frac{1}{5} + \frac{4}{5(5u+6)}\right] du = \frac{1}{5}u + \frac{4}{25}\ln|5u+6| + C$$

So:

$$x = \frac{1}{5}u + \frac{4}{25}\ln|5u+6| + C$$

Step 4: Substitute Back $u = 3x + 2y$

Recall $u = 3x + 2y$. So we substitute:

$$x = \frac{1}{5}(3x+2y) + \frac{4}{25}\ln|5(3x+2y)+6| + C$$

This is an implicit solution

Step 5: Apply the Initial Condition $y(-1) = -1$

Recall $u = 3x + 2y$, so at $x = -1$, $y = -1$:

$$u = 3(-1) + 2(-1) = -5 \Rightarrow 5u + 6 = -25 + 6 = -19$$

Then:

$$-1 = \frac{1}{5}(-5) + \frac{4}{25}\ln(19) + C \Rightarrow C = -\frac{4}{25}\ln 19$$

Final Answer:

$$x = \frac{1}{5}(3x+2y) + \frac{4}{25}\ln|5(3x+2y)+6| - \frac{4}{25}\ln 19$$

4. Solve the given differential equation

$$\frac{dy}{dx} = \tan^2(x+y)$$

Solution

Step 1: Make a Substitution

Let's simplify the expression using substitution.

Let:

$$u = x + y \quad \Rightarrow \quad \frac{du}{dx} = 1 + \frac{dy}{dx}$$

From the original equation:

$$\frac{dy}{dx} = \tan^2(x+y) = \tan^2 u$$

So:

$$\frac{du}{dx} = 1 + \tan^2 u = \sec^2 u$$

Step 2: Separate Variables and Integrate

$$\frac{du}{\sec^2 u} = dx \quad \Rightarrow \quad \cos^2 u\, du = dx$$

So integrate both sides:

$$\int \cos^2 u\, du = \int \frac{1+\cos(2u)}{2}\, du = \frac{1}{2}\left[u + \frac{1}{2}\sin(2u)\right]$$

So:

$$\frac{1}{2}u + \frac{1}{4}\sin(2u) = x + C$$

Step 3: Substitute Back $u = x + y$

$$\frac{1}{2}(x+y) + \frac{1}{4}\sin(2(x+y)) = x + C$$

where C is a constant of integration. This is an implicit solution to the original differential equation

5. Find the solution of period 2π of the equation

$$y' + (\cos x)y = \sin 2x$$

Solution

Step 1: Solve Using the Integrating Factor Method

Integrating Factor

$$\mu(x) = e^{\int \cos x\, dx} = e^{\sin x}$$

Multiply both sides of the ODE by $\mu(x) = e^{\sin x}$:

$$e^{\sin x} y' + e^{\sin x} \cos x\, y = e^{\sin x} \sin 2x$$

Left-hand side becomes:

$$\frac{d}{dx}\left(e^{\sin x} y\right) = e^{\sin x} \sin 2x$$

Step 2: Integrate Both Sides

With $s = \sin x$, $ds = \cos x\, dx$ and $\sin 2x = 2 \sin x \cos x$, the integral is elementary:

$$\int e^{\sin x} \sin 2x\, dx = \int 2s\, e^s\, ds = 2(s-1)e^s = 2(\sin x - 1)\, e^{\sin x}$$

So:

$$y(x) = e^{-\sin x}\left(2(\sin x - 1)e^{\sin x} + C\right) = 2(\sin x - 1) + Ce^{-\sin x}$$

Step 3: Impose Periodicity

Both $2(\sin x - 1)$ and $e^{-\sin x}$ have period 2π, so **every** solution has period 2π, whatever the value of C:

$$y(x) = 2(\sin x - 1) + Ce^{-\sin x}$$

(The periodicity can also be seen without evaluating the integral: $e^{-\sin x}$ is 2π-periodic and $\int_0^{2\pi} e^{\sin t} \sin 2t\, dt = 0$, because the integrand changes sign under $t \mapsto \pi - t$.) Check: $y = 2\sin x - 2$ gives $y' + (\cos x)y = 2\cos x + 2\sin x \cos x - 2\cos x = \sin 2x$

6. Find the solution of period 2π of the equation

$$y' + 3y = \cos x$$

Solution

Step 1: General Solution via Integrating Factor

This is a first-order linear ODE. Let's solve it using the integrating factor method.

Let:

$\mu(x) = e^{\int 3\, dx} = e^{3x}$

Multiply both sides of the equation by e^{3x}:

$$e^{3x}y' + 3e^{3x}y = e^{3x}\cos x \Rightarrow \frac{d}{dx}(e^{3x}y) = e^{3x}\cos x$$

Now integrate both sides:

$$e^{3x}y = \int e^{3x}\cos x\,dx$$

Step 2: Compute the Integral

$$\int e^{3x}\cos x\,dx = \frac{e^{3x}}{10}(3\cos x + \sin x) + C$$

Thus:

$$e^{3x}y = \frac{e^{3x}}{10}(3\cos x + \sin x) + C \Rightarrow y(x) = \frac{1}{10}(3\cos x + \sin x) + Ce^{-3x}$$

Step 3: Impose Periodicity

We are asked to find a solution with period 2π. Let's analyze:

- The term $\frac{1}{10}(3\cos x + \sin x)$ is clearly 2π-periodic
- But the term Ce^{-3x} is not periodic unless $C = 0$

So to obtain a periodic solution, we must take: $C = 0$

Final Answer:

$$y(x) = \frac{1}{10}(3\cos x + \sin x)$$

7. Solve the given differential equation by using an appropriate substitution

$$\frac{dy}{dx} = \sin(x + y)$$

Solution

Step 1: Use Substitution

Let:

$$u = x + y \quad \Rightarrow \quad \frac{du}{dx} = 1 + \frac{dy}{dx}$$

Substitute into the derivative of u:

$$\frac{du}{dx} = 1 + \sin u$$

Step 2: Separate Variables

We now have the separable equation:

$$\frac{du}{1+\sin u} = dx$$

We simplify the left-hand side:

$$\frac{1}{1+\sin u} = \frac{1-\sin u}{(1+\sin u)(1-\sin u)} = \frac{1-\sin u}{1-\sin^2 u} = \frac{1-\sin u}{\cos^2 u}$$

Now integrate:

$$\int \sec^2 u\, du = \tan u, \quad \int \sec u \tan u\, du = \sec u$$

So:

$$\int \frac{du}{1+\sin u} = \tan u - \sec u \;\Rightarrow\; \tan u - \sec u = x + C$$

Step 3: Substitute Back $u = x + y$

$$\tan(x+y) - \sec(x+y) = x + C$$

This is the implicit general solution to the differential equation. In separating variables we divided by $1+\sin u$, so the constant solutions $\sin u = -1$, i.e. $y = -x - \frac{\pi}{2} + 2k\pi$, must be added

8. Find a general solution of the equation

$$ydx + (2x + xy - 3)dy = 0$$

Solution

$$ydx + (2x + xy - 3)dy = 0$$
$$\Downarrow$$
$$\frac{dx}{dy} + \left(1 + \frac{2}{y}\right)x = \frac{3}{y}$$
$$\Downarrow$$
$$\frac{d}{dy}\left[x\, y^2 e^y\right] = 3ye^y$$
$$\Downarrow$$
$$x = \frac{3}{y}\left(1 - \frac{1}{y}\right) + Cy^{-2}e^{-y}$$

or

$$
\begin{aligned}
ydx + (2x + xy - 3)dy &= 0 \\
\Downarrow \quad &\times ye^y \\
y^2e^ydx + ye^y(2x + xy - 3)dy &= 0 \\
\Downarrow & \\
\frac{\partial f}{\partial x} = y^2e^y \quad &\rightarrow \quad f(x,y) = y^2e^yx + h(y) \\
\Downarrow & \\
\frac{\partial f}{\partial y} = ye^y(2x + xy) + \frac{dh}{dy} \quad &\rightarrow \quad \frac{dh}{dy} = -3ye^y \\
\Downarrow & \\
h(y) = -3e^y(y-1) \quad &\rightarrow \quad f(x,y) = y^2e^yx - 3e^y(y-1) = C \\
\Downarrow & \\
x = \frac{3}{y}\left(1 - \frac{1}{y}\right) + Cy^{-2}e^{-y} &
\end{aligned}
$$

9. Solve the differential equation

$$xv\frac{dv}{dx} + v^2 = 32x$$

Solution

$$
\begin{aligned}
xv\frac{dv}{dx} + v^2 &= 32x \\
\Downarrow & \\
\frac{dv}{dx} + \frac{1}{x}v &= \frac{32}{v} \\
\Downarrow \quad u = v^2,\ \frac{du}{dx} &= 2v\frac{dv}{dx} \\
\frac{du}{dx} + \frac{2}{x}u &= 64 \\
\Downarrow & \\
\frac{d}{dx}\left(ux^2\right) &= 64x^2 \\
\Downarrow & \\
u = \frac{64}{3}x + \frac{C}{x^2} \quad &\rightarrow \quad v = \pm\sqrt{\frac{64}{3}x + \frac{C}{x^2}}
\end{aligned}
$$

10. Solve the initial-value problem

$$\frac{dy}{dt} + 2(t+1)y^2 = 0, \quad y(0) = -\frac{1}{8}$$

and give the largest interval I on which the solution is defined:

Solution

$$\frac{dy}{dt} + 2(t+1)y^2 = 0$$
$$\Downarrow$$
$$\frac{dy}{y^2} = -2(t+1)\,dt$$
$$\Downarrow$$
$$y = \frac{1}{t^2+2t+c}, \quad y(0) = -\frac{1}{8}$$
$$\Downarrow$$
$$y = \frac{1}{(t-2)(t+4)}, \quad -4 < t < 2$$

Answers

Answer 1 (Section 2.1)

Isoclines $y = \pm x$ (slope 0): solutions rise where $|y| < |x|$ and fall where $|y| > |x|$.

(a) rises, has a maximum on $y = -x$ and a minimum on $y = x$, then rises again **(b)** $(-1, -1)$ is a local maximum; to the right the curve falls to $-\infty$

(c) falls with slope -4, has a minimum on $y = x$, then rises **(d)** the origin is an inflection point, $y \approx x^3/3$

Answer 2 (Section 2.1)

Critical points $y = -1$ (unstable), $y = 0$ (semi-stable), $y = 1$ (asymptotically stable); inflection points at $y = \pm 1/\sqrt{2}$.

(a) decreases, $y \to -1$ as $x \to -\infty$ and $y \to -\infty$ at a finite x **(b)** increases from -1 to 0

(c) increases from 0 to 1 **(d)** decreases to 1 as $x \to \infty$

Answer 3 (Section 2.2)

(1) $y = -\frac{1}{5}\cos 5x + c$

(2) $y = \frac{1}{3}e^{-3x} + c$

(3) $y = \sin\left(\frac{1}{2}x^2 + c\right)$, plus the singular solutions $y = \pm 1$

Answer 4 (Section 2.2)

(1) Implicit: $\ln|y| = -1 - \dfrac{1}{x} - \ln|x|$; explicit: $y = \dfrac{1}{x}\,e^{-1-1/x}$, $x < 0$

(2) Implicit: $\arctan x = 4t - \dfrac{3\pi}{4}$; explicit: $x = \tan\left(4t - \dfrac{3\pi}{4}\right)$, $\dfrac{\pi}{16} < t < \dfrac{5\pi}{16}$

Answer 5 (Section 2.3)

(1) $y = ce^{-2x}$, $-\infty < x < \infty$

(2) $y = \frac{1}{2}x^2 - \frac{1}{2} + ce^{-x^2}$, $-\infty < x < \infty$

(3) $y = \dfrac{3}{2} + \dfrac{c}{x^2}$, $0 < x < \infty$ (or $-\infty < x < 0$)

(4) $y = \dfrac{e^{-x}}{x}\left(c - \frac{1}{2}\cos 2x\right), \quad 0 < x < \infty$

Answer 6 (Section 2.3)

(1) $x = 2y^2 - \dfrac{49}{5}y, \quad 0 < y < \infty$

(2) $y = \dfrac{x \ln x - x + 21}{x + 1}, \quad 0 < x < \infty$

Answer 7 (Section 2.3)

(1) $x = -y^2 - 2y - 2 + ce^y$

(2) $x = \dfrac{3}{y} - \dfrac{3}{y^2} + \dfrac{c}{y^2}e^{-y}$

Answer 8 (Section 2.3)

$y = \dfrac{10}{x^2}\left[\mathrm{Si}(x) - \mathrm{Si}(1)\right], \quad 0 < x < \infty$

Answer 9 (Section 2.4)

$x^2y - y = c, \quad \text{i.e. } y = \dfrac{c}{x^2 - 1} \quad (x \neq \pm 1)$

Answer 10 (Section 2.4)

(1) Exact: $x^2 - x + \frac{3}{2}y^2 + 7y = c$

(2) Exact: $\frac{5}{2}x^2 + 4xy - \frac{8}{3}y^3 = c$

(3) Exact: $x^2y^2 - 3x + 4y = c$

(4) Not exact ($M_y = -2y \neq N_x = 2x - 2y$); it can be solved by the substitution $y = ux$ of Section 2.5

Answer 11 (Section 2.4)

(1) $\frac{1}{3}x^3 + x^2y + xy^2 - y = \frac{4}{3}$

(2) $4ty + t^2 - 5t + 3y^2 - y = 8$

Answer 12 (Section 2.4)

$e^x(xy + y^2) = c$

Answer 13 (Section 2.5)

$y = 2x - 3\tanh 3x$

Answer 14 (Section 2.5)

$(x + y)^2 = Cx\,e^{y/x}$

Answer 15 (Section 2.5)

$y = \dfrac{5x^2}{x^5 + C}, \quad \text{plus } y = 0 \text{ (lost when substituting } u = 1/y)$

Answer 16 (Section 2.5)

(1) $y = x(c - \ln|x|)$

(2) $\ln|y - x| + \dfrac{x}{y - x} = c, \quad \text{plus } y = x$

(3) $\frac{1}{2}\ln(x^2 + y^2) + \arctan\dfrac{y}{x} = c$

(4) $y = x\sin(\ln x + c)$, plus the singular solutions $y = \pm x$

Answer 17 (Section 2.5)

(1) $y = x\,(8 - 3\ln x)^{1/3}$

(2) $x^2 + y^2 = 2x^4$; explicitly $y = -x\sqrt{2x^2 - 1}$, $x < -\frac{1}{\sqrt{2}}$

Answer 18 (Section 2.5)

(1) $y^3 = 1 + \dfrac{c}{x^3}$

(2) $y^{-3} = 1 + c(1 + t^2)$, plus $y = 0$

(3) $y = 2\arctan\bigl(x + \sqrt{2} - 1\bigr) - x$

Answer 19 (Section 2.6)

$h = 0.1$: $y(1.2) \approx 2.9800$; $h = 0.05$: $y(1.2) \approx 3.1151$

Answer 20 (Section 2.7)

Triples in $t = 5\,\dfrac{\ln 3}{\ln 2} \approx 7.92$ years; quadruples in $t = 5\,\dfrac{\ln 4}{\ln 2} = 10$ years

Answer 21 (Section 2.9)

Both populations are periodic with the same period, $T \approx 45$ (time units)

3 Higher-Order Differential Equations

Learning objectives After this chapter you should be able to

- use superposition, the Wronskian and fundamental sets to build the general solution $y = y_h + y_p$ of a linear equation
- solve homogeneous equations with constant coefficients and Cauchy-Euler equations from the characteristic (auxiliary) equation
- find a second solution by reduction of order, and particular solutions by undetermined coefficients and by variation of parameters
- solve boundary-value problems with influence and Green's functions
- model spring-mass systems and recognize resonance, and treat nonlinear equations by substitution, Taylor series or numerically

3.1 Theory of Linear Equations

- In an **initial-value problem (IVP)**, we seek a solution $y(x)$ of an n^{th} order linear DE so that $y(x)$ satisfies **initial conditions** at x_0
  - n^{th} order linear DE: $a_n(x) \neq 0$

$$\underbrace{a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y}_{L(y)\text{: Linear Operator}} = g(x)$$

  - Initial conditions

$$y(x_0) = y_0,\ y'(x_0) = y_1, \cdots,\ y^{(n-1)}(x_0) = y_{n-1}$$

- **Boundary-value problem (BVP)** consists of solving a linear DE of order 2 or greater, in which the dependent variable y or its derivatives are specified at **different points**
  - For example,

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

  - Boundary conditions

$$y(x_0) = y_0,\ y(x_1) = y_1$$

- The sum, or **superposition** , of two or more solutions of a homogeneous linear DE is also a solution
- Any set of n linearly *independent* solutions $y_1, y_2, \cdots, y_n$ of the n^{th} order homogeneous linear DE on interval I is **a fundamental set of solutions**
- If two functions are linearly dependent, then one is a constant multiple of the other (otherwise, they are linearly independent)
- n solutions $y_1, y_2, \cdots, y_n$ of an n^{th} order homogeneous linear DE on an interval I are linearly independent **if and only if** their **Wronskian** is nonzero for every x in I:

$$W(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0$$

- General solution of n^{th} order *homogeneous* linear DE is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

  where $y_1, y_2, \cdots, y_n$ is a fundamental set of solutions and c_i, $i = 1, 2, \cdots, n$ are arbitrary constants
- General solution of n^{th} order *nonhomogeneous* linear DE is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

  where $y_1, y_2, \cdots, y_n$ is a fundamental set of solutions, y_p is a particular solution, and c_i, $i = 1, 2, \cdots, n$ are arbitrary constants

Example Given that $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$ is the general solution of $x'' + \omega^2 x = 0$ on the interval $(-\infty, \infty)$, show that a solution satisfying the initial conditions $x(0) = x_0$, $x'(0) = x_1$, is given by

$$x(t) = x_0 \cos \omega t + \frac{x_1}{\omega} \sin \omega t$$

Answer 1 (end of chapter) · Solution

Example Determine whether the given set of functions is linearly dependent or linearly independent on the interval $(-\infty, \infty)$

$$f_1(x) = x, \ f_2(x) = x^2, \ f_3(x) = 4x - 3x^2$$

$$f_1(x) = 5, \ f_2(x) = \cos^2 x, \ f_3(x) = \sin^2 x$$

Answer 2 (end of chapter) · Solution

Example Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Form the general solution of the equation

$$y'' - y' - 12y = 0; \quad e^{-3x}, \ e^{4x}, \quad (-\infty, \infty)$$

$$y'' - 2y' + 5y = 0; \quad e^x \cos 2x, \quad e^x \sin 2x, \quad (-\infty, \infty)$$

Answer 3 (end of chapter) · Solution

Example Verify that the given two-parameter family of functions is the general solution of the nonhomogeneous differential equation on the indicated interval

$$y'' - 7y' + 10y = 24e^x; \quad y = c_1 e^{2x} + c_2 e^{5x} + 6e^x; \quad (-\infty, \infty)$$

$$y'' - 4y' + 4y = 2e^{2x} + 4x - 12; \quad y = c_1 e^{2x} + c_2 x e^{2x} + x^2 e^{2x} + x - 2; \quad (-\infty, \infty)$$

Answer 4 (end of chapter) · Solution

3.2 Reduction of Order

- **Reduction of order** can be used to reduce a linear *second-order* DE with known solution y_1 into a linear *first-order* DE, which can be solved for a second solution y_2
- Applying reduction of order $y_2 = u(x)y_1$ to the standard form of a second-order linear homogeneous DE

$$y'' + P(x)y' + Q(x)y = 0$$

gives

$$\begin{gathered} y_1 u'' + (2y_1' + P(x)y_1)\, u' = 0 \\ \Downarrow \ \times y_1, \ u' = w \\ (y_1^2 w)' = -(y_1^2 w)P(x) \\ \Downarrow \\ y_2(x) = y_1(x) \int \frac{\exp\left(-\int P(x)\, dx\right)}{y_1^2(x)}\, dx \end{gathered}$$

Example The indicated function $y_1(x)$ is a solution of the given equation. Use the reduction of order to find a second solution $y_2(x)$

- $y'' - 4y' + 4y = 0; \quad y_1 = e^{2x}$
- $y'' + 16y = 0; \quad y_1 = \cos 4x$
- $y'' - y = 0; \quad y_1 = \cosh x$

Answer 5 (end of chapter) · Solution

Example The indicated function $y_1(x)$ is a solution of the associated homogeneous equation. Use the reduction of order to find a second solution $y_2(x)$ of homogeneous equation and a particular solution $y_p(x)$ of the given nonhomogeneous equation

$$y'' - 4y = 2; \quad y_1 = e^{-2x}$$

Answer 6 (end of chapter) · Solution

3.3 Homogeneous Linear Equations with Constant Coefficients

The general solution of $ay'' + by' + cy = 0$ is found by substituting $y = e^{px}$ and solving the resulting **characteristic equation** $ap^2 + bp + c = 0$ for roots p_1 and p_2

- **Case I** p_1 and p_2 are real and distinct

$$y = c_1 e^{p_1 x} + c_2 e^{p_2 x}$$

- **Case II** p_1 and p_2 are real and equal

$$y = c_1 e^{p_1 x} + c_2 x e^{p_1 x}$$

- **Case III** p_1 and p_2 are complex conjugate: $p_1, p_2 = \alpha \pm i\beta$

$$y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$$

Example Solve $y'' + \omega^2 y = 0$ and $y'' - \omega^2 y = 0$

Answer 7 (end of chapter) · Solution

Example Solve $3y''' + 5y'' + 10y' - 4y = 0$

Answer 8 (end of chapter) · Solution

```
from sympy import Symbol, init_printing
from sympy.solvers import solve
init_printing()

x = Symbol('x')
solve(3*x**3 +5*x**2 +10*x -4, x)
```

$$\left[\frac{1}{3}, \ -1 - \sqrt{3}i, \ -1 + \sqrt{3}i\right]$$

Example Find the general solution of the given second-order differential equation

- $4y'' + y' = 0$
- $y'' - y' - 6y = 0$
- $y'' + 8y' + 16y = 0$

Answer 9 (end of chapter) · Solution

Example Find the general solution of the given higher-order differential equation

- $y''' - 4y'' - 5y' = 0$
- $y''' - 5y'' + 3y' + 9y = 0$

Answer 10 (end of chapter) · Solution

Example Solve the initial-value problem

$y''' + 12y'' + 36y' = 0,\ \ y(0) = 0,\ y'(0) = 1,\ y''(0) = -7$

Answer 11 (end of chapter) · Solution

Try it in Python: Characteristic equations and general solutions with `sympy`: Symbolic solution to ODEs

3.4 Undetermined Coefficients

Method of undetermined coefficients can be used to obtain a particular solution y_p

- The underlying idea is a conjecture about the form of y_p based on the kinds of functions making up the input function $g(x)$
- Limited to nonhomogeneous linear DEs where
  - Coefficients a_i, $i = 1, \cdots, n$, are constants
  - $g(x)$ is a constant, a polynomial function, $e^{\alpha x}$, $\sin \beta x$ or $\cos \beta x$, or finite sums and products of these functions
- There are models of y_p for various functions

	$g(x)$	Form of y_p
1	1 (any constant)	A
2	$5x + 7$	$Ax + B$
3	$3x^2 - 2$	$Ax^2 + Bx + C$
4	$x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5	$\sin 4x$	$A\cos 4x + B\sin 4x$
6	$\cos 4x$	$A\cos 4x + B\sin 4x$
7	e^{5x}	Ae^{5x}
8	$(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9	x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10	$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$

	$g(x)$	Form of y_p
11	$5x^2 \sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12	$xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

- Finally, the general solution is obtained from the **superposition** of y_h and y_p

Example Solve $y'' + 4y = x \cos x$

Answer 12 (end of chapter) · Solution

- If $g(x)$ consists of a sum of, say, m terms of the kind listed in the table, then the assumption for a particular solution y_p consists of the sum of the trial forms
- If y_p contains terms that duplicate terms in y_h, then that y_p must be multiplied by x^n, where n is the smallest positive integer that eliminates that duplication

Example Solve $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$

Answer 13 (end of chapter) · Solution

Example Solve $y^{(4)} + y''' = 1 - x^2 e^{-x}$

Answer 14 (end of chapter) · Solution

Example Solve the given differential equation by undetermined coefficients

- $y'' + 2y' + y = \sin x + 3 \cos 2x$
- $y'' + 2y' - 24y = 16 - (x + 2)e^{4x}$
- $y''' - 6y'' = 3 - \cos x$

Answer 15 (end of chapter) · Solution

Example Solve the given initial-value problem

- $y'' + 4y = -2, \quad y(\pi/8) = 1/2, \quad y'(\pi/8) = 2$
- $5y'' + y' = -6x, \quad y(0) = 0, \; y'(0) = -10$
- $\dfrac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \omega t, \quad x(0) = 0, \; x'(0) = 0$

Answer 16 (end of chapter) · Solution

3.5 Variation of Parameters

The **method of variation of parameters** can be used with linear higher-order DEs

- This method *always* yields a y_p provided the homogeneous equation can be solved
- This method is not limited to the types of input functions constraining the method of undetermined coefficients

To adapt the method of variation of parameters to a linear second-order DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x),$$

we put the above equation in the standard form

$$y'' + P(x)y' + Q(x)y = f(x)$$

To solve,

- Find the homogeneous solutions y_1, y_2
- Seek a particular solution of the form

$$y_p = u_1(x)y_1 + u_2(x)y_2$$

$$y_p'' + P(x)y_p' + Q(x)y_p = f(x)$$

$$\Downarrow\ y_p = u_1y_1 + u_2y_2$$

$$\frac{d}{dx}[y_1u_1' + y_2u_2'] + P(x)[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' = f(x)$$

$$\text{let } y_1u_1' + y_2u_2' = 0,\ \Downarrow \text{ then } y_1'u_1' + y_2'u_2' = f(x)$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$\Downarrow$$

$$u_1 = \int \frac{W_1}{W}\,dx = -\int \frac{y_2}{W}\,f(x)\,dx,\quad u_2 = \int \frac{W_2}{W}\,dx = \int \frac{y_1}{W}\,f(x)\,dx$$

$$\text{where } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},\ W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix},\ W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Example Solve $y'' - y = \dfrac{1}{x}$

Answer 17 (end of chapter) · Solution

The method can be generalized to the standard form of n^{th} order linear DE. A particular solution is

$$y_p = u_1(x)y_1 + u_2(x)y_2 + \cdots + u_n(x)y_n$$

where

$$u_k = \int \frac{W_k}{W}\, dx,\ k = 1, 2, \cdots, n$$

Example Solve each differential equation by variation of parameters

- $y'' + y = \sec x$
- $y'' + y = \sin x$
- $y'' + y = \cos^2 x$
- $3y'' - 6y' + 6y = e^x \sec x$

Answer 18 (end of chapter) · Solution

Example Solve each differential equation by variation of parameters subject to the initial conditions $y(0) = 1,\ y'(0) = 0$

- $4y'' - y = xe^{x/2}$
- $y'' - 2y' + y = e^x \sec^2 x$
- $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$

Answer 19 (end of chapter) · Solution

Example Solve each differential equation by variation of parameters

- $y'' - 4y = \dfrac{e^{2x}}{x}$
- $2y'' + 2y' + y = 4\sqrt{x}$

Answer 20 (end of chapter) · Solution

3.6 Cauchy-Euler Equation

The **Cauchy-Euler equation** is a linear DE of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

where $a_n, a_{n-1}, \cdots, a_0$ are constants and the exponent of the coefficient matches the order of differentiation

$y = x^p$ is a solution of second order Cauchy-Euler equation whenever p is a solution of the **auxiliary equation**

$$a_2 p^2 + (a_1 - a_2)p + a_0 = 0$$

- **Case I:** Distinct Real Roots, p_1, p_2

$$y = c_1 x^{p_1} + c_2 x^{p_2}$$

- **Case II:** Repeated Real Roots, $p_1 = p_2$

$$y = c_1 x^{p_1} + c_2 x^{p_1} \ln x$$

- **Case III:** Complex Conjugate Roots, $p_1, p_2 = \alpha \pm i\beta$

$$y = x^{\alpha} \left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x) \right]$$

Example Solve $x^2 y'' - 3xy' + 3y = 2x^4 e^x$

Answer 21 (end of chapter) · Solution

Example Solve the given differential equation

- $x^2 y'' - 2y = 0$
- $xy'' + y' = 0$
- $x^2 y'' + xy' + 4y = 0$

Answer 22 (end of chapter) · Solution

Example Solve the given differential equation by variation of parameters

- $xy'' - 4y' = x^4$
- $x^2 y'' + xy' - y = \ln x$

Answer 23 (end of chapter) · Solution

3.7 Nonlinear Equations

- Nonlinear equations do not possess superposability

Example Verify that y_1 and y_2 are solutions of the given DE but that $c_1y_1 + c_2y_2$ is, in general, not a solution

$$(y'')^2 = y^2,\ y_1 = e^x,\ y_2 = \cos x$$

Answer 24 (end of chapter) · Solution

- The major difference between linear and nonlinear DEs of order two or higher lies in the realm of solvability. Nonlinear higher-order DEs virtually defy solution. This means that there are no analytical methods whereby either an explicit or implicit solution can be found
- There are still things that can be done; we can always analyze a nonlinear DE qualitatively and numerically
- Nonlinear second-order DE $F(y, y', y'') = 0$ can be reduced to two first-order equations by means of the substitution $u = y'$ and can sometimes be solved using first-order methods

Example Solve $y'' = 2x(y')^2$ and $yy'' = (y')^2$

Answer 25 (end of chapter) · Solution

- In some instances, a solution of a nonlinear IVP can be approximated by a Taylor series centered at x_0

Example Solve $y'' = x + y - y^2,\ y(0) = -1,\ y'(0) = 1$ by using

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

Answer 26 (end of chapter) · Solution

Example The dependent variable y is missing in the given differential equation. Solve the equation by using the substitution $u = y'$

$$y'' + (y')^2 + 1 = 0$$

Answer 27 (end of chapter) · Solution

Example The independent variable x is missing in the given differential equation. Solve the equation by using the substitution $u = y'$

$$yy'' + (y')^2 + 1 = 0$$

Answer 28 (end of chapter) · Solution

Example Consider the initial-value problem

$$y'' + yy' = 0, \quad y(0) = 1, \; y'(0) = -1$$

1. Use the DE and a numerical solver to graph the solution curve
2. Find an explicit solution of the IVP. Use a graphing utility to graph this solution
3. Find an interval of definition for the solution in part 2

Answer 29 (end of chapter) · Solution

Example Show that the substitution $u = y'$ leads to a Bernoulli equation. Solve this equation

$$xy'' = y' + (y')^3$$

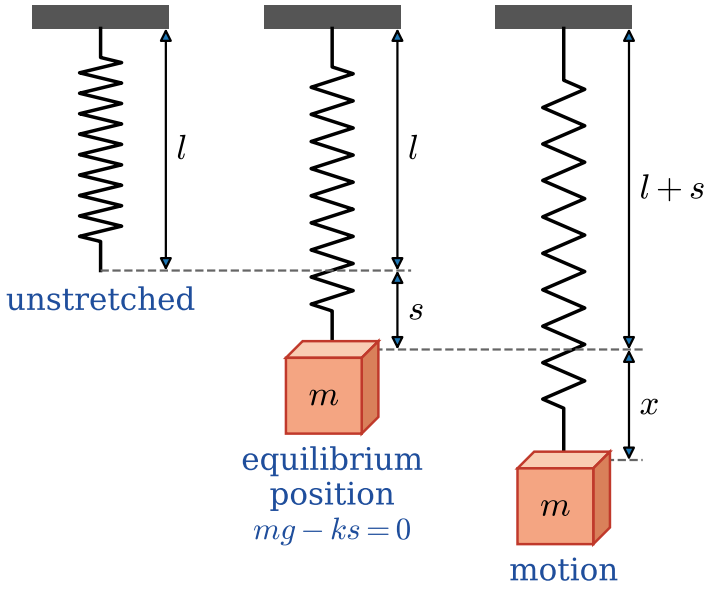
Answer 30 (end of chapter) · Solution

3.8 Linear Models: Initial-Value Problems

Several dynamical systems are modeled with linear 2^{nd}-order DEs with constant coefficients and initial conditions at t_0

- Spring/Mass Systems
  - Free Undamped Motion
  - Free Damped Motion
  - Driven Motion

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -kx - \beta\frac{dx}{dt} + f(t) \\ x(0) &= x_0 \\ \dot{x}(0) &= x_1 \end{aligned}$$



Example Show that the solution of the initial-value problem

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \cos \gamma t,\ x(0) = 0,\ x'(0) = 0$$

is $x(t) = \dfrac{F_0}{\omega^2 - \gamma^2}(\cos \gamma t - \cos \omega t)$

Answer 31 (end of chapter) · Solution

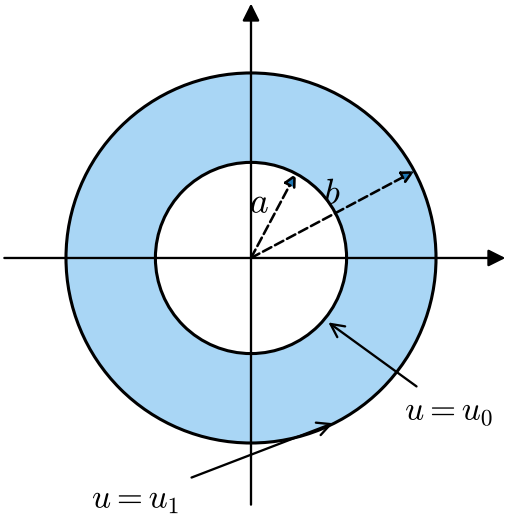
Example Evaluate $\lim\limits_{\gamma \to \omega} \dfrac{F_0}{\omega^2 - \gamma^2}(\cos \gamma t - \cos \omega t)$

Answer 32 (end of chapter) · Solution

Try it in Python: The damped harmonic oscillator solved symbolically, including the critically damped limit $\gamma \to 1$: Symbolic solution to ODEs

3.9 Linear Models: Boundary-Value Problems

Example Temperature in a Ring



The temperature $u(r)$ in circular ring is determined from the boundary-value problem

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} = 0,\ u(a) = u_0,\ u(b) = u_1$$

Answer 33 (end of chapter) · Solution

3.10 Green's Functions

3.10.1 Influence Function

- We shall consider the self-adjoint form

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = -f(x) \tag{SA}$$

  The function $p(x)$ is continuously differentiable and positive, and $q(x)$ and $f(x)$ are continuous for $\alpha < x < \beta$

- The homogeneous second-order differential equation

$$\frac{d}{dx}\left[p(x)\frac{dv}{dx}\right] + q(x)v = 0 \tag{SH}$$

  has exactly two linearly independent solutions $v_1(x)$ and $v_2(x)$: any solution of (SH) can be written in the form

$$v(x) = c_1v_1(x) + c_2v_2(x)$$

  where c_1 and c_2 are constants

- We now consider the function

$$w(x) = v_1(x)\int_\alpha^x v_2(\xi)f(\xi)\,d\xi - v_2(x)\int_\alpha^x v_1(\xi)f(\xi)\,d\xi$$

Differentiating w, we have

$$\begin{aligned}\frac{dw}{dx} &= v_1'(x)\int_\alpha^x v_2(\xi)f(\xi)\,d\xi - v_2'(x)\int_\alpha^x v_1(\xi)f(\xi)\,d\xi + \underbrace{[v_1(x)v_2(x) - v_2(x)v_1(x)]}_{=0} f(x)\\ &= v_1'(x)\int_\alpha^x v_2(\xi)f(\xi)\,d\xi - v_2'(x)\int_\alpha^x v_1(\xi)f(\xi)\,d\xi\end{aligned}$$

- Then

$$\begin{aligned}\frac{d}{dx}\left[p(x)\frac{dw}{dx}\right] &= \underbrace{\frac{d}{dx}\left[p(x)v_1'(x)\right]}_{-q(x)v_1}\int_\alpha^x v_2(\xi)f(\xi)\,d\xi - \underbrace{\frac{d}{dx}\left[p(x)v_2'(x)\right]}_{-q(x)v_2}\int_\alpha^x v_1(\xi)f(\xi)\,d\xi\\ &\quad + \underbrace{p(x)\left[v_1'(x)v_2(x) - v_2'(x)v_1(x)\right]}_{-K} f(x)\\ &= -q(x)w - Kf(x)\end{aligned}$$

where

$$\begin{aligned}&\frac{d}{dx}\left\{p(x)\left[v_1'(x)v_2(x) - v_2'(x)v_1(x)\right]\right\}\\ &\quad = \underbrace{\frac{d}{dx}\left[p(x)v_1'(x)\right]}_{-q(x)v_1(x)} v_2(x) - \underbrace{\frac{d}{dx}\left[p(x)v_2'(x)\right]}_{-q(x)v_2(x)} v_1(x) + p(x)v_1'(x)v_2'(x) - p(x)v_2'(x)v_1'(x) = 0\\ &\quad \Downarrow\\ &\quad K \text{ is constant}\end{aligned}$$

- We have shown that w satisfies the equation:

$$\frac{d}{dx}\left[p(x)\frac{dw}{dx}\right] + q(x)w = -Kf(x)$$

Moreover, as $x \to \alpha$

$$w(\alpha) = w'(\alpha) = 0$$

- Dividing by the constant K, we find that the function

$$u(x) = \int_{\alpha}^{x} R(x,\xi)\, f(\xi)\, d\xi \tag{IN}$$

where

$$R(x,\xi) = \frac{v_1(x)v_2(\xi) - v_2(x)v_1(\xi)}{K} = -\frac{v_1(x)v_2(\xi) - v_2(x)v_1(\xi)}{p(x)\left[v_1'(x)v_2(x) - v_2'(x)v_1(x)\right]}$$

is the solution of the initial value problem

$$\begin{gathered}\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = -f(x) \quad \text{for } x > \alpha \\ u(\alpha) = u'(\alpha) = 0\end{gathered}$$

- Since the denominator of $R(x,\xi)$ is a constant, the function $R(x,\xi)$ satisfies the homogeneous equation (SH) as either a function of x or ξ. In fact,

$$R(x,\xi) = -R(\xi,x)$$

- For a fixed value of ξ, $R(x,\xi)$ is completely characterized as the solution of the homogeneous initial value problem

$$\begin{gathered}\frac{d}{dx}\left[p(x)\frac{dR}{dx}\right] + q(x)R = 0 \quad \text{for } x > \xi \\ R|_{x=\xi} = 0 \\ R'|_{x=\xi} = -\frac{1}{p(\xi)}\end{gathered}$$

- The function $R(x,\xi)$ describes the influence on the value of u at x of a disturbance(impulse) concentrated at ξ. It is sometimes called the **influence function**, or the one-sided Green's function
- If the values of $u(\alpha)$ and $u'(\alpha)$ are prescribed to be other than zero, we must simply add a suitable solution $c_1v_1(x) + c_2v_2(x)$ to the expression (IN)

Example Consider the problem

$$\begin{gathered}u'' + u = -f(x) \quad \text{for } x > 0 \\ u(0) = 1 \\ u'(0) = -1\end{gathered}$$

Solution $u(x) = w(x) + v(x)$

(1) $w(x)$

$$
\begin{aligned}
w'' + w &= -f(x) \quad \text{for } x > 0 \\
w(0) &= 0 \\
w'(0) &= 0
\end{aligned}
$$

For a fixed value of ξ, the influence function $R(x, \xi)$ satisfies

$$
\begin{aligned}
\frac{d^2R}{dx^2} + R &= 0 \quad \text{for } x > \xi \\
R|_{x=\xi} &= 0 \\
R'|_{x=\xi} &= -1
\end{aligned}
$$

Thus $R(x, \xi) = \sin(\xi - x)$ and the solution is

$$
w(x) = \int_0^x \sin(\xi - x)\, f(\xi)\, d\xi
$$

(2) $v(x)$

$$
\begin{aligned}
v'' + v &= 0 \quad \text{for } x > 0 \\
v(0) &= 1 \\
v'(0) &= -1
\end{aligned}
$$

$$
\Downarrow
$$

$$
v(x) = \cos x - \sin x
$$

3.10.2 Green's Function

- Here, we consider a two-point boundary value problem:

$$
\begin{aligned}
\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u &= -f(x) \quad \text{for } \alpha < x < \beta \\
u(\alpha) = u(\beta) &= 0
\end{aligned}
\tag{P1}
$$

  Writing the general solution in the form

$$
u(x) = \int_\alpha^x R(x, \xi)\, f(\xi)\, d\xi + c_1 v_1(x) + c_2 v_2(x)
$$

  we obtain the two equations

$$
\begin{aligned}
c_1 v_1(\alpha) + c_2 v_2(\alpha) &= 0 \\
c_1 v_1(\beta) + c_2 v_2(\beta) &= -\int_\alpha^\beta R(\beta, \xi)\, f(\xi)\, d\xi
\end{aligned}
$$

  These two equations determine a unique pair of constants c_1 and c_2, provided the determinant of their coefficients is not zero; that is, provided

$$D \equiv v_1(\alpha)v_2(\beta) - v_2(\alpha)v_1(\beta) \neq 0$$

- We assume this for the moment. Then

$$\begin{aligned} c_1 &= \frac{v_2(\alpha)}{D}\int_\alpha^\beta R(\beta,\xi)\, f(\xi)\, d\xi \\ &= \frac{v_2(\alpha)}{D}\int_\alpha^x R(\beta,\xi)\, f(\xi)\, d\xi + \frac{v_2(\alpha)}{D}\int_x^\beta R(\beta,\xi)\, f(\xi)\, d\xi \\ c_2 &= -\frac{v_1(\alpha)}{D}\int_\alpha^\beta R(\beta,\xi)\, f(\xi)\, d\xi \\ &= -\frac{v_1(\alpha)}{D}\int_\alpha^x R(\beta,\xi)\, f(\xi)\, d\xi - \frac{v_1(\alpha)}{D}\int_x^\beta R(\beta,\xi)\, f(\xi)\, d\xi \end{aligned}$$

- The solution can then be written as

$$\begin{aligned} u(x) = &\int_\alpha^x \left[R(x,\xi) + \frac{v_2(\alpha)v_1(x) - v_1(\alpha)v_2(x)}{D}R(\beta,\xi)\right] f(\xi)\, d\xi \\ &+ \int_x^\beta \frac{v_2(\alpha)v_1(x) - v_1(\alpha)v_2(x)}{D}R(\beta,\xi)\, f(\xi)\, d\xi \end{aligned}$$

- We find after some algebraic manipulation that

$$\begin{aligned} G(x,\xi) &= R(x,\xi) + \frac{v_2(\alpha)v_1(x) - v_1(\alpha)v_2(x)}{D}R(\beta,\xi) \\ &= \frac{v_1(x)v_2(\xi) - v_2(x)v_1(\xi)}{K}\frac{D}{D} + \frac{v_2(\alpha)v_1(x) - v_1(\alpha)v_2(x)}{D}\frac{v_1(\beta)v_2(\xi) - v_2(\beta)v_1(\xi)}{K} \\ &= \frac{[v_1(\alpha)v_2(\xi) - v_2(\alpha)v_1(\xi)]\,[v_1(x)v_2(\beta) - v_2(x)v_1(\beta)]}{KD} \quad \text{for } \xi \leq x \end{aligned}$$

$$\begin{aligned} G(x,\xi) &= \frac{v_2(\alpha)v_1(x) - v_1(\alpha)v_2(x)}{D}R(\beta,\xi) \\ &= \frac{[v_1(x)v_2(\alpha) - v_2(x)v_1(\alpha)]\,[v_1(\beta)v_2(\xi) - v_2(\beta)v_1(\xi)]}{KD} \quad \text{for } \xi \geq x \end{aligned}$$

- Then the solution of the two-point boundary value problem (P1) can be written in the form

$$u(x) = \int_\alpha^\beta G(x,\xi)\, f(\xi)\, d\xi \qquad \text{(GR)}$$

- The function $G(x,\xi)$ is called the **Green's function** of the problem (P1). It is symmetric. That is

$$G(x,\xi) = G(\xi,x)$$

- To determine the Green's function, we note that for each ξ it satisfies the following boundary value problem

$$\begin{aligned}
\frac{d}{dx}\left[p(x)\frac{dG}{dx}\right] + q(x)G = 0 \quad \text{for } x \neq \xi \\
G|_{x=\alpha} = G|_{x=\beta} = 0 \\
G|_{x=\xi+0} - G|_{x=\xi-0} = 0 \\
G'|_{x=\xi+0} - G'|_{x=\xi-0} = -\frac{1}{p(\xi)}
\end{aligned}$$

Example Consider the problem

$$\begin{aligned}
u'' &= -f(x) \quad \text{for } 0 < x < 1 \\
u(0) &= 0 \\
u(1) &= 0
\end{aligned}$$

Solution

$$\begin{aligned}
G'' = 0 \;&\Rightarrow\; G = ax + b \\
&\Downarrow\; G|_{x=0}=G|_{x=1}=0 \\
G(x,\xi) &= \begin{cases} a_1(\xi)\, x & \text{for } x < \xi \\ a_2(\xi)(1-x) & \text{for } x > \xi \end{cases} \\
&\Downarrow\; G(x,\xi)=G(\xi,x) \\
a_1(\xi) &= A(1-\xi) \\
a_2(\xi) &= A\xi \\
&\Downarrow\; G'|_{x=\xi+0} - G'|_{x=\xi-0} = -1 \\
-A\xi - A(1-\xi) &= -1 \to A = 1 \\
&\Downarrow \\
G(x,\xi) &= \begin{cases} (1-\xi)x & \text{for } x \leq \xi \\ \xi(1-x) & \text{for } x \geq \xi \end{cases}
\end{aligned}$$

The solution is

$$u(x) = \int_0^x \xi(1-x)\, f(\xi)\, d\xi + \int_x^1 (1-\xi)x\, f(\xi)\, d\xi$$

- Two-point boundary value problems with more general boundary conditions can be treated in the same manner. We consider the problem

$$\begin{aligned}
\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = -f(x) \quad \text{for } \alpha < x < \beta \\
-\mu_1 u'(\alpha) + \sigma_1 u(\alpha) = 0 \\
\mu_2 u'(\beta) + \sigma_2 u(\beta) = 0
\end{aligned} \tag{P2}$$

- The Green's function $G(x,\xi)$ is derived as before, provided the condition

$$\begin{aligned} D \equiv & \left[-\mu_1 v_1'(\alpha) + \sigma_1 v_1(\alpha)\right]\left[\mu_2 v_2'(\beta) + \sigma_2 v_2(\beta)\right] \\ & - \left[-\mu_1 v_2'(\alpha) + \sigma_1 v_2(\alpha)\right]\left[\mu_2 v_1'(\beta) + \sigma_2 v_1(\beta)\right] \neq 0 \end{aligned}$$

  is satisfied

- The Green's function $G(x,\xi)$ is the solution of the problem

$$\begin{aligned} \frac{d}{dx}\left[p(x)\frac{dG}{dx}\right] + q(x)G = 0 \quad & \text{for } x \neq \xi \\ -\mu_1 \left.G'\right|_{x=\alpha} + \sigma_1 \left.G\right|_{x=\alpha} = \mu_2 \left.G'\right|_{x=\beta} + \sigma_2 \left.G\right|_{x=\beta} = 0 & \\ \left.G\right|_{x=\xi+0} - \left.G\right|_{x=\xi-0} = 0 & \\ \left.G'\right|_{x=\xi+0} - \left.G'\right|_{x=\xi-0} = -\frac{1}{p(\xi)} & \end{aligned}$$

  It still satisfies the symmetry relation

$$G(x,\xi) = G(\xi,x)$$

Example Consider the problem

$$\begin{aligned} u'' &= -f(x) \quad \text{for } 0 < x < 1 \\ u(0) &= 0 \\ u'(1) + \sigma_2 u(1) &= 1 \end{aligned}$$

Solution $u(x) = w(x) + v(x)$

(1) $w(x)$

$$\begin{aligned} w'' &= -f(x) \quad \text{for } 0 < x < 1 \\ w(0) &= 0 \\ w'(1) + \sigma_2 w(1) &= 0 \end{aligned}$$

$$
\begin{aligned}
G'' &= 0 \\
&\Downarrow \\
G &= ax + b \\
&\Downarrow G|_{x=0} = G'|_{x=1} + \sigma_2 G|_{x=1} = 0 \\
G(x,\xi) &= \begin{cases} a_1(\xi)\, x & \text{for } x < \xi \\ a_2(\xi)\,[1 + \sigma_2(1-x)] & \text{for } x > \xi \end{cases} \\
&\Downarrow G(x,\xi) = G(\xi,x) \\
a_1(\xi) &= A\,[1 + \sigma_2(1-\xi)] \\
a_2(\xi) &= A\xi \\
&\Downarrow G'|_{x=\xi+0} - G'|_{x=\xi-0} = -1 \\
-A\sigma_2\xi - A\,[1 + \sigma_2(1-\xi)] &= -1 \to A = \frac{1}{1+\sigma_2} \\
&\Downarrow \\
G(x,\xi) &= \begin{cases} \frac{[1+\sigma_2(1-\xi)]\,x}{1+\sigma_2} & \text{for } x \le \xi \\ \frac{\xi\,[1+\sigma_2(1-x)]}{1+\sigma_2} & \text{for } x \ge \xi \end{cases}
\end{aligned}
$$

The solution is

$$
w(x) = \int_0^x \frac{\xi\ [1 + \sigma_2(1-x)]}{1+\sigma_2}\, f(\xi)\, d\xi + \int_x^1 \frac{[1 + \sigma_2(1-\xi)]\ x}{1+\sigma_2}\, f(\xi)\, d\xi
$$

(2) $v(x)$

$$
\begin{aligned}
v'' &= 0 \quad \text{for } 0 < x < 1 \\
v(0) &= 0 \\
v'(1) + \sigma_2 v(1) &= 1 \\
&\Downarrow \\
v(x) &= \frac{x}{1+\sigma_2}
\end{aligned}
$$

Example Consider the problem

$$
\begin{aligned}
&r^2u'' + 2ru' - n(n+1)u = -r^2F(r), \ \ 0 < r < R \\
&u(R) = 0, \ |u| < \infty
\end{aligned}
$$

Solution

$$\frac{d}{dr}\left[r^2\frac{dG}{dr}\right]-n(n+1)G=0$$

$$\Downarrow$$

$$G=c_1r^n+c_2r^{-(n+1)}$$

$$\Downarrow G|_{r=0}\ =\ \text{bounded}\ ,\ G|_{r=R}\ =0$$

$$G(r,\rho)=\begin{cases}a_1(\rho)r^n & \text{for } r<\rho\\ a_2(\rho)\left[\left(\frac{r}{R}\right)^{-(n+1)}-\left(\frac{r}{R}\right)^n\right] & \text{for } r>\rho\end{cases}$$

$$\Downarrow G(r,\rho)=G(\rho,r)$$

$$a_1(\rho)=A\left[\left(\frac{\rho}{R}\right)^{-(n+1)}-\left(\frac{\rho}{R}\right)^n\right]$$

$$a_2(\rho)=A\rho^n$$

$$\Downarrow G'|_{r=\rho+0}\ -\ G'|_{r=\rho-0}\ =-1/\rho^2$$

$$\frac{A\rho^n}{R}\left[-(n+1)\left(\frac{\rho}{R}\right)^{-(n+2)}-n\left(\frac{\rho}{R}\right)^{n-1}\right]-An\rho^{n-1}\left[\left(\frac{\rho}{R}\right)^{-(n+1)}-\left(\frac{\rho}{R}\right)^n\right]=-\frac{1}{\rho^2}\ \ \rightarrow\ \ A=\frac{1}{(2n+1)R^{n+1}}$$

$$\Downarrow$$

$$G(r,\rho)=\begin{cases}\frac{1}{(2n+1)R}\left(\frac{r}{R}\right)^n\left[\left(\frac{\rho}{R}\right)^{-(n+1)}-\left(\frac{\rho}{R}\right)^n\right] & \text{for } r\le\rho\\ \frac{1}{(2n+1)R}\left(\frac{\rho}{R}\right)^n\left[\left(\frac{r}{R}\right)^{-(n+1)}-\left(\frac{r}{R}\right)^n\right] & \text{for } r\ge\rho\end{cases}$$

The solution is

$$u(r)=\int_0^R G(r,\rho)\,F(\rho)\rho^2d\rho$$

3.11 Solving Nonlinear Models

- In order to analyze an n^{th} order IVP numerically, we express the n^{th} order ODE as a system of n first-order equations
- For example,

$$\frac{d^2y}{dx^2}=f(x,y,y'),\ \ y(x_0)=y_0,\ y'(x_0)=y_1$$

$$\Downarrow y'=u$$

$$\text{Solve}:\begin{cases}y'=u\\ u'=f(x,y,u)\end{cases}$$

$$\text{subject to}: y(x_0)=y_0,\ u(x_0)=y_1$$

Example Use a numerical solver to obtain the solution curves satisfying the given initial conditions:

$$\frac{d^2x}{dt^2}+\frac{dx}{dt}+x+x^3=0$$

$$x(0) = -3,\ \dot{x}(0) = 4, \quad \text{or} \quad x(0) = 0,\ \dot{x}(0) = -8$$

Solution

$$\begin{aligned} \frac{d^2x}{dt^2} + \frac{dx}{dt} &= -x - x^3 \\ &\Downarrow \\ \dot{x} &= u \\ \dot{u} &= -x - x^3 - u \end{aligned}$$

```
import numpy as np
from scipy.integrate import solve_ivp

def func(t, y):
    return [y[1], -y[0] -y[0]**3 -y[1]]

tf = 14
t_eval = np.linspace(0, tf, 200)

sol1 = solve_ivp(func, [0, tf], [-3, 4], t_eval=t_eval)
sol2 = solve_ivp(func, [0, tf], [0, -8], t_eval=t_eval)
```

```
import matplotlib.pyplot as plt
plt.style.use('ggplot')

plt.figure(figsize=(6, 4))

plt.plot(sol1.t, sol1.y[0], 'b-', label=r'$x_0=-3,\; \dot{x}_0=4$')
plt.plot(sol2.t, sol2.y[0], 'r-', label=r'$x_0=0,\; \dot{x}_0=-8$')
plt.axis((0, tf, -3, 3))
plt.xlabel('t')
plt.ylabel('x')
plt.legend()

plt.show()
```

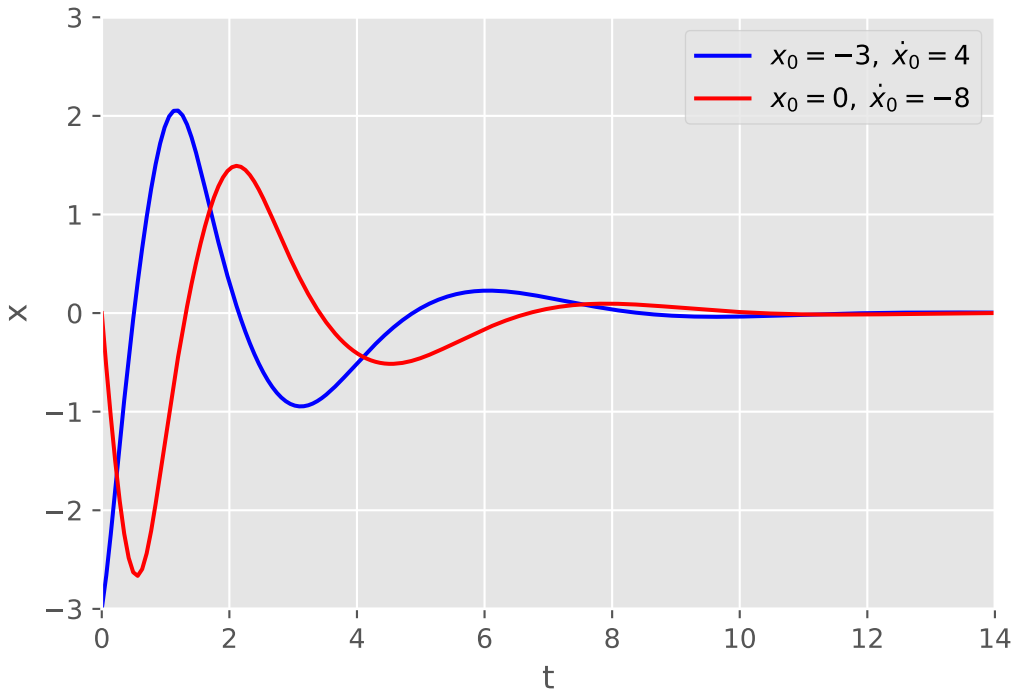
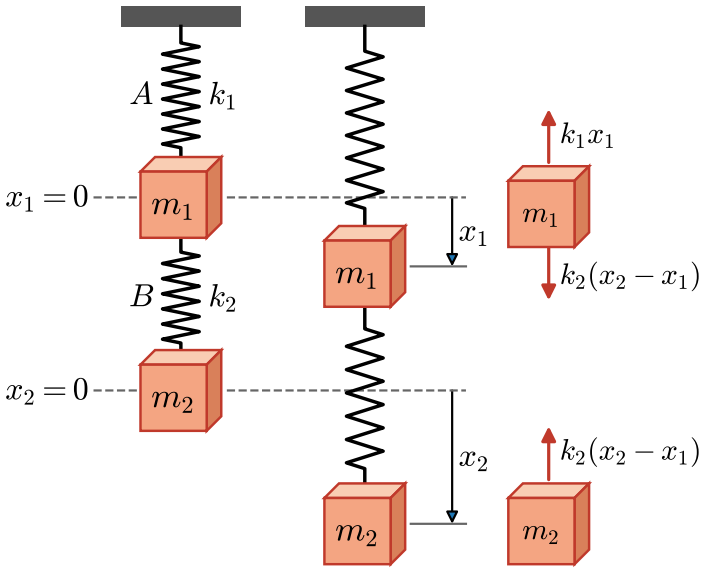


Figure 3.1: $\frac{d^2x}{dt^2} + \frac{dx}{dt} + x + x^3 = 0$

Try it in Python: Nonlinear second-order models as first-order systems with `solve_ivp`: Numerical integration of ODEs using 'scipy'; the double pendulum: Double Pendulum

3.12 Solving System of Linear Equations

When physical systems are coupled, the mathematical model of the system usually consists of a set of coupled DEs



$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1)$$
$$m_2\ddot{x}_2 = -k_2(x_2 - x_1)$$

Linear systems with constant coefficients can be solved by uncoupling the system into distinct linear ODEs in each dependent variable

Example Solve the above equation under the assumption that $k_1 = 6$, $k_2 = 4$, $m_1 = 1$, and $m_2 = 1$ subject to

$$x_1(0) = 0,\ x_1'(0) = 1,\ x_2(0) = 0,\ x_2'(0) = -1$$

```
k1, k2, m1, m2 = 6, 4, 1, 1

def func(t, y):
    return [y[1], -(k1 + k2)/m1 * y[0] + k2/m1 * y[2],
            y[3], k2/m2 * y[0] - k2/m2 * y[2]]

tf = 14
t_eval = np.linspace(0, tf, 200)

sol = solve_ivp(func, [0, tf], [0, 1, 0, -1], t_eval=t_eval)
```

```
plt.figure(figsize=(6, 4))

plt.plot(sol.t, sol.y[0], 'b-', label='$x_1$')
plt.plot(sol.t, sol.y[2], 'r-', label='$x_2$')
plt.axis((0, tf, -0.6, 0.6))
```

```
plt.legend()
plt.xlabel('t')
plt.ylabel('x')

plt.show()
```

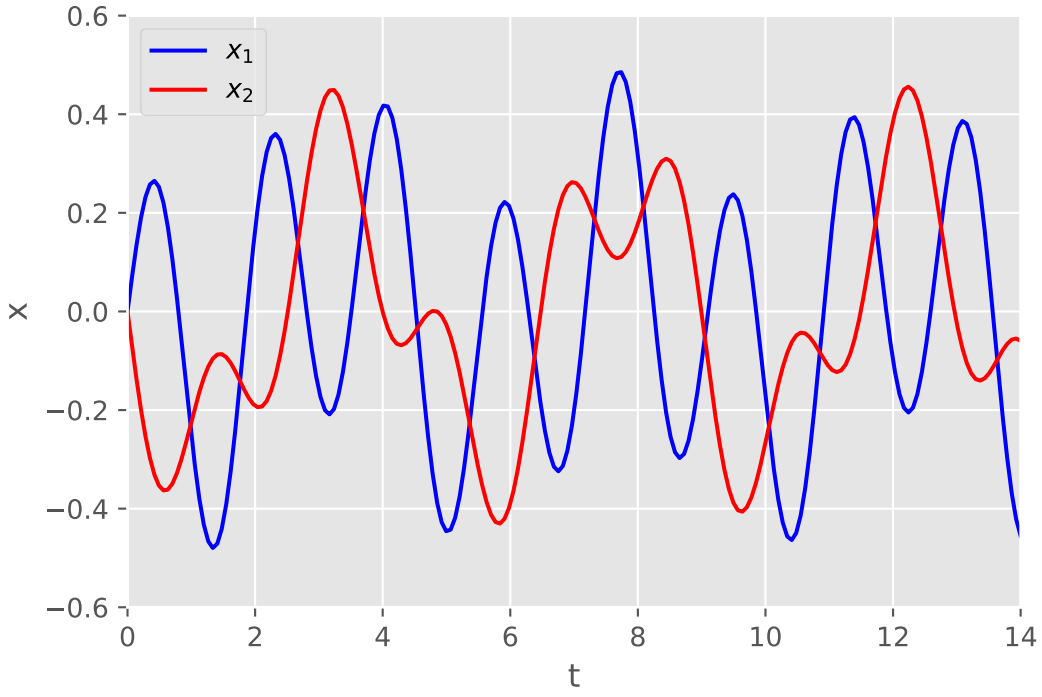


Figure 3.2: Coupled Spring/Mass System

Try it in Python: Coupled spring-mass systems integrated numerically: Numerical integration of ODEs using 'scipy'

Summary

Topic	Form	Result
General solution	linear, order n	$y = c_1y_1 + \cdots + c_ny_n + y_p$; $y_1, \cdots, y_n$ independent iff $W \neq 0$
Reduction of order	$y'' + P(x)y' + Q(x)y = 0$, y_1 known	$y_2 = y_1 \int \frac{e^{-\int P\,dx}}{y_1^2}\,dx$
Constant coefficients	$ay'' + by' + cy = 0$, $ap^2 + bp + c = 0$	$c_1e^{p_1x} + c_2e^{p_2x}$; $(c_1 + c_2x)e^{p_1x}$; $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$
Undetermined coefficients	constant coefficients; g: polynomial, e^{ax}, sin, cos	y_p of the same form as g, multiplied by x^k if it duplicates y_h

Topic	Form	Result
Variation of parameters	$y'' + P(x)y' + Q(x)y = f(x)$	$y_p = u_1 y_1 + u_2 y_2,\ \ u_1' = -\dfrac{y_2 f}{W},$ $u_2' = \dfrac{y_1 f}{W}$
Cauchy-Euler	$ax^2 y'' + bxy' + cy = 0,\ \ y = x^p$	$c_1 x^{p_1} + c_2 x^{p_2};\ \ x^{p_1}(c_1 + c_2 \ln x);$ $x^\alpha [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$
Green's function	$\dfrac{d}{dx}[p\,u'] + qu = -f$ with BCs	$u(x) = \displaystyle\int_\alpha^\beta G(x,\xi) f(\xi)\, d\xi,$ $G(x,\xi) = G(\xi, x)$
Spring-mass	$mx'' + \beta x' + kx = f(t)$	pure resonance: $x'' + \omega^2 x = F_0 \cos \omega t$ gives $\dfrac{F_0}{2\omega}\, t \sin \omega t$
Nonlinear	y or x missing	$u = y';$ if x is missing, $y'' = u\,\dfrac{du}{dy}$

Worked Exercises

1. Solve the given differential equation

$$3y'' - 6y' + 30y = 15 \sin x + e^x \tan 3x$$

Solution

Step 1: Solve the homogeneous equation

$$3y'' - 6y' + 30y = 0$$

Divide through by 3:

$$y'' - 2y' + 10y = 0$$

This is a linear equation with constant coefficients. The characteristic equation is:

$$p^2 - 2p + 10 = 0$$

Solve using the quadratic formula:

$$p = 1 \pm 3i$$

So the general solution to the homogeneous equation is:

$$y_h(x) = e^x \left(c_1 \cos 3x + c_2 \sin 3x\right)$$

Step 2: Find a particular solution

The nonhomogeneous term is:

$$15\sin x + e^x \tan 3x$$

We'll treat each part separately:

(a) $15\sin x$:

We use the method of undetermined coefficients

Guess:

$$y_{p1}(x) = A\cos x + B\sin x$$

Plug into the left-hand side of the homogeneous operator and simplify

$$3y'' - 6y' + 30y = (27A - 6B)\cos x + (6A + 27B)\sin x$$

We want this equal to the right-hand side: $15\sin x$

So $27A - 6B = 0$, $6A + 27B = 15$, which give

$$A = \frac{2}{17}$$
$$B = \frac{9}{17}$$

(b) $e^x \tan 3x$:

$\tan 3x$ is not one of the input functions of the method of undetermined coefficients, so we use the method of variation of parameters

We now solve:

$$y'' - 2y' + 10y = \frac{1}{3}e^x \tan 3x$$

The fundamental solutions to the homogeneous part are:

- $y_1 = e^x \cos 3x$
- $y_2 = e^x \sin 3x$

We apply the method of variation of parameters:

$$y_{p2}(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where:

$$u_1' = -\frac{y_2 f}{W}$$
$$u_2' = \frac{y_1 f}{W}$$

Compute the Wronskian W

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 3e^{2x}$$

Compute u_1', u_2'

$$u_1' = \frac{-e^x \sin 3x \cdot \frac{1}{3} e^x \tan 3x}{3e^{2x}} = -\frac{1}{9} \sin 3x \tan 3x$$
$$u_2' = \frac{e^x \cos 3x \cdot \frac{1}{3} e^x \tan 3x}{3e^{2x}} = \frac{1}{9} \sin 3x$$

Now integrate:

$$u_1(x) = -\frac{1}{9} \int \frac{1 - \cos^2 3x}{\cos 3x}\, dx = \frac{1}{27} \left(\sin 3x - \ln |\sec 3x + \tan 3x|\right)$$
$$u_2(x) = \frac{1}{9} \int \sin 3x\, dx = -\frac{1}{27} \cos 3x$$

so that

$$y_{p2} = u_1 y_1 + u_2 y_2 = -\frac{1}{27} e^x \cos 3x \ln |\sec 3x + \tan 3x|$$

Step 3: General solution

$$y = e^x \left(c_1 \cos 3x + c_2 \sin 3x\right) + \frac{2}{17} \cos x + \frac{9}{17} \sin x - \frac{1}{27} e^x \cos 3x \ln |\sec 3x + \tan 3x|$$

2. Solve the given differential equation

$$y'' - 2y' + y = 4x^2 - 3 + x^{-1} e^x$$

Solution

Step 1: Solve the homogeneous equation

The associated homogeneous equation is:

$$y'' - 2y' + y = 0$$

This is a linear equation with constant coefficients. Its characteristic equation is:

$$p^2 - 2p + 1 = 0 \quad \Rightarrow \quad (p-1)^2 = 0$$

So we have a repeated root: $p = 1$

Thus, the general solution to the homogeneous equation is:

$$y_h(x) = (c_1 + c_2 x) e^x$$

Step 2: Find a particular solution

We now look for a particular solution to:

$$y'' - 2y' + y = 4x^2 - 3 + \frac{e^x}{x}$$

This is a nonhomogeneous term with two parts:

- $f_1(x) = 4x^2 - 3$ (a polynomial)
- $f_2(x) = e^x/x$ (a non-polynomial term)

We'll compute the particular solution as:

$$y_p = y_{p1} + y_{p2}$$

(a) $4x^2 - 3$

Since the right-hand side is a polynomial, we use undetermined coefficients. Try a solution of the form:

$$y_{p1} = Ax^2 + Bx + C$$

Substitute into the differential operator:

$$y_{p1}'' - 2y_{p1}' + y_{p1} = 2A - 2(2Ax + B) + (Ax^2 + Bx + C)$$

Set this equal to the right-hand side:

$$Ax^2 + (-4A + B)x + (2A - 2B + C) = 4x^2 - 3$$

Match coefficients and so:

$$y_{p1}(x) = 4x^2 + 16x + 21$$

(b) $\dfrac{e^x}{x}$

This is not suitable for undetermined coefficients — instead, we use variation of parameters.

We already have two linearly independent solutions to the homogeneous equation:

- $y_1 = e^x$
- $y_2 = xe^x$

Let:

$$y_{p2}(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

Where:

$$u_1{}' = \frac{-y_2(x) \cdot f(x)}{W(y_1, y_2)}$$
$$u_2{}' = \frac{y_1(x) \cdot f(x)}{W(y_1, y_2)}$$

Compute the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{vmatrix} = e^x(1+x)e^x - xe^x \cdot e^x = e^{2x}(1+x-x) = e^{2x}$$

Then:

$$u_1{}' = \frac{-xe^x \cdot \frac{e^x}{x}}{e^{2x}} = \frac{-e^{2x}}{e^{2x}} = -1 \quad \Rightarrow \quad u_1 = -x$$
$$u_2{}' = \frac{e^x \cdot \frac{e^x}{x}}{e^{2x}} = \frac{1}{x} \quad \Rightarrow \quad u_2 = \ln|x|$$

So:

$$y_{p2}(x) = u_1y_1 + u_2y_2 = (-x)e^x + \ln|x| \cdot (xe^x) = xe^x(\ln|x| - 1)$$

Step 3: General solution

The term $-xe^x$ is a solution of the homogeneous equation and can be absorbed into c_2xe^x:

$$y = (c_1 + c_2x)e^x + 4x^2 + 16x + 21 + xe^x \ln|x|$$

3. Find a member of the family of solutions of

$$xy'' + y' + \sqrt{x} = 0$$

whose graph is tangent to the x-axis at $x = 1$

Solution

We are given the second-order linear nonhomogeneous differential equation:

$$xy'' + y' + \sqrt{x} = 0$$

and the condition that the solution's graph is tangent to the x-axis at $x = 1$

This implies:

- The solution satisfies $y(1) = 0$
- The slope is zero at $x = 1$, so $y'(1) = 0$

Step 1: Solve the homogeneous differential equation

We can treat this as a linear second-order ODE:

$$xy'' + y' = -\sqrt{x}$$

This is a linear ODE in standard form:

$$y'' + \frac{1}{x}y' = -\frac{\sqrt{x}}{x} = -x^{-1/2}$$

Let's solve the homogeneous equation first:

$$y'' + \frac{1}{x}y' = 0$$

Use standard method for linear homogeneous equations

Let $y' = u$, then $y'' = u'$, so:

$$u' + \frac{1}{x}u = 0$$

This is a separable first-order equation:

$$\begin{aligned} \frac{du}{u} = -\frac{dx}{x} \;&\Rightarrow\; \ln|u| = -\ln|x| + C \\ \Rightarrow\; u = \frac{c_1}{x} \;&\Rightarrow\; y' = \frac{c_1}{x} \;\Rightarrow\; y = c_1 \ln x + c_2 \end{aligned}$$

So the general solution of the homogeneous equation is:

$$y_h(x) = c_1 \ln x + c_2$$

Step 2: Find a particular solution to the nonhomogeneous equation

We need a particular solution y_p to:

$$y'' + \frac{1}{x}y' = -x^{-1/2}$$

We can use variation of parameters, since we already have two independent solutions to the homogeneous equation:

$$y_1 = \ln x, \quad y_2 = 1$$

Compute the Wronskian:

$$W = \begin{vmatrix} \ln x & 1 \\ \frac{1}{x} & 0 \end{vmatrix} = -\frac{1}{x}$$

The particular solution is:

$$y_p = -\ln x \int \frac{1 \cdot (-x^{-1/2})}{-1/x}\,dx + 1 \cdot \int \frac{\ln x \cdot (-x^{-1/2})}{-1/x}\,dx = -\frac{4}{9}x^{3/2}$$

Step 3: General solution

$$y(x) = c_1 \ln x + c_2 - \frac{4}{9}x^{3/2}$$

We are told the graph is tangent to the x-axis at $x = 1$. Compute $y(1)$ and $y'(x)$:

$$y(1) = c_1 \ln 1 + c_2 - \frac{4}{9}(1)^{3/2} = 0 + c_2 - \frac{4}{9} = 0 \Rightarrow c_2 = \frac{4}{9}$$

$$y'(1) = c_1 - \frac{2}{3} = 0 \Rightarrow c_1 = \frac{2}{3}$$

The solution is

$$y(x) = \frac{2}{3}\ln x + \frac{4}{9} - \frac{4}{9}x^{3/2}$$

4. Find all solutions of the following equation:

$$y'' + y = \tan x, \quad (-\pi/2 < x < \pi/2)$$

Solution

Step 1: Solve the Homogeneous Equation

First, solve the homogeneous part:

$$y_h'' + y_h = 0$$

The characteristic equation is:

$$p^2 + 1 = 0 \Rightarrow p = \pm i$$

So the general solution of the homogeneous equation is:

$$y_h(x) = c_1 \cos x + c_2 \sin x$$

Step 2: Find a Particular Solution $y_p(x)$

We now need a particular solution to the nonhomogeneous equation:

$$y'' + y = \tan x$$

We use the method of variation of parameters. So the particular solution is:

$$y_p = u_1(x)\cos x + u_2(x)\sin x$$

Where:

$$u_1{}'(x) = -\frac{\sin x \cdot \tan x}{W} = -\sin x \tan x, \quad u_2{}'(x) = \frac{\cos x \cdot \tan x}{W} = \cos x \tan x$$

Integrate:

$$u_1(x) = -\int \frac{\sin^2 x}{\cos x} dx, \quad u_2(x) = \int \sin x \, dx = -\cos x$$

Now compute $u_1(x)$:

$$\int \frac{\sin^2 x}{\cos x} dx = \int \frac{1-\cos^2 x}{\cos x} dx = \int \left(\frac{1}{\cos x} - \cos x\right) dx = \int \sec x \, dx - \int \cos x \, dx$$

Therefore:

$$u_1(x) = -\left(\ln|\sec x + \tan x| - \sin x\right) = \sin x - \ln|\sec x + \tan x|$$

Now plug into:

$$y_p = u_1(x)\cos x + u_2(x)\sin x$$

So we get:

$$y_p(x) = -\cos x \ln|\sec x + \tan x|$$

Step 3: General Solution

Combine homogeneous and particular solutions:

$$y(x) = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

5. Solve the given initial-value problem on the interval $(-\infty, 0)$

$$4x^2y'' + y = 0, \;\; y(-1) = 2, \; y'(-1) = 4$$

Solution

Step 1: Identify the Type of Equation

This is a Cauchy-Euler equation. Since the interval is $x < 0$, define:

$$x = -e^t \quad \text{so } t = \ln(-x)$$

Then:

- $y(x) = Y(t)$
- Chain rule:

$$\frac{dy}{dx} = \frac{dY}{dt} \cdot \frac{dt}{dx} = \frac{1}{x}\frac{dY}{dt}, \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dY}{dt}\right) = -\frac{1}{x^2}\frac{dY}{dt} + \frac{1}{x^2}\frac{d^2Y}{dt^2}$$

So:

$$y'' = \frac{1}{x^2}\left(\frac{d^2Y}{dt^2} - \frac{dY}{dt}\right)$$

Now plug into the original equation:

$$4x^2y'' + y = 0 \Rightarrow 4x^2 \cdot \left[\frac{1}{x^2}\left(Y'' - Y'\right)\right] + Y = 0 \Rightarrow 4(Y'' - Y') + Y = 0$$

Thus we get the constant-coefficient equation:

$$4Y'' - 4Y' + Y = 0$$

Step 2: Solve the Constant-Coefficient ODE

$$4Y'' - 4Y' + Y = 0 \Rightarrow 4p^2 - 4p + 1 = 0 \Rightarrow p = \frac{1}{2} \text{ (repeated root)}$$

So the general solution is:

$$Y(t) = (A + Bt)e^{t/2}$$

Recall $x = -e^t \Rightarrow t = \ln(-x)$, so:

$$Y(t) = [A + B\ln(-x)] \cdot (-x)^{1/2}$$

Thus:

$$y(x) = \sqrt{-x}\,[A + B\ln(-x)], \quad x < 0$$

Step 3: Apply Initial Conditions

Step 3.1: Evaluate $y(-1)$

$$y(-1) = \sqrt{1}(A + B\ln 1) = A \Rightarrow A = 2$$

Step 3.2: Compute $y'(x)$

Since $\frac{d}{dx}\ln(-x) = \frac{1}{x}$,

$$y'(x) = -\frac{1}{2\sqrt{-x}}(A + B\ln(-x)) + \sqrt{-x} \cdot \frac{B}{x} = -\frac{A + B\ln(-x)}{2\sqrt{-x}} + \frac{B\sqrt{-x}}{x}$$

Now evaluate at $x = -1$:

$$y'(-1) = -\frac{A + B \cdot 0}{2} + B \cdot \frac{1}{-1} = -\frac{A}{2} - B$$

We already found $A = 2$, so:

$$4 = -1 - B \Rightarrow B = -5$$

Final Answer:

$$y(x) = \sqrt{-x}\,(2 - 5\ln(-x))\,, \quad x < 0$$

This is the unique solution on the interval $(-\infty, 0)$ satisfying the initial conditions

6. Find all solutions of the following equation:

$$y'' + y = \sec x, \;\; (-\pi/2 < x < \pi/2)$$

Solution

Step 1: Solve the Homogeneous Equation

First, solve the homogeneous equation:

$$y_h'' + y_h = 0$$

Its characteristic equation is:

$$p^2 + 1 = 0 \quad \Rightarrow \quad p = \pm i$$

So the general solution to the homogeneous part is:

$$y_h(x) = c_1 \cos x + c_2 \sin x$$

Step 2: Find a Particular Solution Using Variation of Parameters

The Wronskian is:

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Formulas for variation of parameters:

$$y_p = u_1(x) \cos x + u_2(x) \sin x$$

where:

$$u_1'(x) = -\frac{\sin x \cdot \sec x}{W} = -\tan x, \qquad u_2'(x) = \frac{\cos x \cdot \sec x}{W} = 1$$

Integrate both:

$$u_1(x) = -\int \tan x \, dx = \ln|\cos x|, \qquad u_2(x) = \int 1 \, dx = x$$

So:

$$y_p = \cos x \cdot \ln|\cos x| + \sin x \cdot x$$

Step 3: General Solution

The general solution is:

$$y(x) = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x \quad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

This is the full set of solutions to the given equation

7. Find a homogeneous Cauchy-Euler differential equation whose general solution is given:

$$y = c_1 + c_2 x + c_3 x \ln x$$

Solution

Step 1: Recall the Form of a Cauchy-Euler Equation

A homogeneous Cauchy-Euler equation of order 3 has the form:

$$x^3 y''' + a_2 x^2 y'' + a_1 x y' + a_0 y = 0$$

Our goal is to determine such coefficients a_2, a_1, a_0 so that the general solution matches the given one

Step 2: Find Linearly Independent Solutions

The given general solution:

$$y(x) = c_1 + c_2 x + c_3 x \ln x$$

has three linearly independent components:

- $y_1 = 1$
- $y_2 = x$
- $y_3 = x \ln x$

We note:

- $y_1 = 1$ is constant
- $y_2 = x$ is a power of x
- $y_3 = x \ln x$ suggests a repeated root

The general solution of a Cauchy-Euler equation with constant coefficients is of the form:

$$y(x) = x^r, \quad x^r \ln x, \quad \text{etc}$$

Given the presence of $x \ln x$, this suggests the repeated root $r = 1$. So the solution corresponds to the roots of the indicial (auxiliary) equation:

$$(r-0)(r-1)^2 = 0 \Rightarrow r^3 - 2r^2 + r = 0$$

Note that $r^3 - 2r^2 + r$ is the auxiliary (indicial) polynomial, **not** the list of coefficients of the differential equation: for x^r we have $x^3(x^r)''' = r(r-1)(r-2)x^r$, $x^2(x^r)'' = r(r-1)x^r$, $x(x^r)' = rx^r$. The coefficients are found in Step 3

Step 3: Use the General Indicial Form

For a third-order Cauchy-Euler equation, the general form of the indicial equation is:

$$r(r-1)(r-2) + a_2 r(r-1) + a_1 r + a_0 = 0$$

Comparing coefficients:

$$\begin{aligned}
&r^3 : 1 = 1 \\
&r^2 : -3 + a_2 = -2 \Rightarrow a_2 = 1 \\
&r^1 : 2 - a_2 + a_1 = 1 \Rightarrow 2 - 1 + a_1 = 1 \Rightarrow a_1 = 0 \\
&r^0 : a_0 = 0
\end{aligned}$$

So the equation is:

$$x^3 y''' + x^2 y'' = 0$$

8. Solve $2y'' + 2y' + y = 4\sqrt{x}$

Solution

Step 1: Homogeneous part

$$\begin{aligned}
&2y''+2y' + y = 0 \\
&\Downarrow \quad y = e^{px} \\
&2p^2 + 2p + 1 = 0 \quad \rightarrow p = -\frac{1}{2} \pm \frac{1}{2}i \\
&\Downarrow \\
&y_1 = e^{-\frac{1}{2}x} \cos \frac{1}{2}x, \;\; y_2 = e^{-\frac{1}{2}x} \sin \frac{1}{2}x
\end{aligned}$$

Step 2: Nonhomogeneous part: Method of variation of parameters

$$W = \begin{vmatrix} e^{-\frac{1}{2}x} \cos \frac{1}{2}x & e^{-\frac{1}{2}x} \sin \frac{1}{2}x \\ -\frac{1}{2}e^{-\frac{1}{2}x}\left(\cos \frac{1}{2}x + \sin \frac{1}{2}x\right) & \frac{1}{2}e^{-\frac{1}{2}x}\left(\cos \frac{1}{2}x - \sin \frac{1}{2}x\right) \end{vmatrix} = \frac{1}{2}e^{-x}$$

$$W_1 = \begin{vmatrix} 0 & e^{-\frac{1}{2}x}\sin\frac{1}{2}x \\ 2\sqrt{x} & \frac{1}{2}e^{-\frac{1}{2}x}\left(\cos\frac{1}{2}x - \sin\frac{1}{2}x\right) \end{vmatrix} = -2\sqrt{x}\,e^{-\frac{1}{2}x}\sin\frac{1}{2}x$$

$$W_2 = \begin{vmatrix} e^{-\frac{1}{2}x}\cos\frac{1}{2}x & 0 \\ -\frac{1}{2}e^{-\frac{1}{2}x}\left(\cos\frac{1}{2}x + \sin\frac{1}{2}x\right) & 2\sqrt{x} \end{vmatrix} = 2\sqrt{x}\,e^{-\frac{1}{2}x}\cos\frac{1}{2}x$$

$$\Downarrow$$

$$u_1 = \int \frac{W_1}{W}\,dx = -4\int \sqrt{x}\,e^{\frac{1}{2}x}\sin\frac{1}{2}x\,dx$$

$$u_2 = \int \frac{W_2}{W}\,dx = 4\int \sqrt{x}\,e^{\frac{1}{2}x}\cos\frac{1}{2}x\,dx$$

$$\Downarrow$$

$$y = c_1y_1 + c_2y_2 + u_1y_1 + u_2y_2$$

The integrals for u_1 and u_2 are not elementary, so y_p is left in integral form (for example, with lower limit 0 and the upper limit x)

Answers

Answer 1 (Section 3.1)

$x(t) = x_0\cos\omega t + \dfrac{x_1}{\omega}\sin\omega t$

Answer 2 (Section 3.1)

(1) Linearly dependent: $4f_1 - 3f_2 - f_3 = 0$

(2) Linearly dependent: $f_1 - 5f_2 - 5f_3 = 0$

Answer 3 (Section 3.1)

(1) $W = 7e^x \neq 0$; $\quad y = c_1e^{-3x} + c_2e^{4x}$

(2) $W = 2e^{2x} \neq 0$; $\quad y = e^x(c_1\cos 2x + c_2\sin 2x)$

Answer 4 (Section 3.1)

(1) $W(e^{2x}, e^{5x}) = 3e^{7x} \neq 0$ and $y_p = 6e^x$ satisfies the equation

(2) $W(e^{2x}, xe^{2x}) = e^{4x} \neq 0$ and $y_p = x^2e^{2x} + x - 2$ satisfies the equation

Answer 5 (Section 3.2)

(1) $y_2 = xe^{2x}$

(2) $y_2 = \sin 4x$

(3) $y_2 = \sinh x$

Answer 6 (Section 3.2)

$y_2 = e^{2x}$, $\quad y_p = -\dfrac{1}{2}$, $\quad y = c_1e^{2x} + c_2e^{-2x} - \dfrac{1}{2}$

Answer 7 (Section 3.3)

$y'' + \omega^2 y = 0$: $y = c_1 \cos \omega x + c_2 \sin \omega x$

$y'' - \omega^2 y = 0$: $y = c_1 e^{\omega x} + c_2 e^{-\omega x} = C_1 \cosh \omega x + C_2 \sinh \omega x$

Answer 8 (Section 3.3)

$y = c_1 e^{x/3} + e^{-x} \left(c_2 \cos \sqrt{3} x + c_3 \sin \sqrt{3} x \right)$

Answer 9 (Section 3.3)

(1) $y = c_1 + c_2 e^{-x/4}$

(2) $y = c_1 e^{3x} + c_2 e^{-2x}$

(3) $y = (c_1 + c_2 x) e^{-4x}$

Answer 10 (Section 3.3)

(1) $y = c_1 + c_2 e^{5x} + c_3 e^{-x}$

(2) $y = c_1 e^{-x} + (c_2 + c_3 x) e^{3x}$

Answer 11 (Section 3.3)

$y = \dfrac{5}{36} - \dfrac{5}{36} e^{-6x} + \dfrac{1}{6} x e^{-6x}$

Answer 12 (Section 3.4)

$y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{1}{3} x \cos x + \dfrac{2}{9} \sin x$

Answer 13 (Section 3.4)

$y = c_1 e^{3x} + c_2 x e^{3x} + \dfrac{2}{3} x^2 + \dfrac{8}{9} x + \dfrac{2}{3} - 6x^2 e^{3x}$

Answer 14 (Section 3.4)

$y = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x} + \dfrac{1}{6} x^3 + \left(\dfrac{1}{3} x^3 + 3x^2 + 12x \right) e^{-x}$

Answer 15 (Section 3.4)

(1) $y = (c_1 + c_2 x) e^{-x} - \dfrac{1}{2} \cos x - \dfrac{9}{25} \cos 2x + \dfrac{12}{25} \sin 2x$

(2) $y = c_1 e^{-6x} + c_2 e^{4x} - \dfrac{2}{3} - \left(\dfrac{1}{20} x^2 + \dfrac{19}{100} x \right) e^{4x}$

(3) $y = c_1 + c_2 x + c_3 e^{6x} - \dfrac{1}{4} x^2 - \dfrac{6}{37} \cos x + \dfrac{1}{37} \sin x$

Answer 16 (Section 3.4)

(1) $y = \sqrt{2} \sin 2x - \dfrac{1}{2}$

(2) $y = -200 + 200 e^{-x/5} - 3x^2 + 30x$

(3) $x(t) = \dfrac{F_0}{2\omega^2} \sin \omega t - \dfrac{F_0}{2\omega} t \cos \omega t$

Answer 17 (Section 3.5)

$y = c_1 e^x + c_2 e^{-x} + \displaystyle\int_{x_0}^{x} \frac{\sinh(x-t)}{t} \, dt$

Answer 18 (Section 3.5)

(1) $y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$

(2) $y = c_1 \cos x + c_2 \sin x - \dfrac{1}{2} x \cos x$

(3) $y = c_1 \cos x + c_2 \sin x + \dfrac{1}{2} - \dfrac{1}{6} \cos 2x$

(4) $y = e^x (c_1 \cos x + c_2 \sin x) + \dfrac{1}{3} e^x (\cos x \ln|\cos x| + x \sin x)$

Answer 19 (Section 3.5)

(1) $y = \left(\dfrac{1}{8} x^2 - \dfrac{1}{4} x + \dfrac{3}{4} \right) e^{x/2} + \dfrac{1}{4} e^{-x/2}$

(2) $y = (1 - x) e^x - e^x \ln(\cos x), \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

(3) $y = \dfrac{25}{36} e^{2x} + \dfrac{4}{9} e^{-4x} - \dfrac{1}{4} e^{-2x} + \dfrac{1}{9} e^{-x}$

Answer 20 (Section 3.5)

(1) $y = c_1 e^{2x} + c_2 e^{-2x} + \dfrac{1}{4} e^{2x} \ln x - \dfrac{1}{4} e^{-2x} \displaystyle\int_{x_0}^{x} \frac{e^{4t}}{t}\, dt, \quad x > 0$

(2) $y = e^{-x/2} \left(c_1 \cos \frac{x}{2} + c_2 \sin \frac{x}{2} \right) + 4 \displaystyle\int_0^x \sqrt{t}\, e^{-(x-t)/2} \sin \frac{x - t}{2}\, dt$

Answer 21 (Section 3.6)

$y = c_1 x + c_2 x^3 + 2x(x - 1)e^x, \quad x > 0$

Answer 22 (Section 3.6)

(1) $y = c_1 x^2 + c_2 x^{-1}$

(2) $y = c_1 + c_2 \ln x$

(3) $y = c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)$

Answer 23 (Section 3.6)

(1) $y = c_1 + c_2 x^5 + \dfrac{1}{5} x^5 \ln x$

(2) $y = c_1 x + \dfrac{c_2}{x} - \ln x$

Answer 24 (Section 3.7)

y_1, y_2 satisfy the equation, but for $y = c_1 y_1 + c_2 y_2$: $(y'')^2 - y^2 = -4 c_1 c_2\, e^x \cos x \neq 0$ unless $c_1 c_2 = 0$

Answer 25 (Section 3.7)

(1) With $u = y'$: $y = -\dfrac{1}{a} \arctan \dfrac{x}{a} + c_2$ $(c_1 = a^2)$, $y = \dfrac{1}{2a} \ln \left| \dfrac{x + a}{x - a} \right| + c_2$ $(c_1 = -a^2)$, $y = \dfrac{1}{x} + c_2$ $(c_1 = 0)$, and $y = c$

(2) $y = c_2 e^{c_1 x}$

Answer 26 (Section 3.7)

$y(x) \approx -1 + x - x^2 + \dfrac{2}{3} x^3 - \dfrac{1}{3} x^4 + \dfrac{1}{5} x^5$

Answer 27 (Section 3.7)

$y = \ln|\cos(c_1 - x)| + c_2$

Answer 28 (Section 3.7)

$(x + c_2)^2 + y^2 = c_1$ (arcs of circles)

Answer 29 (Section 3.7)

$y = \tan\left(\frac{\pi}{4} - \frac{x}{2}\right), \quad -\frac{\pi}{2} < x < \frac{3\pi}{2}$

Answer 30 (Section 3.7)

$x^2 + (y - c_2)^2 = c_1$, and $y = c$ (from $u = 0$)

Answer 31 (Section 3.8)

$x(t) = \frac{F_0}{\omega^2 - \gamma^2}(\cos\gamma t - \cos\omega t)$

Answer 32 (Section 3.8)

$\lim_{\gamma\to\omega} x(t) = \frac{F_0}{2\omega} t \sin\omega t$ (pure resonance)

Answer 33 (Section 3.9)

$u(r) = u_0 + (u_1 - u_0)\frac{\ln(r/a)}{\ln(b/a)}, \qquad a \le r \le b$

4 The Laplace Transform

Learning objectives After this chapter you should be able to

- compute Laplace transforms from the definition and by linearity, and find inverse transforms with partial fractions
- solve initial-value problems for linear equations with constant coefficients, including piecewise and impulsive forcing
- use the translation theorems, derivatives of transforms, convolution and the transform of a periodic function
- solve systems of linear differential equations by the Laplace transform

4.1 Definition of the Laplace Transform

- Let f be a function defined for $t \geq 0$. Then the *integral*

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s)$$

is the **Laplace Transform** of f provided the integral converges. The result is a function of s

Example Evaluate $\mathcal{L}\{1\}$, $\mathcal{L}\{t\}$, and $\mathcal{L}\{e^{-3t}\}$

Answer 1 (end of chapter) · Solution

Example Evaluate $\mathcal{L}\{f(t)\}$ for $f(t) = \begin{cases} 0, & 0 \leq t < 3 \\ 2, & t \geq 3 \end{cases}$

Answer 2 (end of chapter) · Solution

- $\mathcal{L}$ is a linear transform

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\} = \alpha F(s) + \beta G(s)$$

Example Find $\mathcal{L}\{f(t)\}$ by first using an appropriate trigonometric identity $f(t) = \sin^2 2t$

Answer 3 (end of chapter) · Solution

Example Find $\mathcal{L}\{f(t)\}$

- $f(t) = \begin{cases} -1, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$
- $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$
- $f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

Answer 4 (end of chapter) · Solution

Example Find $\mathcal{L}\{f(t)\}$

- $f(t) = e^{t+7}$
- $f(t) = e^t \cos t$
- $f(t) = t \cos t$
- $f(t) = \sin 3t \cos 3t$
- $f(t) = \sin^4 t$

Answer 5 (end of chapter) · Solution

Try it in Python: Laplace transforms and inverse transforms with `sympy`: Sympy: Laplace Transform

4.2 The Inverse Transform and Transforms of Derivatives

- If $F(s)$ represents the Laplace transform of $f(t)$, then $f(t)$ is the **inverse Laplace transform** of $F(s)$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

- $\mathcal{L}^{-1}$ is a linear transform

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathcal{L}^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\} = \alpha f(t) + \beta g(t)$$

Example Find the given inverse transform

- $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$
- $\mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{48}{s^5}\right\}$
- $\mathcal{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\}$

- $\mathcal{L}^{-1}\left\{\dfrac{1}{4s^2+1}\right\}$
- $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2+2}\right\}$

Answer 6 (end of chapter) · Solution

Example Evaluate $\mathcal{L}^{-1}\left\{\dfrac{-2s+6}{s^2+4}\right\}$

Answer 7 (end of chapter) · Solution

- **Transforms of derivatives**

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= sF(s) - f(0)\\ \mathcal{L}\{f''(t)\} &= s^2F(s) - sf(0) - f'(0)\\ &\vdots\end{aligned}$$

  $\mathcal{L}\left\{\dfrac{d^n f}{dt^n}\right\}$ depends on $F(s) = \mathcal{L}\{f(t)\}$ and the $n-1$ derivatives of $f(t)$ evaluated at $t = 0$
- If f is piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\lim_{s\to\infty} \mathcal{L}\{f(t)\} = 0$$

- The Laplace transform of a linear DE with constant coefficients becomes an algebraic equation in $Y(s)$

Example Use the Laplace transform to solve the IVP

$$\frac{dy}{dt} + 3y = 13\sin 2t,\ y(0) = 6$$

Answer 8 (end of chapter) · Solution

Example Use the Laplace transform to solve the given initial-value problem

- $y'' + y = \sqrt{2}\sin\sqrt{2}t,\ \ y(0) = 10,\ y'(0) = 0$
- $2y''' + 3y'' - 3y' - 2y = e^{-t},\ \ y(0) = 0,\ y'(0) = 0,\ y''(0) = 1$

Answer 9 (end of chapter) · Solution

Try it in Python: Solving an initial-value problem by Laplace transform, step by step in `sympy`: **Example 1** and Solving ODEs using Laplace Transformation

4.3 Translation Theorems

- **First Translation Theorem**

  If $\mathcal{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

Example Evaluate $\mathcal{L}\{e^{-2t}\cos 4t\}$ and $\mathcal{L}^{-1}\left\{\dfrac{2s+5}{(s+3)^2}\right\}$

Answer 10 (end of chapter) · Solution

- **Second Translation Theorem**

  If $\mathcal{L}\{f(t)\} = F(s)$ and $a > 0$, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

- **Alternative Form**

$$\begin{aligned}\mathcal{L}\{g(t)\mathcal{U}(t-a)\} &= \int_a^\infty e^{-st}g(t)\,dt \\ &= \int_0^\infty e^{-s(t'+a)}g(t'+a)\,dt' \\ &= e^{-as}\mathcal{L}\{g(t+a)\}\end{aligned}$$

Example Find either $F(s)$ or $f(t)$

- $\mathcal{L}\{te^{10t}\}$
- $\mathcal{L}\{t^{10}e^{-7t}\}$
- $\mathcal{L}\{e^t\sin 3t\}$
- $\mathcal{L}^{-1}\left\{\dfrac{s}{(s+1)^2}\right\}$
- $\mathcal{L}^{-1}\left\{\dfrac{2s-1}{s^2(s+1)^3}\right\}$

Answer 11 (end of chapter) · Solution

Example Solve $y' + y = f(t)$, $y(0) = 5$, where

$$f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$$

Answer 12 (end of chapter) · Solution

4.4 Additional Operational Properties

- **Derivatives of Transforms:** If $F(s) = \mathcal{L}\{f(t)\}$ and $n = 1, 2, \cdots$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

- **Convolution Theorem:** $f * g = \displaystyle\int_0^t f(\tau) g(t-\tau)\, d\tau$

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

- **Transform of a Periodic Function:** $f(t+T) = f(t)$

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

Example Evaluate $\mathcal{L}\{t \sin \omega t\}$

Answer 13 (end of chapter) · Solution

Example Solve $x'' + 16x = \cos 4t,\ x(0) = 1,\ x'(0) = 1$

Answer 14 (end of chapter) · Solution

Example Evaluate $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+\omega^2)^2}\right\}$

Answer 15 (end of chapter) · Solution

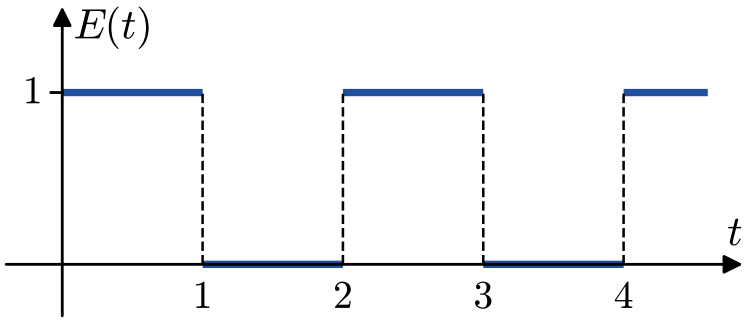
Example Evaluate $\mathcal{L}\left\{\displaystyle\int_0^t f(\tau)\, d\tau\right\}$

Answer 16 (end of chapter) · Solution

Example Solve $f(t) = 3t^2 - e^{-t} - \displaystyle\int_0^t f(\tau)\, e^{t-\tau}\, d\tau$ for $f(t)$

Answer 17 (end of chapter) · Solution

Example Find the Laplace transform of the periodic function



$$\mathcal{L}\{E(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} E(t)\, dt = \frac{1}{s(1+e^{-s})}$$

Example Evaluate the given Laplace transform

- $\mathcal{L}\{te^{-10t}\}$
- $\mathcal{L}\{t\cos 2t\}$
- $\mathcal{L}\{te^{2t}\sin 6t\}$

Answer 18 (end of chapter) · Solution

Example Use the Laplace transform to solve the given initial-value problem

- $y' + y = t\sin t,\ y(0) = 0$
- $y'' + 9y = \cos 3t,\ y(0) = 2,\ y'(0) = 5$

Answer 19 (end of chapter) · Solution

Example Find the convolution $f * g$ of the given functions. After integrating, find the Laplace transform of $f * g$

- $f(t) = 4t,\ g(t) = 3t^2$
- $f(t) = e^{-t},\ g(t) = e^t$

Answer 20 (end of chapter) · Solution

Example Find the Laplace transform

- $\mathcal{L}\{e^{-t} * e^t\cos t\}$

Answer 21 (end of chapter) · Solution

Example Evaluate the given inverse transform

- $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s-1)}\right\}$

Answer 22 (end of chapter) · Solution

Example Use the Laplace transform to solve the given integral or integrodifferential equation

- $f(t) + 2\displaystyle\int_0^t f(\tau)\cos(t-\tau)\,d\tau = 4e^{-t} + \sin t$
- $\dfrac{dy}{dt} = 10 - \displaystyle\int_0^t e^{-4\tau} y(t-\tau)\,d\tau,\ y(0) = 5$

Answer 23 (end of chapter) · Solution

Example The Laplace transform $\mathcal{L}\left\{e^{-t^2}\right\}$ exists, but without finding it solve the initial-value problem

- $y'' + 9y = 3e^{-t^2},\ \ y(0) = 0,\ y'(0) = 0$

Answer 24 (end of chapter) · Solution

Example Solve the integral equation

- $f(t) = e^t + e^t \displaystyle\int_0^t e^{-\tau} f(\tau)\,d\tau$

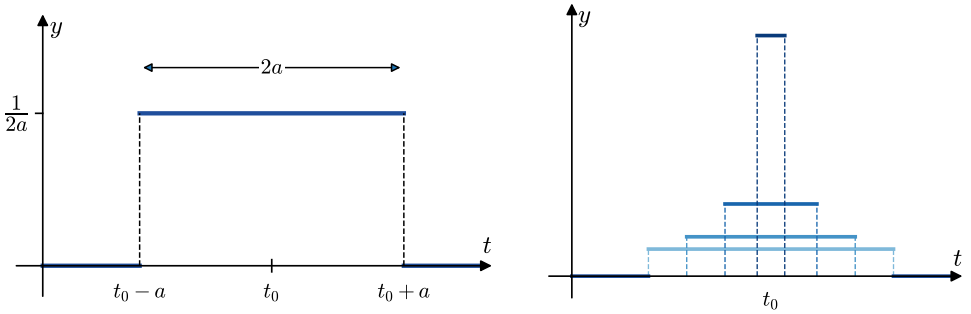
Answer 25 (end of chapter) · Solution

Try it in Python: Derivatives of transforms and the convolution theorem in `sympy`: Derivatives of Transforms, Convolution; an ODE with variable coefficients: **Example 2**

4.5 The Dirac Delta Function

Unit Pulse

$$\delta_a(t-t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \frac{1}{2a}, & t_0 - a \le t \le t_0 + a \\ 0, & t \ge t_0 + a \end{cases}$$



(a) (b) δ_a as $a \to 0$

- **The Dirac Delta Function**

$$\begin{aligned}\delta(t-t_0) &= \lim_{a\to 0}\ \delta_a(t-t_0)\\ &\Downarrow\\ \mathcal{L}\{\delta(t-t_0)\} &= \lim_{a\to 0}\mathcal{L}\{\delta_a(t-t_0)\} = e^{-st_0}\lim_{a\to 0}\left(\frac{e^{sa}-e^{-sa}}{2sa}\right)\\ &= e^{-st_0}\end{aligned}$$

When $t_0 = 0,\ \mathcal{L}\{\delta(t)\} = 1$

Example Solve $y'' + y = 4\delta(t-2\pi)$ subject to $y(0) = 1,\ y'(0) = 0$

Answer 26 (end of chapter) · Solution

Example Use the Laplace transform to solve the given differential equation subject to the indicated initial conditions

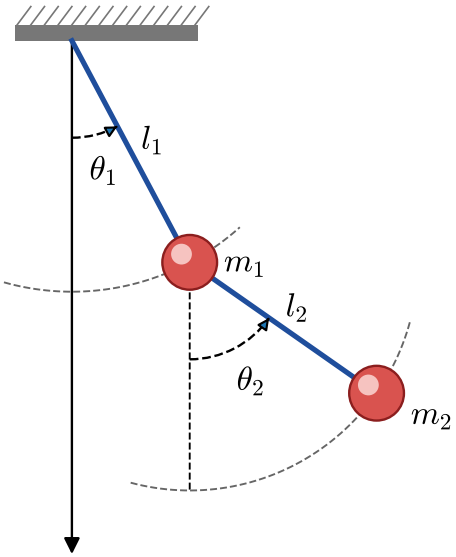
- $y' - 3y = \delta(t-2),\ y(0) = 0$
- $y'' + y = \delta(t-2\pi),\ y(0) = 0,\ y'(0) = 1$
- $y'' + y = \delta(t-\pi/2) + \delta(t-3\pi/2),\ y(0) = 0,\ y'(0) = 0$

Answer 27 (end of chapter) · Solution

4.6 Systems of Linear Differential Equations

- When initial conditions are specified, the Laplace transform reduces a system of linear DEs to a set of simultaneous algebraic equations in the transformed functions

Example Double Pendulum



Linearization For small displacements θ_1 and θ_2,

$$(m_1 + m_2)l_1\ddot{\theta_1} + m_2 l_2\ddot{\theta_2} + (m_1 + m_2)g\,\theta_1 = 0$$
$$l_2\ddot{\theta_2} + l_1\ddot{\theta_1} + g\,\theta_2 = 0$$

Solve the system when

$$m_1 = 3, m_2 = 1, l_1 = l_2 = 5, \text{ and } \ g = 10$$

$$\theta_1(0) = 1, \theta_2(0) = -1, \dot{\theta_1}(0) = 0, \dot{\theta_2}(0) = 0$$

Answer 28 (end of chapter) · Solution

Example Use the Laplace transform to solve the given system of differential equations

$$\dot{x} = -x + y$$
$$\dot{y} = 2x, \ \ x(0) = 0, \ y(0) = 1$$

$$\dot{x} = x - 2y$$
$$\dot{y} = 5x - y, \ \ x(0) = -1, \ y(0) = 2$$

$$2\dot{x} + \dot{y} - 2x = 1$$
$$\dot{x} + \dot{y} - 3x - 3y = 2, \ \ x(0) = 0, \ y(0) = 0$$

Answer 29 (end of chapter) · Solution

Summary

Property	Formula
Definition	$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s)$
Basic transforms	$\mathcal{L}\{1\} = \frac{1}{s},\ \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}},\ \mathcal{L}\{e^{at}\} = \frac{1}{s-a},$ $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2},\ \mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$
Derivatives	$\mathcal{L}\{f'\} = sF(s) - f(0),$ $\mathcal{L}\{f''\} = s^2F(s) - sf(0) - f'(0)$
First translation	$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$
Second translation	$\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = e^{-as}F(s),$ $\mathcal{L}\{g(t)\,\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$
Derivatives of transforms	$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$
Convolution	$\mathcal{L}\{f*g\} = F(s)G(s),$ $f*g = \int_0^t f(\tau)g(t-\tau)\,d\tau$
Periodic function	$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt,$ $f(t+T) = f(t)$
Dirac delta	$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$

Worked Exercises

1. Solve the system

$$\begin{aligned} x' + y &= t \\ 4x + y' &= 0 \end{aligned} \quad \text{at } x(0) = 0,\ y(0) = 2$$

Solution

Step 1: Take Laplace transforms

$$\mathcal{L}\{x'\} + Y(s) = \mathcal{L}\{t\} \Rightarrow sX(s) - x(0) + Y(s) = \frac{1}{s^2} \Rightarrow sX(s) + Y(s) = \frac{1}{s^2} \tag{a}$$

$$4X(s) + \mathcal{L}\{y'\} = 0 \Rightarrow 4X(s) + (sY(s) - y(0)) = 0 \Rightarrow 4X(s) + sY(s) = 2 \tag{b}$$

Step 2: Solve the algebraic system

From equation (a):

$$Y(s) = \frac{1}{s^2} - sX(s)$$

Substitute into equation (b):

$$4X(s) + s\left(\frac{1}{s^2} - sX(s)\right) = 2 \Rightarrow X(s) = \frac{2s-1}{s(4-s^2)}$$

Then, from (a):

$$Y(s) = \frac{1}{s^2} - sX(s) = \frac{1}{s^2} - \frac{2s-1}{4-s^2}$$

Step 3: Partial fraction decomposition

For $X(s)$,

$$X(s) = \frac{-1}{4s} + \frac{3}{8(2-s)} + \frac{5}{8(2+s)}$$

For $Y(s)$,

$$Y(s) = \frac{1}{s^2} + \frac{3}{4(s-2)} + \frac{5}{4(s+2)}$$

Step 4: Inverse Laplace Transform

$$x(t) = -\frac{1}{4} - \frac{3}{8}e^{2t} + \frac{5}{8}e^{-2t}$$
$$y(t) = t + \frac{3}{4}e^{2t} + \frac{5}{4}e^{-2t}$$

2. Solve the system

$$\begin{aligned} x'' + y'' &= e^{2t} \\ 2x' + y'' &= -e^{2t} \end{aligned} \quad \text{at } x(0) = 0,\ y(0) = 0,\ x'(0) = 0,\ y'(0) = 0$$

Solution

Step 1: Apply Laplace Transforms

$$\mathcal{L}\{x'' + y''\} = \mathcal{L}\{e^{2t}\} \Rightarrow s^2X(s) + s^2Y(s) = \frac{1}{s-2} \tag{a}$$

$$\mathcal{L}\{2x' + y''\} = \mathcal{L}\{-e^{2t}\} \Rightarrow 2sX(s) + s^2Y(s) = -\frac{1}{s-2} \tag{b}$$

Step 2: Solve the system algebraically

Subtract (b) from (a):

$$(s^2X + s^2Y) - (2sX + s^2Y) = \frac{1}{s-2} - \left(-\frac{1}{s-2}\right)$$
$$\Rightarrow X(s) = \frac{2}{(s-2)(s^2-2s)}$$

Now substitute into (a) to find $Y(s)$:

$$Y(s) = \frac{1}{s^2(s-2)} - \frac{2}{(s-2)(s^2-2s)}$$

Step 3: Simplify and compute inverse Laplace transforms

$$X(s) = \frac{2}{(s-2)(s^2-2s)} = \frac{1}{2s} - \frac{1}{2(s-2)} + \frac{1}{(s-2)^2}$$

Take inverse Laplace:

$$x(t) = \frac{1}{2} - \frac{1}{2}e^{2t} + te^{2t} = \frac{1}{2}(1-e^{2t}) + te^{2t}$$

Recall:

$$Y(s) = \frac{1}{s^2(s-2)} - \frac{2}{s(s-2)^2} = -\frac{3}{4s} - \frac{1}{2s^2} + \frac{3}{4(s-2)} - \frac{1}{(s-2)^2}$$

Thus:

$$y(t) = -\frac{3}{4} - \frac{1}{2}t + \frac{3}{4}e^{2t} - te^{2t}$$

Check: $x'' + y'' = (2e^{2t} + 4te^{2t}) + (-e^{2t} - 4te^{2t}) = e^{2t}$, $2x' + y'' = 4te^{2t} - e^{2t} - 4te^{2t} = -e^{2t}$, and $y(0) = y'(0) = 0$

3. (a) Show that $y = e^{-t^2}$ is a solution of the initial value problem

$$\frac{dy}{dt} + 2ty = 0, \quad y(0) = 1$$

(b) Find $Y(s) = \mathcal{L}\left[e^{-t^2}\right]$ by applying the Laplace transform to the above differential equation

Hint) $\operatorname{erfc}(t) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_t^\infty e^{-u^2}\, du$

Solution (a)

Step 1: Plug into the differential equation

Let $y(t) = e^{-t^2}$

Then:

$$\frac{dy}{dt} = \frac{d}{dt}(e^{-t^2}) = -2te^{-t^2}$$

Now substitute into the equation:

$$\frac{dy}{dt} + 2ty = -2te^{-t^2} + 2te^{-t^2} = 0$$

Step 2: Check initial condition

$$y(0) = e^0 = 1$$

Solution (b)

Step 1: Take Laplace transforms

Let $Y(s) = \mathcal{L}\{y(t)\}$

Apply Laplace transform:

$$\mathcal{L}\left\{\frac{dy}{dt} + 2ty\right\} = sY(s) - y(0) - 2\frac{dY}{ds} = 0$$

Given $y(0) = 1$, the transformed equation becomes:

$$\frac{dY}{ds} = \frac{1}{2}\left[sY(s) - 1\right]$$

Step 2: Solve the first-order linear ODE

We solve:

$$\frac{dY}{ds} = \frac{1}{2}sY(s) - \frac{1}{2}$$

This is a first-order linear ODE for $Y(s)$. Standard form:

$$\frac{dY}{ds} - \frac{1}{2}sY = -\frac{1}{2}$$

Use integrating factor:

$$\mu(s) = e^{-\frac{1}{4}s^2}$$

Multiply both sides:

$$e^{-\frac{1}{4}s^2}\frac{dY}{ds} - \frac{1}{2}se^{-\frac{1}{4}s^2}Y = -\frac{1}{2}e^{-\frac{1}{4}s^2}$$

LHS becomes derivative of the product:

$$\frac{d}{ds}\left(e^{-\frac{1}{4}s^2}Y\right) = -\frac{1}{2}e^{-\frac{1}{4}s^2}$$

Now integrate both sides:

$$\int \frac{d}{ds}\left(e^{-\frac{1}{4}s^2}Y\right)ds = \int -\frac{1}{2}e^{-\frac{1}{4}s^2}ds$$

There's no elementary antiderivative for $e^{-\frac{1}{4}s^2}$, so we integrate from 0 to s:

$$Y(s) = e^{\frac{1}{4}s^2}\left(C - \frac{1}{2}\int_0^s e^{-\frac{1}{4}u^2}du\right)$$

Step 3: Determine C

The initial condition $y(0) = 1$ has already been used. The constant is fixed instead by the fact that every Laplace transform satisfies $Y(s) \to 0$ as $s \to \infty$. Since $e^{s^2/4} \to \infty$, the bracket must tend to 0:

$$C = \frac{1}{2}\int_0^\infty e^{-\frac{1}{4}u^2}du = \frac{\sqrt{\pi}}{2}$$

Hence, with $u = 2v$,

$$Y(s) = \frac{1}{2}e^{\frac{1}{4}s^2}\int_s^\infty e^{-\frac{1}{4}u^2}du = e^{\frac{1}{4}s^2}\int_{s/2}^\infty e^{-v^2}dv = \frac{\sqrt{\pi}}{2}e^{\frac{s^2}{4}}\,\mathrm{erfc}\left(\frac{s}{2}\right)$$

Therefore:

$$\mathcal{L}\{e^{-t^2}\} = \frac{\sqrt{\pi}}{2}e^{\frac{s^2}{4}}\,\mathrm{erfc}\left(\frac{s}{2}\right)$$

4. One definition of the gamma function $\Gamma(\alpha)$ is given by the improper integral

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\,e^{-t}\,dt,\ \alpha > 0$$

(a) Use this definition to show that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

(b) Find $\mathcal{L}\left[t^\alpha\right]$

Solution (a)

Start with $\Gamma(\alpha+1)$

$$\Gamma(\alpha+1) = \int_0^\infty t^\alpha e^{-t}dt$$

We'll use integration by parts:

- Let $u = t^\alpha \Rightarrow du = \alpha t^{\alpha-1}dt$
- Let $dv = e^{-t}dt \Rightarrow v = -e^{-t}$

Then:

$$\begin{aligned}\Gamma(\alpha+1) &= -t^{\alpha}e^{-t}\big|_0^{\infty} + \int_0^{\infty} \alpha t^{\alpha-1}e^{-t}dt \\ &= 0 + \alpha\int_0^{\infty} t^{\alpha-1}e^{-t}dt \\ &= \alpha\Gamma(\alpha)\end{aligned}$$

Solution (b)

$$\mathcal{L}\{t^{\alpha}\} = \int_0^{\infty} e^{-st}t^{\alpha}dt$$

Make substitution $u = st \Rightarrow t = \dfrac{u}{s},\ dt = \dfrac{du}{s}$

Then:

$$\begin{aligned}\mathcal{L}\{t^{\alpha}\} = \int_0^{\infty} e^{-st}t^{\alpha}dt &= \int_0^{\infty} e^{-u}\left(\frac{u}{s}\right)^{\alpha}\cdot\frac{du}{s} \\ &= \frac{1}{s^{\alpha+1}}\int_0^{\infty} u^{\alpha}e^{-u}du \\ &= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \qquad \alpha > -1,\ s > 0\end{aligned}$$

For $\alpha = n = 0, 1, 2, \cdots,\ \Gamma(n+1) = n!$ and this reduces to $\mathcal{L}\{t^n\} = n!/s^{n+1}$

5. Solve the given differential equation

$$y'' + 2y' + y = (t-1)u(t-1) + \delta(t-5),\ y(0) = 0,\ y'(0) = 0$$

Solution

$$\begin{gathered}y'' + 2y' + y = (t-1)u(t-1) + \delta(t-5),\ \ y(0) = 0,\ y'(0) = 0 \\ \Downarrow \\ (s^2 + 2s + 1)\,Y = \frac{e^{-s}}{s^2} + e^{-5s} \\ \Downarrow \\ Y = \frac{e^{-s}}{s^2(s+1)^2} + \frac{e^{-5s}}{(s+1)^2} = \left[-\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{1}{(s+1)^2}\right]e^{-s} + \frac{e^{-5s}}{(s+1)^2} \\ \Downarrow \\ y(t) = \left[-3 + t + (t+1)e^{-(t-1)}\right]\ u(t-1) + (t-5)e^{-(t-5)}u(t-5)\end{gathered}$$

6. Solve $y'' + 4y' + 3y = e^{t}\delta(t-1),\ \ y(0) = 0,\ y'(0) = 2$

Solution

$$y'' + 4y' + 3y = e^t\delta(t-1), \quad y(0) = 0,\ y'(0) = 2$$
$$\Downarrow$$
$$(s^2 + 4s + 3)Y(s) - 2 = ee^{-s}$$
$$Y(s) = \frac{2}{(s+1)(s+3)} + \frac{e}{(s+1)(s+3)}e^{-s}$$
$$\Downarrow$$
$$y(t) = e^{-t} - e^{-3t} + \frac{e}{2}\left[e^{-(t-1)} - e^{-3(t-1)}\right] u(t-1)$$

7. Use the Laplace transform to find the numerical value of the improper integral

$$\int_0^\infty te^{-2t}\sin 4t\, dt$$

Solution

$$F(s) = \mathcal{L}\left[t\,\sin 4t\right] = \int_0^\infty t\,\sin 4t\, e^{-st}\, dt$$
$$\Downarrow$$
$$F(s) = -\frac{d}{ds}\left(\frac{4}{s^2+4^2}\right) = \frac{8s}{(s^2+4^2)^2}$$
$$\Downarrow$$
$$F(2) = \frac{1}{25}$$

Answers

Answer 1 (Section 4.1)

$\mathcal{L}\{1\} = \frac{1}{s}\ (s>0), \quad \mathcal{L}\{t\} = \frac{1}{s^2}, \quad \mathcal{L}\{e^{-3t}\} = \frac{1}{s+3}\ (s>-3)$

Answer 2 (Section 4.1)

$\mathcal{L}\{f\} = \frac{2e^{-3s}}{s}$

Answer 3 (Section 4.1)

$\mathcal{L}\{\sin^2 2t\} = \frac{8}{s(s^2+16)}$

Answer 4 (Section 4.1)

(1) $\frac{2e^{-s}-1}{s}$

(2) $\frac{1-e^{-s}}{s^2}$

(3) $\frac{1+e^{-\pi s}}{s^2+1}$

Answer 5 (Section 4.1)

$$\mathcal{L}\{e^{t+7}\} = \frac{e^7}{s-1}, \quad \mathcal{L}\{e^t \cos t\} = \frac{s-1}{(s-1)^2+1}, \quad \mathcal{L}\{t\cos t\} = \frac{s^2-1}{(s^2+1)^2}$$

$$\mathcal{L}\{\sin 3t \cos 3t\} = \frac{3}{s^2+36}, \quad \mathcal{L}\{\sin^4 t\} = \frac{24}{s(s^2+4)(s^2+16)}$$

Answer 6 (Section 4.2)

$$\frac{t^2}{2}, \quad t - 2t^4, \quad 1 + 3t + \frac{3}{2}t^2 + \frac{1}{6}t^3, \quad \frac{1}{2}\sin\frac{t}{2}, \quad \cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t$$

Answer 7 (Section 4.2)

$$-2\cos 2t + 3\sin 2t$$

Answer 8 (Section 4.2)

$$y = 8e^{-3t} - 2\cos 2t + 3\sin 2t$$

Answer 9 (Section 4.2)

(1) $y = 10\cos t + 2\sin t - \sqrt{2}\sin\sqrt{2}t$

(2) $y = \frac{1}{2}e^{-t} + \frac{5}{18}e^t - \frac{8}{9}e^{-t/2} + \frac{1}{9}e^{-2t}$

Answer 10 (Section 4.3)

$$\mathcal{L}\{e^{-2t}\cos 4t\} = \frac{s+2}{(s+2)^2+16}, \quad \mathcal{L}^{-1}\left\{\frac{2s+5}{(s+3)^2}\right\} = (2-t)e^{-3t}$$

Answer 11 (Section 4.3)

$$\frac{1}{(s-10)^2}, \quad \frac{10!}{(s+7)^{11}}, \quad \frac{3}{(s-1)^2+9}$$

$$(1-t)e^{-t}, \quad 5 - t - 5e^{-t} - 4te^{-t} - \frac{3}{2}t^2e^{-t}$$

Answer 12 (Section 4.3)

$$y = \begin{cases} 5e^{-t}, & 0 \le t < \pi \\ 5e^{-t} + \frac{3}{2}e^{-(t-\pi)} + \frac{3}{2}(\cos t + \sin t), & t \ge \pi \end{cases}$$

Answer 13 (Section 4.4)

$$\mathcal{L}\{t\sin\omega t\} = \frac{2\omega s}{(s^2+\omega^2)^2}$$

Answer 14 (Section 4.4)

$$x = \cos 4t + \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t$$

Answer 15 (Section 4.4)

$$\frac{\sin\omega t - \omega t\cos\omega t}{2\omega^3}$$

Answer 16 (Section 4.4)

$$\frac{F(s)}{s}$$

Answer 17 (Section 4.4)

$$f(t) = 3t^2 - t^3 + 1 - 2e^{-t}$$

Answer 18 (Section 4.4)

$$\frac{1}{(s+10)^2}, \quad \frac{s^2-4}{(s^2+4)^2}, \quad \frac{12(s-2)}{[(s-2)^2+36]^2}$$

Answer 19 (Section 4.4)

(1) $y = \dfrac{1}{2}\left(t\sin t - t\cos t + \cos t - e^{-t}\right)$

(2) $y = 2\cos 3t + \dfrac{5}{3}\sin 3t + \dfrac{1}{6}t\sin 3t$

Answer 20 (Section 4.4)

(1) $f * g = t^4, \quad \mathcal{L}\{f*g\} = \dfrac{24}{s^5}$

(2) $f * g = \sinh t, \quad \mathcal{L}\{f*g\} = \dfrac{1}{s^2-1}$

Answer 21 (Section 4.4)

$$\frac{1}{s+1}\cdot\frac{s-1}{(s-1)^2+1}$$

Answer 22 (Section 4.4)

$$e^t - 1 - t - \frac{t^2}{2}$$

Answer 23 (Section 4.4)

(1) $f(t) = (4 - 7t + 4t^2)e^{-t}$

(2) $y = 40 - e^{-2t}\left(35\cosh\sqrt{3}t + 20\sqrt{3}\sinh\sqrt{3}t\right)$

Answer 24 (Section 4.4)

$$y(t) = \int_0^t e^{-\tau^2}\sin 3(t-\tau)\,d\tau$$

Answer 25 (Section 4.4)

$f(t) = e^{2t}$

Answer 26 (Section 4.5)

$y = \cos t + 4\sin t\,\mathcal{U}(t-2\pi)$

Answer 27 (Section 4.5)

(1) $y = e^{3(t-2)}\,\mathcal{U}(t-2)$

(2) $y = \sin t + \sin t\,\mathcal{U}(t-2\pi)$

(3) $y = -\cos t\,\mathcal{U}(t-\frac{\pi}{2}) + \cos t\,\mathcal{U}(t-\frac{3\pi}{2})$

Answer 28 (Section 4.6)

$$\theta_1(t) = \frac{1}{4}\cos\frac{2}{\sqrt{3}}t + \frac{3}{4}\cos 2t, \qquad \theta_2(t) = \frac{1}{2}\cos\frac{2}{\sqrt{3}}t - \frac{3}{2}\cos 2t$$

Answer 29 (Section 4.6)

(1) $x = \dfrac{1}{3}\left(e^t - e^{-2t}\right), \quad y = \dfrac{2}{3}e^t + \dfrac{1}{3}e^{-2t}$

(2) $x = -\cos 3t - \dfrac{5}{3}\sin 3t, \quad y = 2\cos 3t - \dfrac{7}{3}\sin 3t$

(3) $x = -\dfrac{1}{2} + \dfrac{5}{2}e^{2t} - 2e^{3t}, \quad y = -\dfrac{1}{6} - \dfrac{5}{2}e^{2t} + \dfrac{8}{3}e^{3t}$

5 Series Solutions of Linear Differential Equations

Learning objectives After this chapter you should be able to

- find power series solutions about an ordinary point, and the interval on which they converge
- classify singular points, and solve about a regular singular point by the method of Frobenius using the indicial equation
- solve Bessel's and Legendre's equations and equations reducible to them
- use the properties of J_ν, Y_ν, I_ν, K_ν and P_n: recurrence relations, Rodrigues' formula and orthogonality

5.1 Solutions about Ordinary Points

- A power series in $x-a$ is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

  Such a series is also said to be a **power series centered at** a

- A power series is **convergent** if its sequence of partial sums converges
  - Convergence of power series can often be determined by the ratio test. Suppose that $c_n \neq 0$ for all n, and that

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$$

    If $L<1$, the series converges absolutely, if $L>1$, the series diverges, and if $L=1$, the test is inconclusive
  - Every power series has a **radius of convergence**, R. If $R>0$, a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a|<R$
  - A function f is **analytic** at point a if it can be represented by a power series in $x-a$ with a positive radius of convergence
  - Power series can be combined through the operations of addition, multiplication, and division

- Consider the linear second-order DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \quad a_2(x) \neq 0$$

– Divide by $a_2(x)$ to put into standard form

$$y'' + P(x)y' + Q(x)y = 0$$

– Point x_0 is an **ordinary point** of the DE if both $P(x)$ and $Q(x)$ are analytic at x_0. A point that is not an ordinary point is a **singular point** of the equation

– If $x = x_0$ is an ordinary point of the DE, we can always find **two linearly independent solutions** in the form of a power series centered at x_0

– A series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point

Example: Solve $y'' + xy = 0$

- Since there are no finite singular points, two power series solutions are guaranteed, centered at 0, convergent for $|x| < \infty$
- Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ and the second derivative $y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$ into the DE gives

$$\begin{aligned} y'' + xy &= \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} \\ &= 2c_2 + \sum_{k=1}^{\infty} \left[(k+1)(k+2)c_{k+2} + c_{k-1}\right] x^k = 0 \end{aligned}$$

- The coefficient of each power of x must be set equal to zero:

$$c_2 = 0 \text{ and } c_{k+2} = -\frac{c_{k-1}}{(k+1)(k+2)},\ k = 1, 2, 3, \cdots$$

$$\begin{aligned} c_3 &= -\frac{c_0}{2 \cdot 3} \\ c_4 &= -\frac{c_1}{3 \cdot 4} \\ c_5 &= -\frac{c_2}{4 \cdot 5} = 0 \\ c_6 &= -\frac{c_3}{5 \cdot 6} = \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6} \\ c_7 &= -\frac{c_4}{6 \cdot 7} = \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7} \\ c_8 &= -\frac{c_5}{7 \cdot 8} = 0 \\ &\vdots \end{aligned}$$

- After grouping the terms containing c_0 and the terms containing c_1, we obtain $y = c_0 y_1(x) + c_1 y_2(x)$

$$\begin{aligned} y_1(x) &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)(3n)} x^{3n} \\ y_2(x) &= x + \sum_{n=1}^{\infty} \frac{(-1)^n}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n)(3n+1)} x^{3n+1} \end{aligned}$$

Try it in Python: Taylor series of functions with `sympy.series`: Series

5.2 Solutions about Singular Points

- Consider the linear second-order DE

$$(x-x_0)^2y'' + (x-x_0)p(x)y' + q(x)y = 0$$

  - Point x_0 is a **regular singular point** of the DE if both $p(x)$ and $q(x)$ are analytic at x_0
  - A singular point that is not regular is an **irregular singular point** of the equation

- To solve a DE about a regular singular point, we employ **Frobenius' Theorem**

  - If x_0 is a regular singular point of the standard DE, there exists **at least one nonzero solution** of the form

$$y = (x-x_0)^r \sum_{n=0}^{\infty} c_n(x-x_0)^n = \sum_{n=0}^{\infty} c_n(x-x_0)^{n+r}$$

    where r is a constant, and the series converges at least on some interval, $0 < x - x_0 < R$

  - After substituting $y = \sum_{n=0}^{\infty} c_n(x-x_0)^{n+r}$ into a DE and simplifying, the **indicial equation** is obtained, a quadratic equation in r that results from equating the total coefficient of the lowest power of x to zero

  - The **indicial roots** are the solutions to the quadratic equation and are then substituted into a recurrence relation

Suppose that $x = x_0$ is a regular singular point of a DE and the indicial roots are r_1 and r_2: $r_1 \geq r_2$

- **Case I**

  r_1 and r_2 are distinct and do not differ by an integer,

$$y_1(x) = \sum_{n=0}^{\infty} c_n(x-x_0)^{n+r_1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^{n+r_2}$$

- **Case II**

  $r_1 - r_2 = N$, where N is a positive integer,

$$y_1(x) = \sum_{n=0}^{\infty} c_n(x-x_0)^{n+r_1},\ c_0 \neq 0$$
$$y_2(x) = Cy_1(x)\ln(x-x_0) + \sum_{n=0}^{\infty} b_n(x-x_0)^{n+r_2},\ b_0 \neq 0$$

- **Case III**

  $r_1 = r_2$,

$$y_1(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r_1},\ c_0 \neq 0$$
$$y_2(x) = y_1(x)\ln(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_1}$$

Example: Solve $2xy'' + (1 + x)y' + y = 0$

- Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ gives

$$\begin{aligned} &2xy'' + (1 + x)y' + y \\ &\quad = 2\sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n + r)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n + r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} \\ &\quad = x^r \left[r(2r - 1)c_0 x^{-1} + \sum_{k=0}^{\infty}[(k + r + 1)(2k + 2r + 1)c_{k+1} + (k + r + 1)c_k]x^k \right] = 0 \end{aligned}$$

  which implies

$$r(2r - 1) = 0$$
$$(k + r + 1)(2k + 2r + 1)c_{k+1} + (k + r + 1)c_k = 0, \quad k = 0, 1, \cdots$$

- We see that the indicial roots are $r_1 = \frac{1}{2}$ and $r_2 = 0$

$$r_1 = \frac{1}{2},\ c_{k+1} = -\frac{c_k}{2(k + 1)},\ k = 0, 1, 2, \cdots$$
$$r_2 = 0, \quad c_{k+1} = -\frac{c_k}{2k + 1},\ k = 0, 1, 2, \cdots$$

- For $r_1 = \frac{1}{2}$,

$$\begin{aligned} c_1 &= -\frac{c_0}{2 \cdot 1} \\ c_2 &= -\frac{c_1}{2 \cdot 2} = \frac{c_0}{2^2 \cdot 2!} \\ c_3 &= -\frac{c_2}{2 \cdot 3} = \frac{-c_0}{2^3 \cdot 3!} \\ c_4 &= -\frac{c_3}{2 \cdot 4} = \frac{c_0}{2^4 \cdot 4!} \\ &\vdots \\ c_n &= \frac{(-1)^n c_0}{2^n n!} \end{aligned}$$

- For $r_2 = 0$,

$$\begin{aligned} c_1 &= -\frac{c_0}{1} \\ c_2 &= -\frac{c_1}{3} = \frac{c_0}{1\cdot 3} \\ c_3 &= -\frac{c_2}{5} = \frac{-c_0}{1\cdot 3\cdot 5} \\ c_4 &= -\frac{c_3}{7} = \frac{c_0}{1\cdot 3\cdot 5\cdot 7} \\ &\vdots \\ c_n &= \frac{(-1)^n c_0}{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)} \end{aligned}$$

- The series solutions are

$$\begin{aligned} y_1(x) &= x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!} x^n\right] \\ y_2(x) &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)} x^n \end{aligned}$$

5.3 Special Functions

The following DEs occur frequently in advanced studies in applied mathematics, physics, and engineering

- **Bessel's equation of order ν,** solutions are **Bessel functions**

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

- **Legendre's equation of order n,** solutions are **Legendre polynomials**

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

5.3.1 Bessel Functions

- Because $x = 0$ is a regular singular point of Bessel's equation, there exists at least one solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$

$$\begin{aligned} &x^2 y'' + xy' + (x^2 - \nu^2)y \\ &= \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} c_n (n+r)x^{n+r} \\ &\quad + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= c_0(r^2 - \nu^2)x^r + x^r \sum_{n=1}^{\infty} c_n[(n+r)^2 - \nu^2]x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2} = 0 \end{aligned}$$

- The indicial equation is $r^2 - \nu^2 = 0$ so that the indicial roots are $r_1 = \nu$ and $r_2 = -\nu$. When $r_1 = \nu \geq 0$,

$$
\begin{aligned}
& x^\nu \sum_{n=1}^{\infty} c_n n(n+2\nu) x^n + x^\nu \sum_{n=0}^{\infty} c_n x^{n+2} \\
& \qquad = x^\nu \left[(1+2\nu) c_1 x + \sum_{k=0}^{\infty} \left[(k+2)(k+2+2\nu) c_{k+2} + c_k \right] x^{k+2} \right] = 0 \\
& \qquad \Downarrow \\
& c_1 = 0, \quad c_{k+2} = \frac{-c_k}{(k+2)(k+2+2\nu)}, \quad k = 0, 1, 2, \cdots \\
& \qquad \Downarrow \\
& c_3 = c_5 = c_7 = \cdots = 0, \\
& c_{2n} = -\frac{c_{2n-2}}{2^2 n(n+\nu)} \quad \leftarrow \quad k+2 = 2n
\end{aligned}
$$

- Thus

$$
\begin{aligned}
c_2 &= -\frac{c_0}{2^2 \cdot 1 \cdot (1+\nu)} \\
c_4 &= -\frac{c_2}{2^2 \cdot 2 \cdot (2+\nu)} \\
&= \frac{c_0}{2^4 \cdot 1 \cdot 2 (1+\nu)(2+\nu)} \\
c_6 &= -\frac{c_4}{2^2 \cdot 3 \cdot (3+\nu)} \\
&= -\frac{c_0}{2^6 \cdot 1 \cdot 2 \cdot 3 (1+\nu)(2+\nu)(3+\nu)} \\
&\vdots \\
c_{2n} &= \frac{(-1)^n c_0}{2^{2n} n! (1+\nu)(2+\nu) \cdots (n+\nu)}
\end{aligned}
$$

- It is standard practice to choose

$$
\begin{aligned}
c_0 &= \frac{1}{2^\nu \Gamma(1+\nu)} \\
&\text{thus,} \\
c_{2n} &= \frac{(-1)^n}{2^{2n+\nu} n! \, \Gamma(1+\nu+n)}
\end{aligned}
$$

Note $\Gamma(\nu) = \int_0^\infty x^{\nu-1} e^{-x} \, dx, \;\; \Gamma(\nu+1) = \nu \Gamma(\nu)$

```
import numpy as np
from scipy.special import gamma, factorial

import matplotlib.pyplot as plt

x = np.linspace(-3.5, 5.5, 2251)
y = gamma(x)
```

```
def gamma_plot():
    fig = plt.figure(figsize=(6, 4))

    plt.plot(x, y, 'b', alpha=0.6, label=r'$\Gamma(x)$')
    k = np.arange(1, 7)
    plt.plot(k, factorial(k -1), 'ko', alpha=0.6,
         label=r'$(x-1)!,\; x = 1, 2, 3\cdots$')
    plt.xlim(-3.5, 5.5)
    plt.ylim(-10, 25)
    plt.grid('true')
    plt.xlabel('x')
    plt.ylabel(r'$\Gamma(x)$')
    plt.legend(loc='lower right')

    plt.show()

gamma_plot()
```

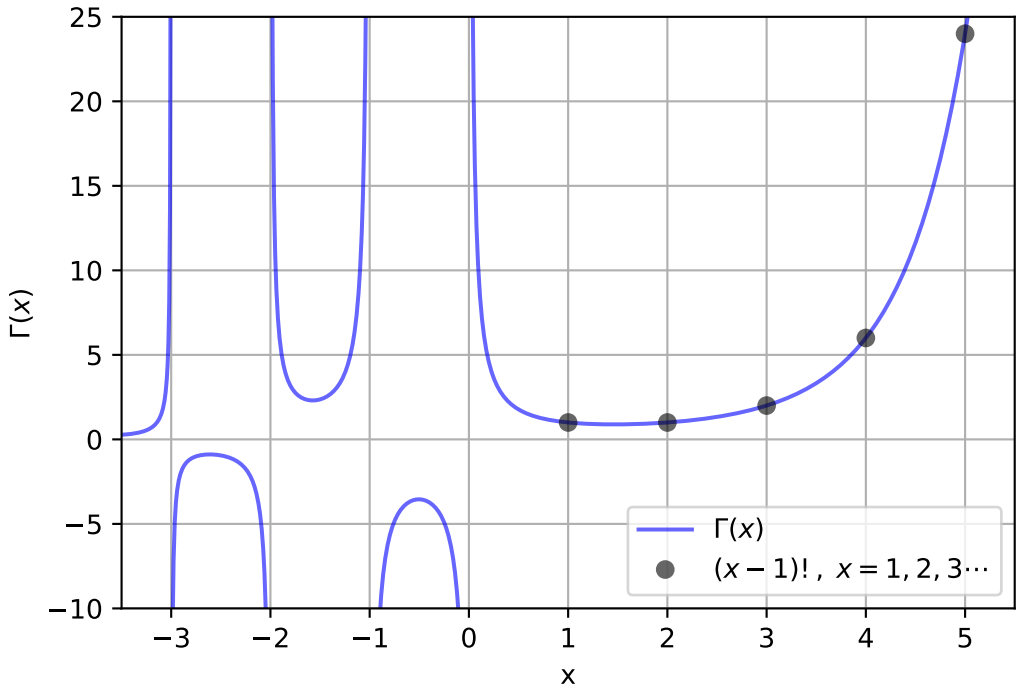


Figure 5.1: Gamma Function

- **Bessel Functions of the First Kind**

  The series solution $y_1 = \sum_{n=0}^{\infty} c_{2n} x^{2n+\nu}$ is usually denoted by $J_\nu(x)$

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

  Also, for the second exponent $r_2 = -\nu$

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}$$

- When $\nu = 0$, $J_0(x)$
- When $r_1 - r_2 = 2\nu$ is not a positive integer, $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent
- When $r_1 - r_2 = 2\nu$ is a positive integer, there are two possibilities
  * When $\nu = m =$ positive integer, $J_{-m}(x)$ is a constant multiple of $J_m(x)$: $J_m(x) = (-1)^m J_{-m}(x)$
  * When ν is half an odd positive integer, $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent

Example Solve $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$

Answer 1 (end of chapter) · Solution

- **Bessel Functions of the Second Kind**

  If $\nu \neq$ integer

  $$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$

  $J_\nu(x)$ and $Y_\nu(x)$ are linearly independent solutions of

  $$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

  As $\nu \to m$ (an integer),

  $$Y_m(x) = \lim_{\nu \to m} Y_\nu(x)$$

  $J_m(x)$ and $Y_m(x)$ are linearly independent solutions of

  $$x^2y'' + xy' + (x^2 - m^2)y = 0$$

- Hence for *any* value of ν, the general solution of Bessel equation can be written as

  $$y = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

  $Y_\nu(x)$ is called the **Bessel function of the second kind** of order ν

Example Solve $x^2y'' + xy' + (x^2 - 9)y = 0$

Answer 2 (end of chapter) · Solution

5.3.1.1 Properties of Bessel Functions

- When m is integer
  - $J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \left(\frac{x}{2}\right)^{2n+m}$
  - $J_{-m}(x) = (-1)^m J_m(x)$
  - $J_m(-x) = (-1)^m J_m(x)$
  - $J_m(0) = \begin{cases} 0, & m > 0 \\ 1, & m = 0 \end{cases}$
  - $\lim_{x \to 0^+} Y_m(x) = -\infty$

```
from scipy.special import jv, yv

plt.style.use('ggplot')

fig = plt.figure(figsize=(6, 8))

ax1 = fig.add_subplot(211)

x = np.linspace(0, 20, 200)
for m in range(5):
    y = jv(m, x)
    ax1.plot(x, y, label=f'$J_{m}(x)$')

ax1.axis((0, 20, -0.6, 1))

ax1.set_ylabel('$J_m(x)$')
ax1.legend()

ax2 = fig.add_subplot(212)

for m in range(5):
    y = yv(m, x)
    ax2.plot(x, y, label=f'$Y_{m}(x)$')

ax2.axis((0, 20, -3, 1))

ax2.set_xlabel('x')
ax2.set_ylabel('$Y_m(x)$')
ax2.legend()

plt.show()
```

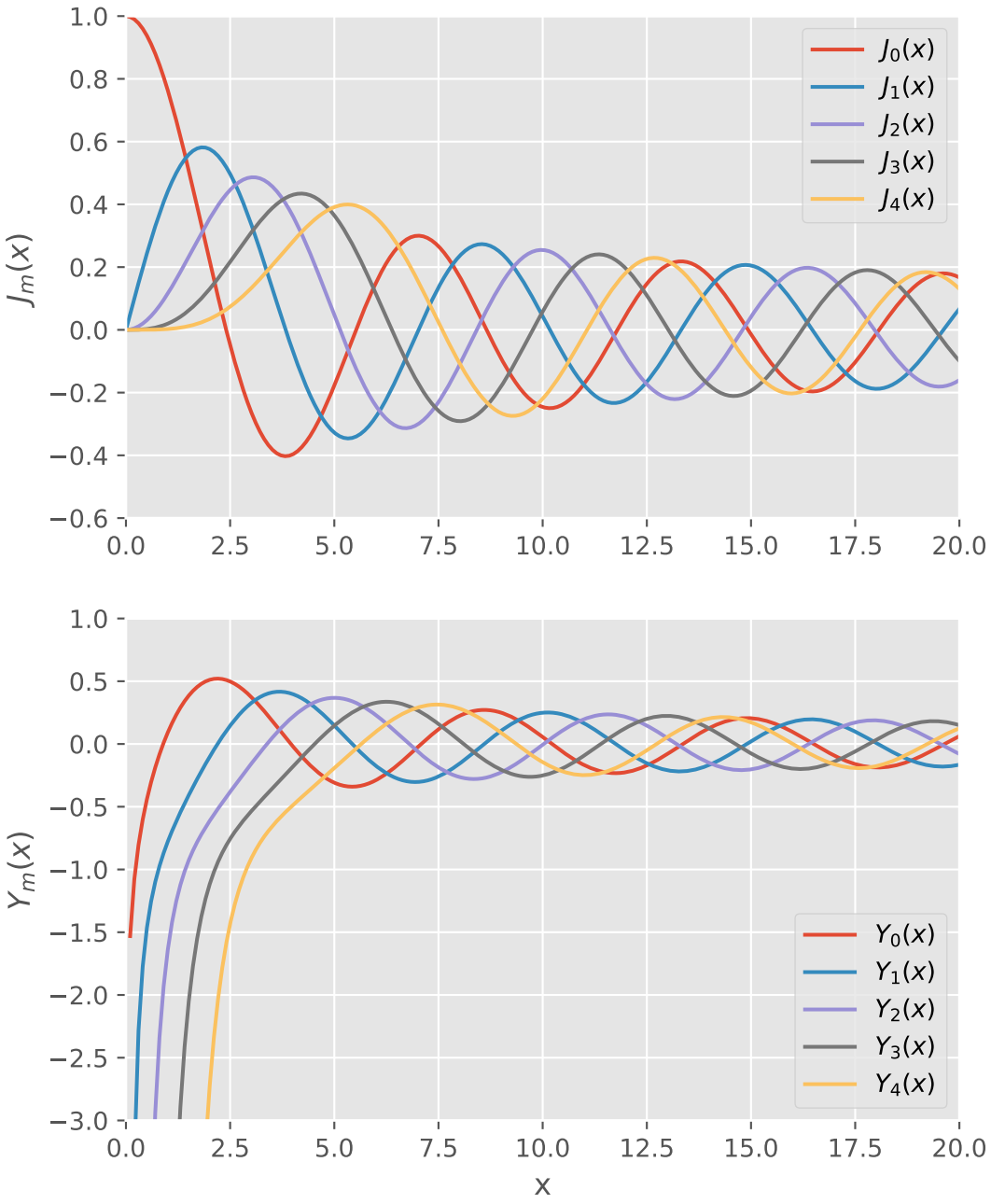


Figure 5.2: Bessel functions

- **Differential Recurrence Relation**

$$
\begin{aligned}
xJ'_\nu(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n(2n+\nu)}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
&= \nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} + x \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} \\
&= \nu J_\nu(x) - x \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(2+\nu+k)} \left(\frac{x}{2}\right)^{2k+\nu+1} \\
&= \nu J_\nu(x) - x J_{\nu+1}(x) \\
&\Downarrow \times x^{-\nu-1} \\
\frac{d}{dx}[x^{-\nu} J_\nu(x)] &= -x^{-\nu} J_{\nu+1}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
xJ'_\nu(x) &= -\nu J_\nu(x) + x J_{\nu-1}(x) \\
&\Downarrow \times x^{\nu-1} \\
\frac{d}{dx}[x^{\nu} J_\nu(x)] &= x^{\nu} J_{\nu-1}(x)
\end{aligned}
$$

Example $J'_0(x) = -J_1(x)$

Answer 3 (end of chapter) · Solution

5.3.1.2 DEs Solvable in Terms of Bessel Functions

- Parametric Bessel equation of order ν

$$
\begin{aligned}
x^2y'' + xy' + (\alpha^2 x^2 - \nu^2)y &= 0 \\
&\Downarrow t = \alpha x,\ \alpha > 0 \\
t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y &= 0 \\
&\Downarrow \\
y &= c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)
\end{aligned}
$$

- Modified Bessel equation of order ν

$$
\begin{aligned}
x^2y'' + xy' - (x^2 + \nu^2)y &= 0 \\
&\Downarrow t = ix \\
t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y &= 0 \\
&\Downarrow I_\nu(x) = i^{-\nu} J_\nu(ix),\ K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi},\ K_m(x) = \lim_{\nu \to m} K_\nu(x) \\
y &= c_1 I_\nu(x) + c_2 K_\nu(x)
\end{aligned}
$$

- Yet another equation

$$y'' + \frac{1-2a}{x}y' + \left(b^2c^2x^{2c-2} + \frac{a^2-p^2c^2}{x^2}\right)y = 0,\ p \geq 0$$

$$\Big\Downarrow\ z = bx^c,\ y(x) = \left(\frac{z}{b}\right)^{a/c} w(z)$$

$$y = x^a\left[c_1 J_p(bx^c) + c_2 Y_p(bx^c)\right]$$

- The aging spring

$$m\ddot{x} + ke^{-\alpha t}x = 0,\ \alpha > 0$$

$$\Big\Downarrow\ s = \frac{2}{\alpha}\sqrt{\frac{k}{m}}e^{-\alpha t/2}$$

$$s^2\frac{d^2x}{ds^2} + s\frac{dx}{ds} + s^2x = 0$$

- **Spherical Bessel Functions:**

  When solving the Helmholtz equation in spherical coordinates by separation of variables, the radial equation has the form

$$x^2y'' + 2xy' + (x^2 - n(n+1))\,y = 0$$

  The two linearly independent solutions to this equation are called the spherical Bessel functions j_n and y_n

$$j_n(x) = \sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(x)$$

$$y_n(x) = \sqrt{\frac{\pi}{2x}}Y_{n+\frac{1}{2}}(x) = (-1)^{n+1}\sqrt{\frac{\pi}{2x}}J_{-n-\frac{1}{2}}(x)$$

  when $\nu = n + \frac{1}{2}$ is half an odd integer, that is, $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots$

  Let's consider the case when $\nu = \frac{1}{2}$,

$$J_{1/2}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(1+\frac{1}{2}+n)}\left(\frac{x}{2}\right)^{2n+1/2}$$

  where

$$
\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi} \\
\Gamma\left(\frac{3}{2}\right) &= \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} \\
\Gamma\left(\frac{5}{2}\right) &= \Gamma\left(1+\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2^2}\sqrt{\pi} = \frac{3\cdot 2}{2^3}\sqrt{\pi} = \frac{3!}{2^3}\sqrt{\pi} \\
\Gamma\left(\frac{7}{2}\right) &= \Gamma\left(1+\frac{5}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5\cdot 3}{2^3}\sqrt{\pi} = \frac{5\cdot 4\cdot 3\cdot 2}{2^3\cdot 4\cdot 2}\sqrt{\pi} = \frac{5!}{2^5 2!}\sqrt{\pi} \\
&\vdots \\
\Gamma\left(1+\frac{1}{2}+n\right) &= \frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}
\end{aligned}
$$

Hence

$$
J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}} \left(\frac{x}{2}\right)^{2n+1/2} = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sqrt{\frac{2}{\pi x}} \sin x
$$

Example Show that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

Answer 4 (end of chapter) · Solution

```
from scipy.special import spherical_jn

plt.style.use('ggplot')

fig, ax = plt.subplots(figsize=(6, 4))

x = np.linspace(np.finfo(np.float32).eps, 10, 200)
for n in range(5):
    y = spherical_jn(n, x)
    ax.plot(np.append(0, x), np.append(0, y),
        label=f'$j_{n}(x)$')

ax.axis((0, 10, -0.4, 1.0))

ax.set_xlabel('$x$')
ax.set_ylabel('$j_n(x)$')
ax.legend()

plt.show()
```

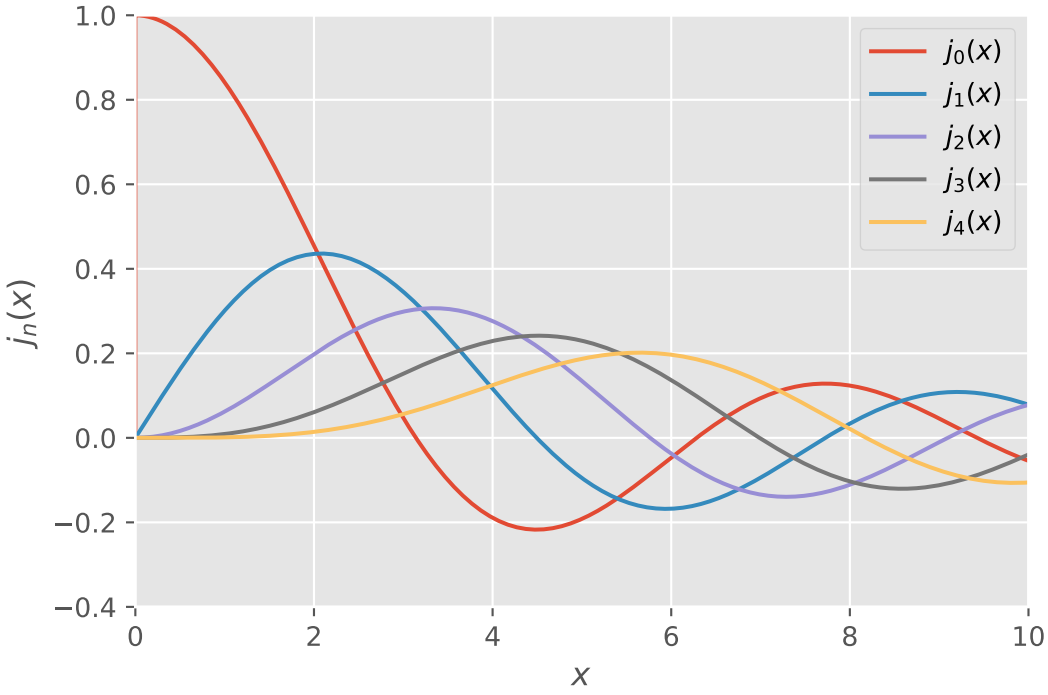


Figure 5.3: Spherical Bessel Function

Example Find the general solution of the given differential equation on $(0, \infty)$

- $x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0$
- $xy'' + y' + xy = 0$
- $x^2y'' + xy' + (9x^2 - 4)y = 0$
- $x^2y'' + xy' - (16x^2 + \frac{4}{9})y = 0$

Answer 5 (end of chapter) · Solution

Example Find the general solution of the given differential equation on $(0, \infty)$

- $x^2y'' + 2xy' + \alpha^2x^2y = 0;\ y = x^{-1/2}u(x)$
- $xy'' + 2y' + 4y = 0$

Answer 6 (end of chapter) · Solution

Example Verify that $I_\nu(x) = i^{-\nu}J_\nu(ix)$ is a real function

Answer 7 (end of chapter) · Solution

Example Express the general solution of the given differential equation in terms of the modified Bessel functions

$$xy'' + y' - 7x^3y = 0$$

Answer 8 (end of chapter) · Solution

Example First express the general solution of the given differential equation in terms of Bessel functions and then express the general solution in terms of elementary functions

$$x^2y'' + 4xy' + (x^2 + 2)y = 0$$

Answer 9 (end of chapter) · Solution

Example Derive the following result

$$\int_0^x rJ_0(r)\,dr = xJ_1(x)$$

$$J_0'(x) = J_{-1}(x) = -J_1(x)$$

Answer 10 (end of chapter) · Solution

Example Use the solution of the aging spring equation $mx'' + ke^{-at}x = 0$ to discuss the behavior of $x(t)$ as $t \to \infty$ in the three cases

- $c_1 \neq 0,\ c_2 = 0$
- $c_1 = 0,\ c_2 \neq 0$
- $c_1 \neq 0,\ c_2 \neq 0$

Answer 11 (end of chapter) · Solution

5.3.2 Legendre Polynomials

- Since $x = 0$ is an ordinary point of Legendre's equation, we substitute the series $y = \sum_{j=0}^{\infty} c_j x^j$ to get

$$(1-x^2)y'' - 2xy' + n(n+1)y =$$

$$[n(n+1)c_0 + 2c_2] + [(n-1)(n+2)c_1 + 6c_3]x + \sum_{j=2}^{\infty}[(j+2)(j+1)c_{j+2} + (n-j)(n+j+1)c_j]x^j = 0$$

$$\Downarrow$$

$$c_2 = -\frac{n(n+1)}{2}c_0$$

$$c_3 = -\frac{(n-1)(n+2)}{6}c_1$$

$$\vdots$$

$$c_{j+2} = -\frac{(n-j)(n+j+1)}{(j+2)(j+1)}c_j, \quad j = 2, 3, 4, \cdots$$

- The recurrence relation yields

$$c_2 = -\frac{n(n+1)}{2!}c_0$$

$$c_4 = -\frac{(n-2)(n+3)}{4\cdot 3}c_2 = \frac{(n-2)n(n+1)(n+3)}{4!}c_0$$

$$c_6 = -\frac{(n-4)(n+5)}{6\cdot 5}c_4 = -\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}c_0$$

$$\vdots$$

$$c_3 = -\frac{(n-1)(n+2)}{3!}c_1$$

$$c_5 = -\frac{(n-3)(n+4)}{5\cdot 4}c_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}c_1$$

$$c_7 = -\frac{(n-5)(n+6)}{7\cdot 6}c_5 = -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}c_1$$

$$\vdots$$

- Thus for at least $|x| < 1$, we obtain two linearly independent power series solutions:

$$y_1(x) = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 \cdots\right]$$

$$y_2(x) = c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)}{7!}\right.$$

  - If n is an even integer, the series $y_1(x)$ reduces to a polynomial of degree n with only even powers of x and the series $y_2(x)$ is an infinite series that diverges at $x = \pm 1$
  - If n is an odd integer, the series $y_2(x)$ reduces to a polynomial of degree n with only odd powers of x and the series $y_1(x)$ is an infinite series that diverges at $x = \pm 1$

- Hence, for a nonnegative integer n, one of the two solutions is a **polynomial**

  For example, if $n = 4$, then

$$y_1(x) = c_0\left[1 - \frac{4\cdot 5}{2!}x^2 + \frac{2\cdot 4\cdot 5\cdot 7}{4!}x^4\right] = c_0\left[1 - 10x^2 + \frac{35}{3}x^4\right]$$

- It is traditional to choose specific values for c_0 or c_1, depending on whether n is an even or odd positive integer, respectively
  - For $n = 0$, $c_0 = 1$, and for $n = 2, 4, 6, \cdots$,

    $$c_0 = (-1)^{n/2} \frac{1 \cdot 3 \cdots (n-1)}{2 \cdot 4 \cdots n}$$
  - For $n = 1$, $c_1 = 1$, and for $n = 3, 5, 7, \cdots$,

    $$c_1 = (-1)^{(n-1)/2} \frac{1 \cdot 3 \cdots n}{2 \cdot 4 \cdots (n-1)}$$
- For example, when $n = 4$, we have

$$y_1(x) = (-1)^{4/2} \frac{1 \cdot 3}{2 \cdot 4} \left[1 - 10x^2 + \frac{35}{3} x^4 \right] = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

- **Legendre Polynomials**

  These specific n-th degree polynomials are called **Legendre polynomials** and denoted by $P_n(x)$

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x)
\end{aligned}$$

```
from scipy.special import eval_legendre

fig, ax = plt.subplots(figsize=(5, 5))

x = np.linspace(-1, 1, 100)
for n in range(6):
    y = eval_legendre(n, x)
    ax.plot(x, y, label=f'$P_{n}(x)$')

ax.axis((-1.1, 1.1, -1.1, 1.1))
ax.set_xlabel('x')
ax.set_ylabel('$P_n(x)$')
ax.legend(loc='lower left', bbox_to_anchor=(1, 0))

plt.show()
```

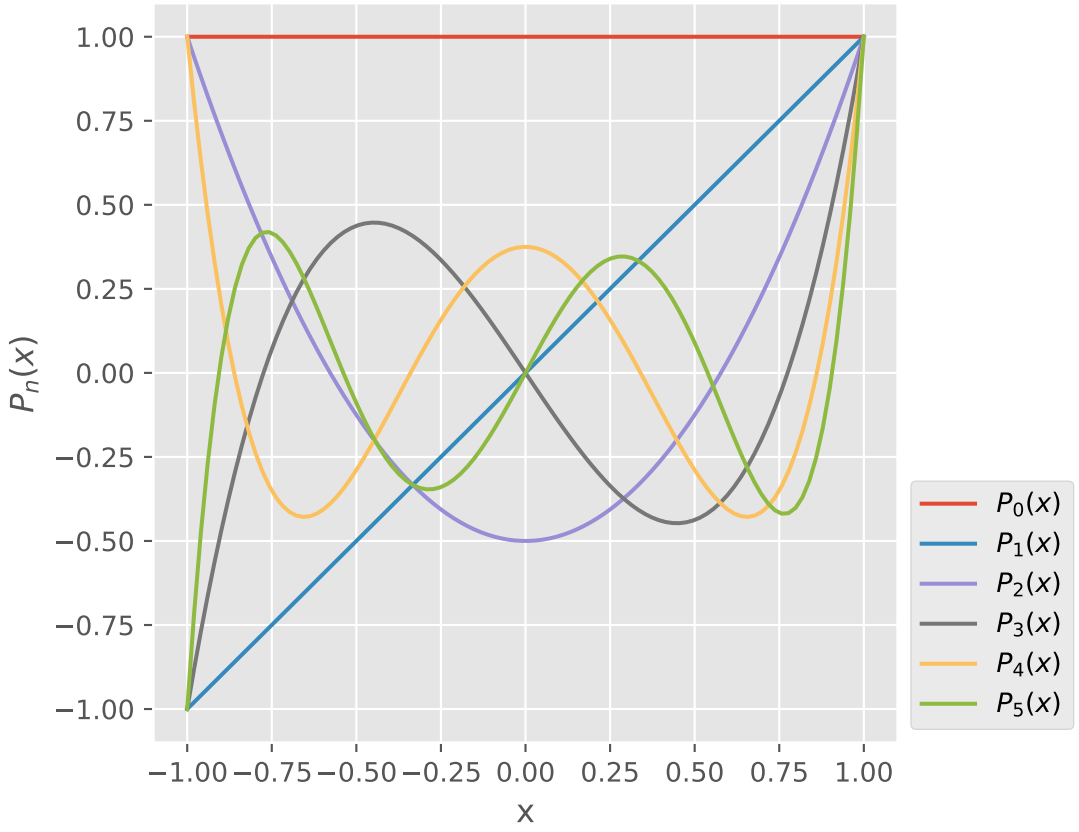


Figure 5.4: Legendre polynomials

- **Properties of Legendre Polynomials**
  - $P_n(-x) = (-1)^n P_n(x)$
  - $P_n(1) = 1$
  - $P_n(-1) = (-1)^n$
  - $P_n(0) = 0, \;\; n$ odd
  - $P_n'(0) = 0, \;\; n$ even
- **Rodrigues' Formula**

  The Legendre polynomials can also be represented using **Rodrigues' Formula**,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n \tag{RF}$$

  This can be demonstrated through the following observations
  - The right hand side of (RF) is an n-th order polynomial
  - Treating $\left(x^2 - 1\right)^n = (x-1)^n (x+1)^n$ as a product and using Leibniz' rule to differentiate n times, we have

$$\frac{d^n}{dx^n}(x-1)^n(x+1)^n = n!(x+1)^n + \text{terms with } (x-1) \text{ as a factor}$$

    so that $P_n(1) = \dfrac{n!2^n}{2^n n!} = 1$

– If $h(x) = (1-x^2)^n$, then $h'(x) = -2nx(1-x^2)^{n-1}$, so that

$$(1-x^2)\,h' + 2nx\,h = 0$$

Now differentiate $n+1$ times, using Leibniz, to get

$$(1-x^2)\,h^{(n+2)} - 2(n+1)xh^{(n+1)} - 2\frac{(n+1)n}{2}h^{(n)} + 2nxh^{(n+1)} + 2n(n+1)h^{(n)} = 0$$
$$\Downarrow$$
$$(1-x^2)\,h^{(n+2)} - 2xh^{(n+1)} + n(n+1)h^{(n)} = 0$$

– As the equation is linear and $P_n(x) \propto h^{(n)}(x)$, $P_n(x)$ satisfies the Legendre equation of order n

$$(1-x^2)\,P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

- **Integral Relations**

$$\int_{-1}^{1} P_k(x)P_l(x)\,dx = \delta_{k,l}\frac{2}{2l+1}, \quad k \le l$$

$$\int_{-1}^{1} xP_s(x)P_r(x)\,dx = \delta_{s+1,r}\frac{2r}{(2r-1)(2r+1)}, \; s \le r$$

- **Recurrence Relation**

The Legendre polynomials satisfy the following recurrence relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

Consider the polynomial $xP_n(x)$. It has degree $n+1$ and is thus in the linear span of $P_0, \cdots, P_{n+1}$. We can hence write $xP_n(x)$ as a linear combination of the first $n+2$ Legendre polynomials:

$$xP_n(x) = c_0P_0(x) + c_1P_1(x) + \cdots + c_{n+1}P_{n+1}(x)$$

– Thus

$$\int_{-1}^{1} xP_n(x)P_k(x)\,dx = c_k\int_{-1}^{1} P_k^2(x)\,dx$$
$$\rightarrow c_k = \frac{2k+1}{2}\int_{-1}^{1} xP_n(x)P_k(x)\,dx$$

These integrals vanish unless $k = n \pm 1$ and for this case, we can use

$$c_{n-1} = \frac{2n-1}{2}\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx = \frac{n}{2n+1}$$

$$c_{n+1} = \frac{2n+3}{2}\int_{-1}^{1} xP_n(x)P_{n+1}(x)\,dx = \frac{n+1}{2n+1}$$

– Hence

$$\begin{aligned} xP_n(x) &= c_{n-1}P_{n-1}(x) + c_{n+1}P_{n+1}(x) \\ &= \frac{n}{2n+1}P_{n-1}(x) + \frac{n+1}{2n+1}P_{n+1}(x) \end{aligned}$$

This is what we wanted to prove

Example Use the recurrence relation and $P_0(x) = 1$, $P_1(x) = x$, to generate the next six Legendre polynomials

Answer 12 (end of chapter) · Solution

Example Show that the differential equation

$$\sin\theta\frac{d^2y}{d\theta^2} + \cos\theta\frac{dy}{d\theta} + n(n+1)\,\sin\theta\, y = 0$$

can be transformed into Legendre's equation by means of the substitution $x = \cos\theta$

Answer 13 (end of chapter) · Solution

Example Find the first three positive values of λ for which the problem

$$(1-x^2)y'' - 2xy' + \lambda y = 0$$

$$y(0) = 0,\ y(x),\ y'(x) \text{ bounded on } [-1,1]$$

has nontrivial solutions

Answer 14 (end of chapter) · Solution

- The differential equation

$$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0$$

is known as the **associated Legendre equation**. When $m = 0$, this equation reduces to Legendre's equation. A solution of the associated equation is

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)$$

where $P_n(x)$, $n = 0, 1, 2, \cdots$ are the Legendre polynomials. The solutions $P_n^m(x)$ for $m = 0, 1, 2, \cdots$, are called **associated Legendre functions**

Example Answer the following questions

- Find the associated Legendre functions $P_0^0(x)$, $P_1^0(x)$, $P_1^1(x)$, $P_2^1(x)$, $P_2^2(x)$, $P_3^1(x)$, $P_3^2(x)$, and $P_3^3(x)$
- What can you say about $P_n^m(x)$ when m is an even non-negative integer?
- What can you say about $P_n^m(x)$ when m is a non-negative integer and $m > n$?
- Verify that $y = P_1^1(x)$ satisfies the associated Legendre equation when $n = 1$ and $m = 1$

Answer 15 (end of chapter) · Solution

Try it in Python: Legendre and Chebyshev polynomials as `numpy.polynomial` classes: Polynomials

Summary

Topic	Key result
Ordinary point	P, Q analytic at x_0: two independent solutions $\sum c_n(x-x_0)^n$, convergent at least up to the nearest singular point
Regular singular point	$(x-x_0)P$ and $(x-x_0)^2Q$ analytic: $y = \sum c_n(x-x_0)^{n+r}$, r from the indicial equation
Frobenius cases ($r_1 \geq r_2$)	$r_1 - r_2 \notin \mathbb{Z}$: two series; $r_1 - r_2 = N$: $y_2 = Cy_1 \ln(x-x_0) + \sum b_n(x-x_0)^{n+r_2}$; $r_1 = r_2$: $y_2 = y_1 \ln(x-x_0) + \sum b_n(x-x_0)^{n+r_1}$
Bessel	$x^2y'' + xy' + (x^2-\nu^2)y = 0$: $y = c_1J_\nu(x) + c_2Y_\nu(x)$ ($J_{-\nu}$ instead of Y_ν if ν is not an integer)
Parametric and modified Bessel	$x^2y'' + xy' + (\alpha^2x^2-\nu^2)y = 0$: $c_1J_\nu(\alpha x) + c_2Y_\nu(\alpha x)$; $x^2y'' + xy' - (x^2+\nu^2)y = 0$: $c_1I_\nu(x) + c_2K_\nu(x)$
Bessel recurrences	$\frac{d}{dx}[x^\nu J_\nu] = x^\nu J_{\nu-1}$, $\frac{d}{dx}[x^{-\nu}J_\nu] = -x^{-\nu}J_{\nu+1}$
Legendre	$(1-x^2)y'' - 2xy' + n(n+1)y = 0$: $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$
Legendre properties	$(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$, $\int_{-1}^{1} P_kP_l\,dx = \delta_{kl}\frac{2}{2l+1}$

Worked Exercises

1. The differential equation

$$xy'' + (1-x)y' + \alpha y = 0$$

is known as Laguerre's equation

(a) Find two linearly independent solutions for $0 \leq x < \infty$

(b) Show that there is a polynomial solution of degree n, in case $\alpha = n$ is a non-negative integer

(c) These solutions are naturally called Laguerre polynomials and are denoted by $L_n(x)$. Rodrigues' formula for the Laguerre polynomials is

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}x^n e^{-x}$$

Use this formula to find the Laguerre polynomials corresponding to $n = 0, 1, 2, 3, 4$

Solution (a)

Let's apply the Frobenius method to solve the differential equation near the regular singular point at $x = 0$

Step 1: Frobenius Series Assumption

Assume a solution of the form:

$$y(x) = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r}, \quad a_0 \neq 0$$

Then,

$$y'(x) = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$$

$$y''(x) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2}$$

Step 2: Substitute into the Equation

Substitute into the original equation:

$$\begin{aligned} &xy'' + (1-x)y' + \alpha y \\ &= \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1} + \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} - \sum_{k=0}^{\infty} a_k (k+r) x^{k+r} + \sum_{k=0}^{\infty} \alpha a_k x^{k+r} = 0 \end{aligned}$$

Group and simplify:

$$\sum_{k=0}^{\infty} a_k(k+r)^2 x^{k+r-1} + \sum_{k=0}^{\infty} a_k(\alpha - k - r)x^{k+r} = 0$$

Step 3: Indicial Equation

The lowest power of x is x^{r-1}. Its coefficient is:

$$a_0 r^2$$

Set this to zero:

$$r^2 = 0 \quad \Rightarrow \quad r = 0$$

Step 4: Recurrence Relation

Now use $r = 0$, so:

$$\sum_{k=0}^{\infty} \left[a_{k+1}(k+1)^2 + a_k(\alpha - k)\right] x^k = 0$$

So the recurrence relation is:

$$a_{k+1} = -\frac{(\alpha - k)}{(k+1)^2} a_k$$

Choose $a_0 = 1$ (arbitrary constant), this gives a general form:

$$a_k = \frac{(-1)^k \alpha(\alpha - 1)\cdots(\alpha - k + 1)}{(k!)^2}$$

Thus the first solution is

$$y_1(x) = \sum_{k=0}^{\infty} a_k x^k$$

When the indicial equation has a repeated root $r = 0$, the second solution has the form:

$$y_2(x) = y_1(x) \ln x + \sum_{k=0}^{\infty} b_k x^k$$

We plug this into the original equation. This leads to

$$2y_1' - y_1 + \sum_{k=0}^{\infty} \left[(k+1)^2 b_{k+1} + (\alpha - k)b_k\right] x^k = 0$$

Solution (b)

When $\alpha = n$,

$$a_k = \frac{(-1)^k n!}{(n-k)!(k!)^2} \quad \text{for } k \le n$$

For $k > n$, we get $a_k = 0$, so the series terminates. The series becomes a polynomial of degree n:

$$y_1(x) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!(k!)^2} x^k = L_n(x)$$

The second linearly independent solution is of the form:

$$y_2(x) = L_n(x) \ln x + \sum_{k=0}^{\infty} d_k x^k$$

Solution (c)

To find the Laguerre polynomials $L_n(x)$ for $n = 0, 1, 2, 3, 4$, we will use Rodrigues' formula.

$$L_0(x) = \frac{e^x}{0!} \frac{d^0}{dx^0} \left(x^0 e^{-x}\right) = e^x \cdot e^{-x} = 1$$

$$L_1(x) = \frac{e^x}{1!} \frac{d}{dx} \left(x e^{-x}\right) = e^x \cdot (1-x) e^{-x} = 1 - x$$

$$L_2(x) = \frac{e^x}{2!} \frac{d^2}{dx^2} \left(x^2 e^{-x}\right) = 1 - 2x + \frac{1}{2} x^2$$

$$L_3(x) = 1 - 3x + \frac{3}{2} x^2 - \frac{1}{6} x^3$$

$$L_4(x) = 1 - 4x + 3x^2 - \frac{2}{3} x^3 + \frac{1}{24} x^4$$

2. The differential equation

$$(1 - x^2) y'' - x y' + \alpha^2 y = 0$$

where α is a parameter, is known as Chebyshev's equation

(a) Find two power series solutions centered at the ordinary point 0

(b) When $\alpha = n$ is a nonnegative integer, show that Chebyshev's differential equation always possesses a polynomial solution of degree n

(c) These solutions are naturally called Chebyshev polynomials and are denoted by $T_n(x)$. Rodrigues' formula for the Chebyshev polynomials is

$$T_n(x) = (-1)^n \frac{2^n n!}{(2n)!} \sqrt{1-x^2} \frac{d^n}{dx^n}(1-x^2)^{n-1/2}$$

Use this formula to find the Chebyshev polynomials corresponding to $n = 0, 1, 2, 3$

Solution (a)

We are to find two linearly independent power series solutions centered at $x = 0$, which is an ordinary point (i.e., the coefficients of y'' and y' are analytic at $x = 0$)

Step 1: Assume a Power Series Solution

Let's assume a solution of the form:

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then compute the derivatives:

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}, \quad y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Now substitute y, y', y'' into the differential equation:

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - \sum_{k=2}^{\infty} k(k-1) a_k x^k - \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} \alpha^2 a_k x^k = 0$$

Shift the first term to match powers of x^k and combine all terms and simplify the coefficients:

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} + (\alpha^2 - k^2) a_k \right] x^k = 0$$

Set each coefficient of x^k to 0, we get the recurrence relation:

$$a_{k+2} = -\frac{\alpha^2 - k^2}{(k+2)(k+1)} a_k$$

Step 2: Build Two Linearly Independent Solutions

We choose a_0 and a_1 arbitrarily to construct two independent solutions

- First solution: even powers only (let $a_0 = 1$, $a_1 = 0$)

  Then the series becomes:

  $$y_1(x) = a_0 + a_2 x^2 + a_4 x^4 + \cdots$$

  Use recurrence:

$$a_2 = -\frac{\alpha^2 - 0^2}{2 \cdot 1} a_0 = -\frac{\alpha^2}{2}$$

$$a_4 = -\frac{\alpha^2 - 2^2}{4 \cdot 3} a_2 = -\frac{\alpha^2 - 4}{12} a_2 = \frac{(\alpha^2 - 4)(\alpha^2)}{24}$$

And so on

- Second solution: odd powers only (let $a_0 = 0$, $a_1 = 1$)

  Then the series becomes:

$$y_2(x) = a_1 x + a_3 x^3 + a_5 x^5 + \cdots$$

  Use recurrence:

$$a_3 = -\frac{\alpha^2 - 1^2}{3 \cdot 2} a_1 = -\frac{\alpha^2 - 1}{6}$$

$$a_5 = -\frac{\alpha^2 - 3^2}{5 \cdot 4} a_3 = \frac{(\alpha^2 - 9)(\alpha^2 - 1)}{120}$$

  And so on

Two linearly independent power series solutions centered at $x = 0$ are:

$$y_1(x) = a_0 \left[1 - \frac{\alpha^2}{2} x^2 + \frac{(\alpha^2 - 4)\alpha^2}{24} x^4 - \cdots\right]$$

$$y_2(x) = a_1 \left[x - \frac{\alpha^2 - 1}{6} x^3 + \frac{(\alpha^2 - 1)(\alpha^2 - 9)}{120} x^5 - \cdots\right]$$

In general,

$$a_{2k} = (-1)^k \frac{\left(\alpha^2 - (2k-2)^2\right)\left(\alpha^2 - (2k-4)^2\right) \cdots \left(\alpha^2 - 2^2\right)\alpha^2}{(2k)!} a_0 = \hat{a}_{2k} a_0$$

$$a_{2k+1} = (-1)^k \frac{\left(\alpha^2 - (2k-1)^2\right)\left(\alpha^2 - (2k-3)^2\right) \cdots \left(\alpha^2 - 1\right)}{(2k+1)!} a_1 = \hat{a}_{2k+1} a_1$$

$$\Downarrow$$

$$y = a_0 \sum_{k=0}^{\infty} \hat{a}_{2k} x^{2k} + a_1 \sum_{k=0}^{\infty} \hat{a}_{2k+1} x^{2k+1} = a_0 y_1(x) + a_1 y_2(x), \quad \hat{a}_0 = \hat{a}_1 = 1$$

Solution (b)

Let $\alpha = n$, a nonnegative integer. Then the recurrence becomes:

$$a_{k+2} = -\frac{n^2 - k^2}{(k+2)(k+1)} a_k$$

Observe that when $k = n$, we get:

$$a_{n+2} = -\frac{n^2 - n^2}{(n+2)(n+1)} a_n = 0$$

If we start with $a_0 \neq 0$, and define the even-power solution:

$$y(x) = a_0 + a_2 x^2 + a_4 x^4 + \cdots$$

then for even n, the recurrence will produce:

$$a_{n+2} = 0 \Rightarrow \text{ the series terminates at } x^n$$

So, the solution is a polynomial of degree n

Similarly, if $a_1 \neq 0$, the odd-power series will terminate at x^n when n is odd. Thus, we get a polynomial solution of degree exactly n

These solutions (suitably normalized) are known as the Chebyshev polynomials of the first kind, usually denoted $T_n(x)$. The other series (odd powers when n is even, even powers when n is odd) does not terminate, so only one of the two solutions is a polynomial

Solution (c)

Let's define:

$$f_n(x) = (1-x^2)^{n-\frac{1}{2}}, \quad \text{then} \quad T_n(x) = (-1)^n \frac{2^n n!}{(2n)!} \sqrt{1-x^2} \cdot \frac{d^n}{dx^n} f_n(x)$$

- $n = 0$

  We have: $f_0(x) = (1-x^2)^{-1/2}$, but $\frac{d^0}{dx^0} f_0 = f_0$

  $T_0(x) = \frac{1}{1} \cdot \sqrt{1-x^2} \cdot (1-x^2)^{-1/2} = 1$

- $n = 1$

  $f_1(x) = (1-x^2)^{1-\frac{1}{2}} = (1-x^2)^{1/2}$

  Differentiate:

  $\frac{d}{dx}(1-x^2)^{1/2} = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = -\frac{x}{\sqrt{1-x^2}}$

  Now plug into the formula:

  $T_1(x) = (-1)^1 \cdot \frac{2^1 \cdot 1!}{2!} \cdot \sqrt{1-x^2} \cdot \left(-\frac{x}{\sqrt{1-x^2}}\right) = x$

- $n = 2$

  $f_2(x) = (1 - x^2)^{3/2}$

  Differentiate twice:

  First derivative:

  $f_2'(x) = \frac{3}{2}(1 - x^2)^{1/2} \cdot (-2x) = -3x(1 - x^2)^{1/2}$

  Second derivative:

  $f_2''(x) = \frac{d}{dx}[-3x(1 - x^2)^{1/2}] = -3(1 - x^2)^{1/2} + 3x^2(1 - x^2)^{-1/2}$

  Now plug into Rodrigues' formula:

  $$\begin{aligned} T_2(x) &= \frac{2^2 \cdot 2!}{(4)!} \cdot \sqrt{1 - x^2} \cdot f_2''(x) \\ &= \frac{4 \cdot 2}{24} \cdot \sqrt{1 - x^2} \cdot \left[-3(1 - x^2)^{1/2} + 3x^2(1 - x^2)^{-1/2}\right] \\ &= 2x^2 - 1 \end{aligned}$$

- $n = 3$

  $f_3(x) = (1 - x^2)^{5/2}$

  We'll compute the 3rd derivative and plug it into the formula:

  $$\begin{aligned} T_3(x) &= (-1)^3 \cdot \frac{2^3 \cdot 3!}{(6)!} \cdot \sqrt{1 - x^2} \cdot f_3'''(x) \\ &= -\frac{8 \cdot 6}{720} \cdot \sqrt{1 - x^2} \cdot \left[45x(1 - x^2)^{1/2} - 15x^3(1 - x^2)^{-1/2}\right] \\ &= 4x^3 - 3x \end{aligned}$$

3. If n is an integer, you can use the substitution $R(x) = (\alpha x)^{-1/2} Z(x)$ to solve the differential equation

$$x^2 \frac{d^2R}{dx^2} + 2x \frac{dR}{dx} + \left[\alpha^2 x^2 - n(n+1)\right] R = 0$$

Find the general solution of the differential equation on the interval $(0, \infty)$

Solution

Step 1: Apply the substitution

Let:

$$R(x) = (\alpha x)^{-1/2} Z(x) = \frac{1}{\sqrt{\alpha x}} Z(x)$$

We compute derivatives of $R(x)$:

- First derivative:

  Using the product rule:

$$\frac{dR}{dx} = \frac{d}{dx}\left(\frac{1}{\sqrt{\alpha x}}Z(x)\right) = \frac{d}{dx}\left(\alpha^{-1/2}x^{-1/2}Z(x)\right)$$

  Now differentiate:

$$\frac{dR}{dx} = \alpha^{-1/2}\left(-\frac{1}{2}x^{-3/2}Z(x) + x^{-1/2}Z\text{'}(x)\right)$$

- Second derivative:

  Differentiate again:

$$\frac{d^2R}{dx^2} = \alpha^{-1/2}\left[\frac{3}{4}x^{-5/2}Z(x) - x^{-3/2}Z\text{'}(x) + x^{-1/2}Z\text{''}(x)\right]$$

Step 2: Plug into the original equation

Now substitute R, R', R'' into the equation:

$$x^2R'' + 2xR' + \left(\alpha^2x^2 - n(n+1)\right)R = 0$$

Thus the equation becomes:

$$\alpha^{-1/2}\left[x^{3/2}Z'' + x^{1/2}Z' + \left(\alpha^2x^{3/2} - \left(n(n+1) + \frac{1}{4}\right)x^{-1/2}\right)Z\right] = 0$$

Factor out $x^{1/2}$:

$$\alpha^{-1/2}x^{1/2}\left[xZ'' + Z' + \left(\alpha^2x - \frac{n(n+1)+\frac{1}{4}}{x}\right)Z\right] = 0$$

So the reduced equation is:

$$xZ'' + Z' + \left(\alpha^2x - \frac{n(n+1)+\frac{1}{4}}{x}\right)Z = 0$$

Step 3: Change variable to standard form

Let's define:

$$u = \alpha x \;\Rightarrow\; \frac{dZ}{dx} = \alpha\frac{dZ}{du}, \quad \frac{d^2Z}{dx^2} = \alpha^2\frac{d^2Z}{du^2}$$

Plug into the reduced equation and simplify:

$$uZ'' + Z' + \left(u - \frac{n(n+1)+\frac{1}{4}}{u}\right)Z = 0$$

Now multiply through by u:

$$u^2 Z'' + uZ' + \left(u^2 - \left(n(n+1) + \tfrac{1}{4}\right)\right) Z = 0$$

Step 4: Identify the equation

This is Bessel's equation of order ν, where:

$$\nu^2 = n(n+1) + \tfrac{1}{4} \;\Rightarrow\; \nu = \sqrt{n(n+1) + \tfrac{1}{4}} = n + \tfrac{1}{2}$$

(because n is an integer, and this square root simplifies nicely)

So we now recognize:

$$u^2 Z'' + uZ' + (u^2 - (n + \tfrac{1}{2})^2) Z = 0$$

Final Answer

The general solution is:

$$R(x) = \frac{1}{\sqrt{\alpha x}} \left[A J_{n+1/2}(\alpha x) + B Y_{n+1/2}(\alpha x) \right]$$

4. Use the indicated change of variable to find the general solution of the given differential equation on the interval $(0, \infty)$

$$x^2 y'' + \left(\alpha^2 x^2 - \nu^2 + \frac{1}{4}\right) y = 0, \;\; y = \sqrt{x}\, u(x)$$

Solution

Step 1: Compute derivatives of $y = \sqrt{x} u(x)$

- First derivative:

$$y' = \frac{1}{2} x^{-1/2} u(x) + x^{1/2} u'(x)$$

- Second derivative:

$$y'' = -\frac{1}{4} x^{-3/2} u(x) + x^{-1/2} u'(x) + x^{1/2} u''(x)$$

Step 2: Plug into the original equation

Now the full equation becomes:

$$\left(-\frac{1}{4} x^{1/2} u + x^{3/2} u' + x^{5/2} u''\right) + \left(\alpha^2 x^2 - \nu^2 + \frac{1}{4}\right) x^{1/2} u = 0$$

Group like terms and factor out $x^{1/2}$:

$$x^{1/2}\left(x^2u'' + xu' + (\alpha^2x^2 - \nu^2)u\right) = 0$$

Since $x^{1/2} > 0$ on $(0, \infty)$, we divide both sides:

$$x^2u'' + xu' + (\alpha^2x^2 - \nu^2)u = 0$$

This is a parameterized Bessel differential equation

Final Answer

The general solution is:

$$y(x) = \sqrt{x}\left[AJ_\nu(\alpha x) + BY_\nu(\alpha x)\right]$$

5. When $n = 0$, Legendre's differential equation has the polynomial solution $y = P_0(x) = 1$. Use reduction of order to find a second Legendre function satisfying the DE on the interval $(-1, 1)$

Solution

Step 1: Use reduction of order

Let:

$$y_2(x) = v(x)y_1(x) = v(x) \cdot 1 = v(x)$$

So we're seeking a second solution $y_2(x) = v(x)$, where $y_1(x) = 1$ is a known solution

Step 2: Plug into the equation

Start from the reduced equation:

$$(1 - x^2)y'' - 2xy' = 0$$

Since $y = v(x)$, we compute:

$$y' = v', \quad y'' = v''$$

So the equation becomes:

$$(1 - x^2)v'' - 2xv' = 0$$

This is a first-order ODE in v' if we let $w = v'$. Then:

$$(1 - x^2)w' - 2xw = 0 \quad \text{(a linear first-order equation in } w)$$

Step 3: Solve for w

We rewrite the equation as:

$$w' - \frac{2x}{1 - x^2}w = 0$$

This is a linear first-order ODE. Use integrating factor:

$$\mu(x) = \exp\left(-\int \frac{2x}{1-x^2}dx\right)$$

Let $u = 1 - x^2 \Rightarrow du = -2xdx$

Then:

$$\int \frac{2x}{1-x^2}dx = -\int \frac{1}{u}du = -\ln|1-x^2|$$

So:

$$\mu(x) = e^{\ln|1-x^2|} = |1-x^2|$$

but on $(-1,1)$, $1 - x^2 > 0$, so

$$\mu(x) = 1 - x^2$$

Now solve:

$$\frac{d}{dx}(w \cdot \mu(x)) = 0 \Rightarrow w(1-x^2) = C \Rightarrow w = \frac{C}{1-x^2}$$

Recall $w = v'$, so:

$$v\text{'} = \frac{C}{1-x^2} \Rightarrow v(x) = C\int \frac{1}{1-x^2}dx$$

So:

$$v(x) = \frac{C}{2}\ln\left|\frac{1+x}{1-x}\right| + D$$

Step 4: General solution

We already had $y_1(x) = 1$, and now we have:

$$y_2(x) = v(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad \text{(valid on } (-1,1) \text{ where the log is real)}$$

6. When $n = 1$, Legendre's differential equation has the polynomial solution $y = P_1(x) = x$. Use reduction of order to find a second Legendre function satisfying the DE on the interval $(-1, 1)$

Solution

Step 1: Use reduction of order

Let:

$$y_2(x) = v(x) \cdot y_1(x) = v(x) \cdot x$$

Plug into the differential equation:

$$(1-x^2)(v''x+2v')-2x(v'x+v)+2(vx)=0$$

Expand and simplify:

$$x(1-x^2)v''+(2-4x^2)v'=0$$

Step 2: Let $w=v'$, then solve

Let $w=v'$, so $w'=v''$. Then:

$$x(1-x^2)w'+(2-4x^2)w=0$$

We rewrite as:

$$w'+\frac{2-4x^2}{x(1-x^2)}w=0$$

Step 3: Solve a first-order linear ODE

Partial fractions give $\dfrac{2-4x^2}{x(1-x^2)}=\dfrac{2}{x}-\dfrac{1}{1-x}+\dfrac{1}{1+x}$, so the integrating factor is

$$\mu(x)=\exp\left(2\ln|x|+\ln(1-x)+\ln(1+x)\right)=x^2(1-x^2),\qquad -1<x<1$$

Now solve the equation:

$$\frac{d}{dx}\left(w\cdot\mu(x)\right)=0\Rightarrow w\cdot\mu(x)=C\Rightarrow w=\frac{C}{x^2(1-x^2)}$$

Now integrate to find v:

$$\begin{aligned}v&=\int w\,dx=C\int\frac{dx}{x^2(1-x^2)}=C\int\left(\frac{1}{x^2}+\frac{1}{1-x^2}\right)dx\\&=C\left(-\frac{1}{x}+\frac{1}{2}\ln\frac{1+x}{1-x}\right)+D\end{aligned}$$

Final Answer:

Taking $C=1$, $D=0$, the second linearly independent solution for $n=1$ on $(-1,1)$ is

$$y_2(x)=x\,v(x)=\frac{x}{2}\ln\frac{1+x}{1-x}-1\quad\text{(up to constant multiples)}$$

This is the Legendre function of the second kind $Q_1(x)$. Check: $y_2'=\frac{1}{2}\ln\frac{1+x}{1-x}+\frac{x}{1-x^2}$, $y_2''=\frac{2}{(1-x^2)^2}$, and $(1-x^2)y_2''-2xy_2'+2y_2=\frac{2}{1-x^2}-\frac{2x^2}{1-x^2}-2=0$

7. Find the general solution of the differential equation on the interval $(0,\infty)$

$$xy''+y=0$$

Hint) Use the substitution $y(x) = \sqrt{x}u(x)$, $2\sqrt{x} = z$

Solution

By using $y(x) = \sqrt{x}u(x)$ and $2\sqrt{x} = z$, the differential equation becomes

$$\begin{aligned}
y' &= \frac{1}{2\sqrt{x}}u + \sqrt{x}u' \\
y'' &= -\frac{1}{4x\sqrt{x}}u + \frac{1}{\sqrt{x}}u' + \sqrt{x}u'' \\
&\Downarrow \\
x\sqrt{x}u'' + \sqrt{x}u' + \left(\sqrt{x} - \frac{1}{4\sqrt{x}}\right)u &= 0 \\
u' &= \frac{du}{dz}\frac{dz}{dx} = \frac{1}{\sqrt{x}}\frac{du}{dz} \\
u'' &= \frac{d}{dz}\left(\frac{du}{dx}\right)\frac{dz}{dx} = \frac{1}{x}\frac{d^2u}{dz^2} - \frac{1}{2x\sqrt{x}}\frac{du}{dz} \\
&\Downarrow \\
z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + (z^2 - 1)u &= 0
\end{aligned}$$

The general solution of the above differential equation on the interval $(0, \infty)$ is

$$u(z) = c_1 J_1(z) + c_2 Y_1(z)$$

Thus the general solution on the interval $(0, \infty)$ is

$$y(x) = c_1\sqrt{x}J_1(2\sqrt{x}) + c_2\sqrt{x}Y_1(2\sqrt{x})$$

8. The equation

$$y'' - 2xy' + 2\alpha y = 0$$

where α is a constant, is called the *Hermite* equation.

(a) Find two linearly independent solutions for $-\infty < x < \infty$

(b) Show that there is a polynomial solution of degree n, in case $\alpha = n$ is a non-negative integer

Solution

(a)

$$
\begin{aligned}
&y'' - 2xy' + 2\alpha y = 0 \\
&\quad \Downarrow \quad y = \sum_{k=0}^{\infty} c_k x^k, \;\; x = 0 \text{ is an ordinary point} \\
&\sum_{k=2}^{\infty} c_k k(k-1)x^{k-2} - 2\sum_{k=1}^{\infty} c_k k x^k + 2\alpha \sum_{k=0}^{\infty} c_k x^k = 0 \\
&2c_2 + 2\alpha c_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} - 2(k-\alpha)c_k\right] x^k = 0 \\
&\quad \Downarrow \\
&c_2 = -\alpha c_0, \;\; c_{k+2} = -\frac{2(\alpha-k)}{(k+2)(k+1)} c_k \\
&\quad \Downarrow \\
&c_{2k} = (-2)^k \frac{(\alpha-(2k-2))\,(\alpha-(2k-4))\cdots(\alpha-2)\,\alpha}{(2k)!} c_0 = \hat{c}_{2k} c_0 \\
&c_{2k+1} = (-2)^k \frac{(\alpha-(2k-1))\,(\alpha-(2k-3))\cdots(\alpha-1)}{(2k+1)!} c_1 = \hat{c}_{2k+1} c_1 \\
&\quad \Downarrow \\
&y = c_0 \sum_{k=0}^{\infty} \hat{c}_{2k} x^{2k} + c_1 \sum_{k=0}^{\infty} \hat{c}_{2k+1} x^{2k+1} = c_0 y_0(x) + c_1 y_1(x), \quad \hat{c}_0 = \hat{c}_1 = 1
\end{aligned}
$$

(b) If n is an even integer, the first series terminates, whereas $y_1(x)$ is an infinite series. For example, if $n = 4$, then

$$
y_0(x) = c_0 \left[1 - 2\frac{4}{2!}x^2 + 2^2 \frac{(4-2)4}{4!} x^4\right] = c_0 \left[1 - 4x^2 + \frac{4}{3}x^4\right]
$$

Similarly, when n is an odd integer, the series for $y_1(x)$ terminates with x^n. For example, if $n = 5$, then

$$
y_1(x) = c_1 \left[x - 2\frac{(5-1)}{3!}x^3 + 4\frac{(5-3)(5-1)}{5!}x^5\right] = c_1 \left[x - \frac{4}{3}x^3 + \frac{4}{15}x^5\right]
$$

That is, when n is a nonnegative integer, we obtain an n-th degree polynomial solution of *Hermite* equation

9. Find the general solution of the differential equation on the interval $(0, \infty)$

$$
x^2 y'' + 2xy' + \left(\alpha^2 x^2 - n(n+1)\right) y = 0, \;\; n \text{ is an integer}
$$

Hint) Use the substitution $y(x) = \dfrac{1}{\sqrt{x}} z(x)$

Solution

By using $y(x) = \dfrac{1}{\sqrt{x}} z(x)$, the differential equation becomes

$$
x^2 z'' + xz' + \left[\alpha^2 x^2 - \left(n + \frac{1}{2}\right)^2\right] z = 0
$$

The general solution of the above differential equation on the interval $(0, \infty)$ is

$$z(x) = c_1 J_{n+\frac{1}{2}}(\alpha x) + c_2 Y_{n+\frac{1}{2}}(\alpha x)$$

Thus the general solution on the interval $(0, \infty)$ is

$$y(x) = c_1 \frac{1}{\sqrt{x}} J_{n+\frac{1}{2}}(\alpha x) + c_2 \frac{1}{\sqrt{x}} Y_{n+\frac{1}{2}}(\alpha x)$$

Answers

Answer 1 (Section 5.3)

$y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\,(c_1 \sin x + c_2 \cos x)$

Answer 2 (Section 5.3)

$y = c_1 J_3(x) + c_2 Y_3(x)$

Answer 3 (Section 5.3)

Put $\nu = 0$ in $\dfrac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x)$

Answer 4 (Section 5.3)

With $\Gamma\left(\frac{1}{2} + n\right) = \dfrac{(2n)!}{2^{2n} n!}\sqrt{\pi}$, the series becomes $\sqrt{\dfrac{2}{\pi x}} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sqrt{\frac{2}{\pi x}} \cos x$

Answer 5 (Section 5.3)

(1) $y = c_1 J_{1/3}(x) + c_2 J_{-1/3}(x)$

(2) $y = c_1 J_0(x) + c_2 Y_0(x)$

(3) $y = c_1 J_2(3x) + c_2 Y_2(3x)$

(4) $y = c_1 I_{2/3}(4x) + c_2 K_{2/3}(4x)$

Answer 6 (Section 5.3)

(1) $y = x^{-1/2}\left[c_1 J_{1/2}(\alpha x) + c_2 J_{-1/2}(\alpha x)\right] = C_1 \dfrac{\sin \alpha x}{x} + C_2 \dfrac{\cos \alpha x}{x}$

(2) $y = x^{-1/2}\left[c_1 J_1(4\sqrt{x}) + c_2 Y_1(4\sqrt{x})\right]$

Answer 7 (Section 5.3)

$I_\nu(x) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n+\nu}$, a series of real (and positive) terms for $x > 0$

Answer 8 (Section 5.3)

$y = c_1 I_0\left(\dfrac{\sqrt{7}}{2} x^2\right) + c_2 K_0\left(\dfrac{\sqrt{7}}{2} x^2\right)$

Answer 9 (Section 5.3)

$y = x^{-3/2}\left[c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)\right] = C_1 \dfrac{\sin x}{x^2} + C_2 \dfrac{\cos x}{x^2}$

Answer 10 (Section 5.3)

Integrate $\dfrac{d}{dr}\left[rJ_1(r)\right] = rJ_0(r)$ from 0 to x; $J_0' = J_{-1}$ follows from $\dfrac{d}{dx}\left[x^\nu J_\nu\right] = x^\nu J_{\nu-1}$ with $\nu = 0$, and $J_{-1} = -J_1$

Answer 11 (Section 5.3)

As $t \to \infty$ the argument $s = \frac{2}{\alpha}\sqrt{k/m}\, e^{-\alpha t/2} \to 0$:

(1) $x(t) \to c_1$ (the mass comes to rest) **(2)** $|x(t)| \to \infty$, growing almost linearly in t **(3)** the Y_0 term dominates, so $|x(t)| \to \infty$

Answer 12 (Section 5.3)

$P_2 = \frac{1}{2}(3x^2 - 1)$, $P_3 = \frac{1}{2}(5x^3 - 3x)$, $P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$, $P_5 = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

$P_6 = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$, $P_7 = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$

Answer 13 (Section 5.3)

With $\dfrac{dy}{d\theta} = -\sin\theta\, y_x$ and $\dfrac{d^2y}{d\theta^2} = \sin^2\theta\, y_{xx} - \cos\theta\, y_x$, the equation becomes $\sin\theta\left[(1-x^2)y_{xx} - 2xy_x + n(n+1)y\right] 0$

Answer 14 (Section 5.3)

$\lambda = 2,\ 12,\ 30$ ($n = 1, 3, 5$; the solutions are P_1, P_3, P_5)

Answer 15 (Section 5.3)

$P_0^0 = 1$, $P_1^0 = x$, $P_1^1 = (1-x^2)^{1/2}$, $P_2^1 = 3x(1-x^2)^{1/2}$, $P_2^2 = 3(1-x^2)$

$P_3^1 = \frac{3}{2}(5x^2 - 1)(1-x^2)^{1/2}$, $P_3^2 = 15x(1-x^2)$, $P_3^3 = 15(1-x^2)^{3/2}$

m even: P_n^m is a polynomial of degree n; $m > n$: $P_n^m \equiv 0$

6 Matrices

Learning objectives After this chapter you should be able to

- perform matrix algebra, and solve linear systems by Gaussian and Gauss-Jordan elimination
- use the rank to decide whether a system is consistent and how many parameters its solution has
- evaluate determinants by cofactor expansion and their properties, find inverses, and use Cramer's rule and the LU factorization
- find eigenvalues and eigenvectors, compute matrix powers with the Cayley-Hamilton theorem, and approximate eigenvalues by the power method
- diagonalize matrices, orthogonally diagonalize symmetric matrices, and apply matrices to quadratic forms, codes, least squares and compartmental models

6.1 Prerequisites

- Vector Spaces (Section 9.6)
- Gram-Schmidt Orthogonalization Process (Section 9.7)

6.2 Matrix Algebra

- A **matrix** is any rectangular array of numbers or functions

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

  - The numbers or functions in the array are **entries** or **elements**
  - An $n \times n$ matrix is a **square** matrix of **order** n

- **Column** and **row vectors** are $n \times 1$ and $1 \times n$ matrices

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

- **Equality of Matrices**

$$\mathbf{A} = (a_{ij})_{m\times n} \quad \text{and} \quad \mathbf{B} = (b_{ij})_{m\times n} \quad \text{are equal}$$

$$\text{if } a_{ij} = b_{ij} \quad \text{for each } i \text{ and } j$$

- **Matrix Addition**

$$\mathbf{A} + \mathbf{B} = \left(a_{ij} + b_{ij}\right)_{m \times n}$$

- **Scalar Multiplication**

$$k\mathbf{A} = \left(ka_{ij}\right)_{m \times n}$$

- **Properties of Matrix Addition and Scalar Multiplication**

  Suppose $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are $m \times n$ matrices and k_1 and k_2 are scalars. Then

$$\begin{gathered}
\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \\
\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \\
(k_1 k_2)\,\mathbf{A} = k_1\,(k_2 \mathbf{A}) \\
k_1\,(\mathbf{A} + \mathbf{B}) = k_1 \mathbf{A} + k_1 \mathbf{B} \\
(k_1 + k_2)\,\mathbf{A} = k_1 \mathbf{A} + k_2 \mathbf{A}
\end{gathered}$$

- **Matrix multiplication**

$$\mathbf{AB} = \left(\sum_{k=1}^{p} a_{ik} b_{kj}\right)_{m \times n}$$

  where $\mathbf{A}$ is an $m \times p$ matrix, $\mathbf{B}$ is a $p \times n$ matrix, and $\mathbf{AB}$ is the $m \times n$ matrix

  - In general, $\mathbf{AB} \neq \mathbf{BA}$
  - **Associative Law:** $\mathbf{A}\,(\mathbf{BC}) = (\mathbf{AB})\,\mathbf{C}$
  - **Distributive Law:** $\mathbf{A}\,(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

- **Transpose of a Matrix**

$$\mathbf{A}^T = \begin{pmatrix}
a_{11} & a_{21} & \cdots & a_{m1} \\
a_{12} & a_{22} & \ddots & a_{m2} \\
\vdots & \ddots & \ddots & \vdots \\
a_{1n} & a_{2n} & \cdots & a_{mn}
\end{pmatrix}$$

- **Properties of Transpose**

$$\begin{gathered}
\left(\mathbf{A}^T\right)^T = \mathbf{A} \\
(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T \\
(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T \\
(k\mathbf{A})^T = k\mathbf{A}^T
\end{gathered}$$

- **Special Matrices**

  - In a **zero matrix**, all entries are zeros
  - In a **triangular matrix**, all entries above or below the main diagonal are zeros (lower triangular or upper triangular)

- In a **diagonal matrix**, all entries not on the main diagonal are zeros
- A **scalar matrix** is a diagonal one where all entries on the main diagonal are equal. If those entries are 1's, it is an **identity matrix**, $\mathbf{I}$ (or $\mathbf{I}_n$ when there is a need to emphasize the order of the matrix)
- An $n \times n$ matrix $\mathbf{A}$ is **symmetric** if $\mathbf{A}^T = \mathbf{A}$

Example If $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ -5 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 6 \\ 3 & 2 \end{pmatrix}$,

find (a) $\mathbf{AB}$, (b) $\mathbf{BA}$, (c) $\mathbf{A}^2$, (d) $\mathbf{B}^2$

Answer 1 (end of chapter) · Solution

Example Show that if $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{AA}^T$ is symmetric

Answer 2 (end of chapter) · Solution

Example In matrix theory, many of the familiar properties of the real number system are not valid. If a and b are real numbers, then $ab = 0$ implies that $a = 0$ or $b = 0$. Find two matrices such that $\mathbf{AB} = \mathbf{0}$ but $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$

Answer 3 (end of chapter) · Solution

Example Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. Explain why, in general, the given formula is not valid

$$(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$$

Answer 4 (end of chapter) · Solution

Example Find the resulting vector $\mathbf{b}$ if the given vector $\mathbf{a} = \langle 1, 1 \rangle$ is rotated through the indicated angle $\theta = \pi/2$

Answer 5 (end of chapter) · Solution

Example Verify that the quadratic form $ax^2 + bxy + cy^2$ is the same as

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Answer 6 (end of chapter) · Solution

Try it in Python: Matrix operations with `numpy`: Matrix and vector operations; exact (symbolic) matrices with `sympy`: Linear algebra

6.3 Systems of Linear Algebraic Equations

- **General Form**

  A system of m linear equations in n unknowns has the general form

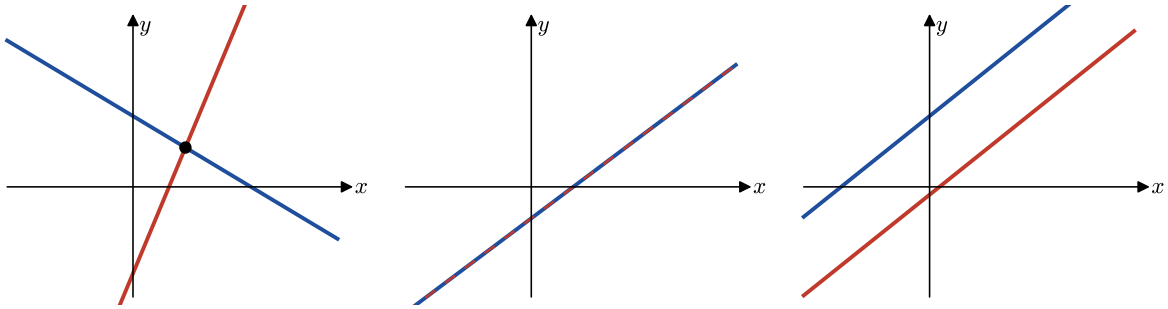
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

- The **coefficients** of the unknowns can be abbreviated as a_{ij}. The numbers $b_1, b_2, \cdots, b_m$ are called the **constants** of the system. If all the constants are zero, the system is said to be **homogeneous**, otherwise it is **nonhomogeneous**

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

- A linear system of equations is said to be **consistent** if it has at least one solution and **inconsistent** if it has no solutions. If a linear system is consistent, it has either
  - a unique solution (that is, precisely one solution), or
  - infinitely many solutions



(a) Consistent unique solution

(b) Consistent infinitely many solutions

(c) Inconsistent no solution

- **Augmented Matrix**

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ddots & a_{2n} & b_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

- A system can be solved with **elementary operations** (**row reduction** for matrices) on an augmented matrix

ElementaryOperations	Meaning
R_{ij}	Interchange rows i and j
cR_i	Multiply the row i by the nonzero constant c
$cR_i + R_j$	Multiply the row i by c and add to the row j

- In the **Gaussian elimination**, we row-reduce the augmented matrix until we arrive at a row-equivalent augmented matrix in **row-echelon form**
  - The first nonzero entry in a nonzero row is a 1
  - In consecutive nonzero rows, the first entry 1 in the lower row appears to the right of the 1 in the higher row
  - Rows consisting of all zeros are at the bottom of the matrix

Example Solve

$$\begin{pmatrix} 2 & 6 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ 9 \end{pmatrix}$$

Using row operations on the augmented matrix, we obtain

$$\left(\begin{array}{ccc|c} 2 & 6 & 1 & 7 \\ 1 & 2 & -1 & -1 \\ 5 & 7 & -4 & 9 \end{array}\right) \overset{R_{12}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 6 & 1 & 7 \\ 5 & 7 & -4 & 9 \end{array}\right) \overset{\substack{-2R_1+R_2 \\ -5R_1+R_3}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 2 & 3 & 9 \\ 0 & -3 & 1 & 14 \end{array}\right)$$

$$\overset{\frac{1}{2}R_2}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & -3 & 1 & 14 \end{array}\right) \overset{3R_2+R_3}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & \frac{11}{2} & \frac{55}{2} \end{array}\right)$$

$$\overset{\frac{2}{11}R_3}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & 1 & 5 \end{array}\right)$$

The last matrix is in row-echelon form. We can make the last matrix above to be in reduced row-echelon form

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & 1 & 5 \end{array}\right) \overset{-2R_2+R_1}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & -4 & -10 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & 1 & 5 \end{array}\right) \overset{\substack{4R_3+R_1 \\ -\frac{3}{2}R_3+R_2}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

We see that the solution is $x_1 = 10,\ x_2 = -3,\ x_3 = 5$

Example Solve

$$\begin{pmatrix} 1 & 3 & -2 \\ 4 & 1 & 3 \\ 2 & -5 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -7 \\ 5 \\ 19 \end{pmatrix}$$

Using row operations on the augmented matrix, we obtain

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 4 & 1 & 3 & 5 \\ 2 & -5 & 7 & 19 \end{array}\right) \begin{array}{l} -4R_1 + R_2 \\ -2R_1 + R_3 \\ \Longrightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & -11 & 11 & 33 \\ 0 & -11 & 11 & 33 \end{array}\right) \begin{array}{l} -R_2 + R_3 \\ -\frac{1}{11}R_2 \\ \Longrightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & 1 & -1 & -3 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

In this case, the last matrix implies that the original system of three equations is really equivalent to two equations

$$\begin{array}{l} \\ -3R_2+R_1 \\ \Longrightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -3 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

If we let $x_3 = t$, $x_1 = -t + 2$ and $x_2 = t - 3$, then we see that the system has infinitely many solutions

Example Solve

$$\begin{pmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 8 \end{pmatrix}$$

Using row operations on the augmented matrix, we obtain

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & -1 & -6 \\ 2 & -3 & 8 \end{array}\right) \begin{array}{l} -4R_1 + R_2 \\ -2R_1 + R_3 \\ \Longrightarrow \end{array} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -5 & -10 \\ 0 & -5 & 6 \end{array}\right) \begin{array}{l} -R_2 + R_3 \\ -\frac{1}{5}R_2 \\ \Longrightarrow \end{array} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \\ \hline 0 & 0 & 16 \end{array}\right)$$

The system has no solution

- A **homogeneous system** of linear equations is **always consistent**. The solution consisting of all zeros is called the **trivial solution**. A homogeneous system either possesses only the trivial solution or possesses the trivial solution along with infinitely many nontrivial solutions
- **A homogeneous system possesses nontrivial solutions if the number m of equations is less than the number n of unknowns** $(m < n)$

Example Find the positive integers x_1, x_2, x_3, and x_4 so that

$$x_1C_2H_6 + x_2O_2 \to x_3CO_2 + x_4H_2O$$

Because the number of atoms of each element must be the same on each side of the last equation, we have:

Atom	
C	$2x_1 = x_3$
H	$6x_1 = 2x_4$
O	$2x_2 = 2x_3 + x_4$

$$\left(\begin{array}{cccc|c} 2 & 0 & -1 & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right) \begin{array}{c} R_{12} \\ R_{23} \\ \Longrightarrow \end{array} \left(\begin{array}{cccc|c} 6 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 2 & 0 & -1 & 0 & 0 \end{array}\right) \begin{array}{c} \text{row operations} \\ \Longrightarrow \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{7}{6} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \end{array}\right)$$

Then when we let $x_4 = t$, $x_1 = \frac{1}{3}t$, $x_2 = \frac{7}{6}t$, $x_3 = \frac{2}{3}t$. If we pick $t = 6$, $x_1 = 2$, $x_2 = 7$, $x_3 = 4$, $x_4 = 6$

Example Use either Gaussian elimination or Gauss-Jordan elimination to solve the given system or show that no solution exists

$$\begin{aligned} x_1 - x_2 &= 11 \\ 4x_1 + 3x_2 &= -5 \end{aligned}$$

$$\begin{aligned} 9x_1 + 3x_2 &= -5 \\ 2x_1 + x_2 &= -1 \end{aligned}$$

$$\begin{aligned} x_1 - x_2 - x_3 &= -3 \\ 2x_1 + 3x_2 + 5x_3 &= 7 \\ x_1 - 2x_2 + 3x_3 &= -11 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 - x_2 - x_3 &= 8 \\ x_1 - x_2 + x_3 &= 3 \\ -x_1 + x_2 + x_3 &= 4 \end{aligned}$$

Answer 7 (end of chapter) · Solution

Example Balance the given chemical equation:

$$C_5H_8 + O_2 \rightarrow CO_2 + H_2O$$

$$Cu + HNO_3 \rightarrow Cu(NO_3)_2 + H_2O + NO$$

Answer 8 (end of chapter) · Solution

Example Compute the given product for an arbitrary 3×3 matrix $\mathbf{A}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix} \mathbf{A}$$

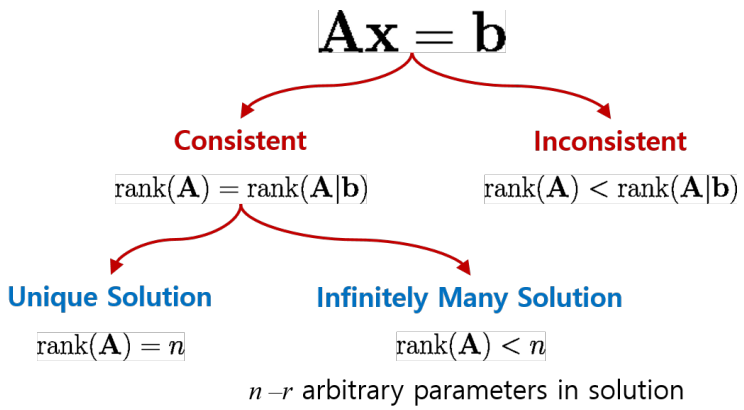
Answer 9 (end of chapter) · Solution

Try it in Python: Solving linear systems numerically and symbolically, condition numbers, and least squares: System of Linear equations

6.4 Rank of a Matrix

- The **rank** of an $m \times n$ matrix $\mathbf{A}$, $\text{rank}(\mathbf{A})$, is **the maximum number of linearly independent row vectors**. If a matrix $\mathbf{A}$ is row equivalent to a row-echelon form $\mathbf{B}$, then
  - the row space of $\mathbf{A}$ = the row space of $\mathbf{B}$
  - the nonzero rows of $\mathbf{B}$ form a basis for the row space of $\mathbf{A}$, and
  - $\text{rank}(\mathbf{A})$ = the number of nonzero rows in $\mathbf{B}$
- **Consistency of $\mathbf{Ax} = \mathbf{b}$**
  - A linear system of equations $\mathbf{Ax} = \mathbf{b}$ is consistent if and only if $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b})$
  - Suppose a linear system $\mathbf{Ax} = \mathbf{b}$ with m equations and n unknowns is consistent. If $\text{rank}(\mathbf{A}) = r \leq n$, then the solution of the system contains $n - r$ parameters. This means that we have the unique solution when $r = n$

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ddots & a_{2n} & b_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \xrightarrow{\text{row operations}} \left(\begin{array}{cccc|ccc|c} 1 & a'_{12} & \cdots & a'_{1r} & a'_{1r+1} & \cdots & a'_{1n} & b'_1 \\ 0 & 1 & \ddots & \vdots & a'_{2r+1} & \ddots & a'_{2n} & b'_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & a'_{rr+1} & \cdots & a'_{rn} & b'_r \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$



Example Find the rank of the given matrix

$$\begin{pmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 1 & 0 & 5 & 1 \\ 2 & 1 & \frac{2}{3} & 3 & \frac{1}{3} \\ 6 & 6 & 6 & 12 & 0 \end{pmatrix}$$

Answer 10 (end of chapter) · Solution

Example Determine whether the given set of vectors is linearly dependent or linearly independent

$$\mathbf{u}_1 = \langle 1, 2, 3 \rangle,\ \mathbf{u}_2 = \langle 1, 0, 1 \rangle,\ \mathbf{u}_3 = \langle 1, -1, 5 \rangle$$

$$\mathbf{u}_1 = \langle 1, -1, 3, -1 \rangle,\ \mathbf{u}_2 = \langle 1, -1, 4, 2 \rangle,\ \mathbf{u}_3 = \langle 1, -1, 5, 7 \rangle$$

Answer 11 (end of chapter) · Solution

Example Suppose the system $\mathbf{Ax} = \mathbf{b}$ is consistent and $\mathbf{A}$ is a 5×8 matrix and $\text{rank}\,\mathbf{A} = 3$. How many parameters does the solution of the system have?

Answer 12 (end of chapter) · Solution

Example Let $\mathbf{A}$ be a nonzero 4×6 matrix

1. What is the maximum rank that $\mathbf{A}$ can have?

2. If $\text{rank}(\mathbf{A}|\mathbf{b}) = 2$, then for what value(s) of $\text{rank}(\mathbf{A})$ is the system $\mathbf{Ax} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$, inconsistent? Consistent?

3. If $\text{rank}(\mathbf{A}) = 3$, then how many parameters does the solution of the system $\mathbf{Ax} = \mathbf{0}$ have?

Answer 13 (end of chapter) · Solution

Example Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be the first, second, and third column vectors, respectively, of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 7 \\ 1 & 0 & 2 \\ -1 & 5 & 13 \end{pmatrix}$$

What can we conclude about $\text{rank}(\mathbf{A})$ from the observation $2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$?

Answer 14 (end of chapter) · Solution

6.5 Determinants

- **Determinant of a 2×2 Matrix**

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- **Determinant of a 3×3 Matrix**

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- **Determinant of a $n \times n$ Matrix**

$$\det \mathbf{A} = \sum (-1)^h a_{1l_1} a_{2l_2} \cdots a_{nl_n}$$

  where the summation is over all permutations $l_1, l_2, \cdots, l_n$ of $1, 2, \cdots, n$ and the sign accords with the parity of the permutation

- **Cofactor and Minor**

  The **cofactor of** a_{ij} is the determinant

$$C_{ij} = (-1)^{i+j} M_{ij}$$

  where M_{ij} is the determinant of the submatrix obtained by deleting the i-th row and the j-th column of $\mathbf{A}$. The determinant M_{ij} is called a **minor determinant**

- **Cofactor Expansion of a Determinant: Laplace Development**

  Let $\mathbf{A} = \left(a_{ij}\right)_{n\times n}$ be an $n \times n$ matrix. For each $1 \le i \le n$, **the cofactor expansion of** $\det(\mathbf{A})$ **along the** i**-th row** is

$$\det(\mathbf{A}) = \sum_{k=1}^{n} a_{ik}C_{ik}$$

  For each $1 \le j \le n$, **the cofactor expansion of** $\det(\mathbf{A})$ **along the** j**-th column** is

$$\det(\mathbf{A}) = \sum_{k=1}^{n} a_{kj}C_{kj}$$

Example Suppose

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & -1 & 2 \\ -2 & 3 & 5 \end{pmatrix}$$

Evaluate the indicated minor determinant or cofactor

1. M_{12} **2.** M_{32} **3.** C_{13} **4.** C_{22}

Answer 15 (end of chapter) · Solution

Example Evaluate the determinant of the matrix

$$\begin{pmatrix} 3 & 5 \\ -1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{pmatrix}$$

Answer 16 (end of chapter) · Solution

Example Evaluate the determinant of the given matrix by cofactor expansion

$$\begin{pmatrix} -2 & -1 & 4 \\ -3 & 6 & 1 \\ -3 & 4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ 2 & 3 & 4 \end{pmatrix}$$

Answer 17 (end of chapter) · Solution

6.6 Properties of Determinants

- If $\mathbf{A}^T$ is the transpose of the $n \times n$ matrix $\mathbf{A}$, then $\det(\mathbf{A}^T) = \det(\mathbf{A})$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

- If any two rows (columns) of an $n \times n$ matrix $\mathbf{A}$ are the same, then $\det(\mathbf{A}) = 0$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

- If all the entries in a row (column) of an $n \times n$ matrix $\mathbf{A}$ are zero, then $\det(\mathbf{A}) = 0$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

- If $\mathbf{B}$ is the matrix obtained by interchanging any two rows (columns) of an $n \times n$ matrix $\mathbf{A}$, then $\det(\mathbf{B}) = -\det(\mathbf{A})$

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- If $\mathbf{B}$ is the matrix obtained by multiplying a row (column) by a nonzero real number k, then $\det(\mathbf{B}) = k \det(\mathbf{A})$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$ matrices, then $\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$

$$\left| \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \right| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}$$

- Suppose $\mathbf{B}$ is the matrix obtained from an $n \times n$ matrix $\mathbf{A}$ by multiplying a row(column) by a nonzero k and adding the result to another row(column). Then $\det(\mathbf{B}) = \det(\mathbf{A})$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ka_{21} + a_{31} & ka_{22} + a_{32} & ka_{23} + a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- If $\mathbf{A}$ is an $n \times n$ triangular matrix, then $\det(\mathbf{A}) = a_{11}a_{22}\cdots a_{nn}$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}$$

Alien Cofactors

- Suppose $\mathbf{A}$ is an $n \times n$ matrix. If $a_{i1}, a_{i2}, \cdots, a_{in}$ are the entries in the i-th row and $C_{p1}, C_{p2}, \cdots, C_{pn}$ are the cofactors of the entries in the p-th row, then

$$\sum_{k=1}^{n} a_{ik}C_{pk} = 0 \text{ for } i \neq p$$

- If $a_{1j}, a_{2j}, \cdots, a_{nj}$ are the entries in the j-th column and C_{1p}, C_{2p}, $\cdots$, C_{np} are the cofactors of the entries in the p-th column, then

$$\sum_{k=1}^{n} a_{kj}C_{kp} = 0 \text{ for } j \neq p$$

Example State the appropriate theorem(s) in this section that justifies the given equality

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} -5 & 6 \\ 2 & -8 \end{vmatrix} = \begin{vmatrix} 1 & 6 \\ -6 & -8 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 18 \\ 5 & 9 & -12 \end{vmatrix} = 6\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 5 & 9 & -4 \end{vmatrix}$$

Answer 18 (end of chapter) · Solution

Example Evaluate the determinant of the given matrix using the result

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 5$$

1. $\mathbf{A} = \begin{pmatrix} a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 \\ c_3 & c_2 & c_1 \end{pmatrix}$

2. $\mathbf{A} = \begin{pmatrix} 2a_1 & a_2 & a_3 \\ 6b_1 & 3b_2 & 3b_3 \\ 2c_1 & c_2 & c_3 \end{pmatrix}$

3. $\mathbf{A} = \begin{pmatrix} 4a_1 - 2a_3 & a_2 & a_3 \\ 4b_1 - 2b_3 & b_2 & b_3 \\ 2c_1 - c_3 & \frac{1}{2}c_2 & \frac{1}{2}c_3 \end{pmatrix}$

Answer 19 (end of chapter) · Solution

Example Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{pmatrix}$$

Without expanding, show that $\det \mathbf{A} = 0$

Answer 20 (end of chapter) · Solution

Example Evaluate

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$$

Answer 21 (end of chapter) · Solution

6.7 Inverse of a Matrix

- If $\mathbf{A}$ is an $n \times n$ matrix and there exists an $n \times n$ matrix $\mathbf{B}$ such that

  $$\mathbf{AB} = \mathbf{BA} = \mathbf{I}$$

  then $\mathbf{A}$ is said to be **nonsingular** or **invertible** and $\mathbf{B}$ is the **inverse** of $\mathbf{A}$
- An $n \times n$ matrix that has no inverse is called **singular**. If $\mathbf{A}$ is nonsingular, its inverse is denoted by $\mathbf{B} = \mathbf{A}^{-1}$
- **Properties of the Inverse**

  Let $\mathbf{A}$ and $\mathbf{B}$ be nonsingular matrices. Then

  $$\begin{aligned} \left(\mathbf{A}^{-1}\right)^{-1} &= \mathbf{A} \\ (\mathbf{AB})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\ \left(\mathbf{A}^{T}\right)^{-1} &= \left(\mathbf{A}^{-1}\right)^{T} \end{aligned}$$

- **Adjoint Matrix**

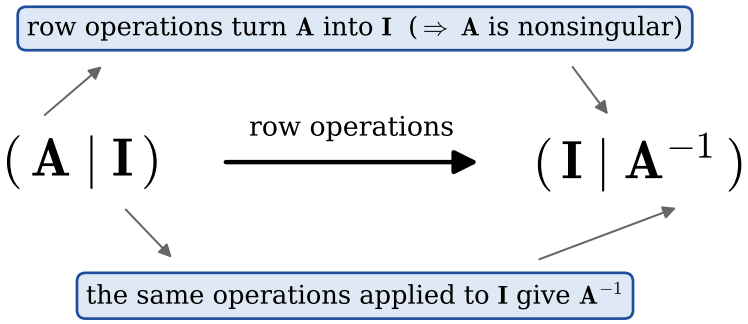
$$\text{adj}(\mathbf{A}) = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T$$

- **Finding the Inverse**

  Let $\mathbf{A}$ be an $n \times n$ matrix. If $\det(\mathbf{A}) \neq 0$ (**nonsingular**), then

$$\mathbf{A}^{-1} = \frac{\text{adj}(\mathbf{A})}{\det(\mathbf{A})}$$

  or



- **Using the Inverse to Solve Systems**

  The coefficient matrix $\mathbf{A}$ is $n \times n$. In particular, if $\mathbf{A}$ is nonsingular, the system $\mathbf{Ax} = \mathbf{b}$ can be solved by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

  A homogeneous system of n linear equations in n unknowns $\mathbf{Ax} = \mathbf{0}$ has

  – only the trivial solution if and only if $\mathbf{A}$ is nonsingular

  – a nontrivial solution if and only if $\mathbf{A}$ is singular

Example Verify that the matrix $\mathbf{B}$ is the inverse of the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$$

Answer 22 (end of chapter) · Solution

Example Determine whether the given matrix is singular or nonsingular. If it is nonsingular, find the inverse using $\mathbf{A}^{-1} = \frac{\text{adj}\,\mathbf{A}}{\det \mathbf{A}}$

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 1 & 4 \\ 3 & 2 & -2 & 1 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix}$$

Answer 23 (end of chapter) · Solution

Example Find the inverse of the given matrix or show that no inverse exists

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 & 3 \\ 2 & 1 & 0 \\ -1 & -2 & 0 \end{pmatrix}$$

Answer 24 (end of chapter) · Solution

Example If $\mathbf{A}$ is nonsingular, then $\left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T$, verify this for

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix}$$

Answer 25 (end of chapter) · Solution

Example Find the inverse of the rotation matrix

$$\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

What does $\mathbf{a} = \mathbf{M}^{-1}\mathbf{b}$ represent?

Answer 26 (end of chapter) · Solution

Example A nonsingular matrix $\mathbf{A}$ is said to be **orthogonal** if $\mathbf{A}^{-1} = \mathbf{A}^T$

1. Verify that the rotation matrix is orthogonal

2. Verify that

$$\mathbf{A} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & 0 & -2 \\ \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & -\sqrt{3} & 1 \end{pmatrix}$$

is an orthogonal matrix

Answer 27 (end of chapter) · Solution

Example Answer the questions based on the supposition

- Show that if $\mathbf{A}$ is an orthogonal matrix, then $\det \mathbf{A} = \pm 1$
- Suppose $\mathbf{A}$ and $\mathbf{B}$ are nonsingular $n \times n$ matrices. Then show that $\mathbf{AB}$ is nonsingular

- Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices and that either $\mathbf{A}$ or $\mathbf{B}$ is singular. Then show that $\mathbf{AB}$ is singular
- Suppose $\mathbf{A}$ is a nonsingular matrix. Then show that $\det \mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}}$
- Suppose $\mathbf{A}^2 = \mathbf{A}$. Then show that either $\mathbf{A} = \mathbf{I}$ or $\mathbf{A}$ is singular
- Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices. $\mathbf{A}$ is nonsingular, and $\mathbf{AB} = \mathbf{0}$. Then show that $\mathbf{B} = \mathbf{0}$
- Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices. $\mathbf{A}$ is nonsingular, and $\mathbf{AB} = \mathbf{AC}$. Then show that $\mathbf{B} = \mathbf{C}$
- Suppose $\mathbf{A}$ and $\mathbf{B}$ are nonsingular $n \times n$ matrices. Is $\mathbf{A} + \mathbf{B}$ necessarily nonsingular?
- Suppose $\mathbf{A}$ is a nonsingular matrix. Then show that $\mathbf{A}^T$ is nonsingular
- Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ nonzero matrices and $\mathbf{AB} = \mathbf{0}$. Then show that both $\mathbf{A}$ and $\mathbf{B}$ are singular

Answer 28 (end of chapter) · Solution

6.8 Cramer's Rule

If $\det(\mathbf{A}) \neq 0$, the solution of the system is given by

$$x_k = \frac{\det(\mathbf{A}_k)}{\det(\mathbf{A})}, \quad k = 1, 2, \cdots, n$$

where

$$\mathbf{A}_k = \begin{pmatrix} a_{11} & \cdots & a_{1k-1} & b_1 & a_{1k+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2k-1} & b_2 & a_{2k+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk-1} & b_n & a_{nk+1} & \cdots & a_{nn} \end{pmatrix}$$

Example Solve the given system of equations by Cramer's rule

$$\begin{aligned} -3x_1 + x_2 &= 3 \\ 2x_1 - 4x_2 &= -6 \end{aligned}$$

$$\begin{aligned} 0.1x_1 - 0.4x_2 &= 0.13 \\ x_1 - x_2 &= 0.4 \end{aligned}$$

$$\begin{aligned} 2x + y &= 1 \\ 3x + 2y &= -2 \end{aligned}$$

Answer 29 (end of chapter) · Solution

Example Consider the system

$$x_1 + x_2 = 1$$
$$x_1 + \epsilon x_2 = 2$$

When ϵ is close to 1, the lines that make up the system are almost parallel

1. Use Cramer's rule to show that a solution of the system is

$$x_1 = 1 - \frac{1}{\epsilon - 1}, \quad x_2 = \frac{1}{\epsilon - 1}$$

2. The system is said to be ill-conditioned since small changes in the input data(for example, the coefficients) cause a significant or large change in the output or solution. Verify this by finding the solution of the system for $\epsilon = 1.01$ and then for $\epsilon = 0.99$

Answer 30 (end of chapter) · Solution

6.9 The Eigenvalue Problem

- Let $\mathbf{A}$ be an $n \times n$ matrix. A number λ is said to be an **eigenvalue** of $\mathbf{A}$ if there exists a nonzero solution vector $\mathbf{k}$ of the linear system

  $$\mathbf{Ak} = \lambda\mathbf{k}$$

  and the solution vector $\mathbf{k}$ is said to be an **eigenvector** corresponding to the eigenvalue λ
- The problem of solving $\mathbf{Ak} = \lambda\mathbf{k}$ for nonzero vectors $\mathbf{k}$ is called the **eigenvalue problem** for $\mathbf{A}$
- We must solve the **characteristic equation** $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to find an eigenvalue λ
- To find an eigenvector $\mathbf{k}$ corresponding to an eigenvalue λ, we solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{k} = \mathbf{0}$ by applying Gauss elimination to $(\mathbf{A} - \lambda\mathbf{I}|\mathbf{0})$

Example Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

To find the eigenvalues, we solve

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} = 0$$

It follows that the characteristic equation is

$$-\lambda^3 - \lambda^2 + 12\lambda = -\lambda(\lambda + 4)(\lambda - 3) = 0$$

Hence the eigenvalues are $\lambda_1 = -4,\ \lambda_2 = 0,\ \lambda_3 = 3$

For $\lambda_1 = -4,$ we have

$$(\mathbf{A} + 4\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} 5 & 2 & 1 & 0 \\ 6 & 3 & 0 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

Thus $k_1 = -k_3,\ k_2 = 2k_3$. Choosing $k_3 = 1$ gives the eigenvector

$$\mathbf{k}_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 0,$ we have

$$(\mathbf{A} - 0\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 6 & -1 & 0 & 0 \\ -1 & -2 & -1 & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{13} & 0 \\ 0 & 1 & \frac{6}{13} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

Thus $k_1 = -\frac{1}{13}k_3,\ k_2 = -\frac{6}{13}k_3$. Choosing $k_3 = 1$ gives the eigenvector

$$\mathbf{k}_2 = \begin{pmatrix} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{pmatrix}$$

For $\lambda_3 = 3,$ we have

$$(\mathbf{A} - 3\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 6 & -4 & 0 & 0 \\ -1 & -2 & -4 & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

Thus $k_1 = -k_3,\ k_2 = -\frac{3}{2}k_3$. Choosing $k_3 = 1$ gives the eigenvector

$$\mathbf{k}_3 = \begin{pmatrix} -1 \\ -\frac{3}{2} \\ 1 \end{pmatrix}$$

```
import sympy
sympy.init_printing()

A = sympy.Matrix([[1, 2, 1], [6, -1, 0], [-1, -2, -1]])

A.eigenvects()
```

$$\left[\left(-4,\ 1,\ \left[\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right]\right),\ \left(0,\ 1,\ \left[\begin{bmatrix} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{bmatrix}\right]\right),\ \left(3,\ 1,\ \left[\begin{bmatrix} -1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}\right]\right)\right]$$

Example Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{pmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{pmatrix}$$

The characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 9-\lambda & 1 & 1 \\ 1 & 9-\lambda & 1 \\ 1 & 1 & 9-\lambda \end{vmatrix} = -(\lambda - 11)(\lambda - 8)^2 = 0$$

shows that $\lambda_1 = 11$ and that $\lambda_2 = \lambda_3 = 8$ is an eigenvalue of multiplicity 2

For $\lambda_1 = 11$, we have

$$(\mathbf{A} - 11\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

Thus $k_1 = k_3$, $k_2 = k_3$. Choosing $k_3 = 1$ gives the eigenvector

$$\mathbf{k}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 8$, we have

$$(\mathbf{A} - 8\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Here $k_1 + k_2 + k_3 = 0$, we are free to select two of the variables arbitrarily

Choosing, on the one hand, $k_2 = 1$, $k_3 = 0$, and on the other, $k_2 = 0$, $k_3 = 1$, we obtain two linearly independent eigenvectors

$$\mathbf{k}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{k}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

corresponding to a single eigenvalue

If instead we choose $k_2 = 1$, $k_3 = 1$ and then $k_2 = 1$, $k_3 = -1$, we obtain, respectively, two entirely different but orthogonal eigenvectors

$$\mathbf{k}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{k}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

```
A = sympy.Matrix([[9, 1, 1], [1, 9, 1], [1, 1, 9]])
A.eigenvects()
```

$$\left[\left(8,\ 2,\ \left[\begin{bmatrix}-1\\1\\0\end{bmatrix},\ \begin{bmatrix}-1\\0\\1\end{bmatrix}\right]\right),\ \left(11,\ 1,\ \left[\begin{bmatrix}1\\1\\1\end{bmatrix}\right]\right)\right]$$

Example Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{pmatrix} 3 & 4 \\ -1 & 7 \end{pmatrix}$$

From the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 4 \\ -1 & 7-\lambda \end{vmatrix} = (\lambda - 5)^2 = 0$$

we see $\lambda_1 = \lambda_2 = 5$ is an eigenvalue of algebraic multiplicity 2

To find the eigenvector(s) corresponding to $\lambda_1 = 5$, we resort to the system $(\mathbf{A} - 5\mathbf{I}|\mathbf{0})$

$$(\mathbf{A} - 5\mathbf{I}|\mathbf{0}) = \left(\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Thus $k_1 = 2k_2$. If we choose $k_2 = 1$, we find the single eigenvector

$$\mathbf{k}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We define the geometric multiplicity of an eigenvalue to be the number of linearly independent eigenvectors for the eigenvalue

When the geometric multiplicity of an eigenvalue is less than the algebraic multiplicity, we say the matrix is *defective*. In the case of defective matrices, we must search for additional system

$$(\mathbf{A} - 5\mathbf{I}|\mathbf{k}_1) = \left(\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

Thus $k_1 - 2k_2 = -1$. If we choose $k_2 = 0$, we find the generalized eigenvector

$$\mathbf{k}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

```
A = sympy.Matrix([[3, 4], [-1, 7]])
A.eigenvects()
```

$$\left[\left(5,\ 2,\ \left[\begin{bmatrix}2\\1\end{bmatrix}\right]\right)\right]$$

```
lambda1 = A.eigenvects()[0][0]
k1 = A.eigenvects()[0][2][0]
A1 = A -lambda1 *sympy.eye(2)
A1.pinv_solve(k1)
```

$$\begin{bmatrix} \frac{4w_{00}}{5} + \frac{2w_{10}}{5} - \frac{1}{5} \\ \frac{2w_{00}}{5} + \frac{w_{10}}{5} + \frac{2}{5} \end{bmatrix}$$

- Let $\mathbf{A}$ be a square matrix with real entries. If $\lambda = \alpha + i\beta$, $\beta \neq 0$, is a complex eigenvalue of $\mathbf{A}$ with eigenvector $\mathbf{k}$, then $\bar{\lambda}$ is also an eigenvalue, with eigenvector $\bar{\mathbf{k}}$:

$$\mathbf{A}\bar{\mathbf{k}} = \bar{\lambda}\bar{\mathbf{k}}$$

- $\lambda = 0$ is an eigenvalue of $\mathbf{A}$ if and only if $\mathbf{A}$ is singular
- If λ is an eigenvalue of nonsingular $\mathbf{A}$ with eigenvector $\mathbf{k}$, $1/\lambda$ is an eigenvalue of $\mathbf{A}^{-1}$ with the same eigenvector $\mathbf{k}$
- The eigenvalues of an upper triangular, lower triangular, and diagonal matrix are the main diagonal entries

Example Find the eigenvalues and eigenvectors of the given matrix. State whether the matrix is singular or nonsingular

$$\begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}, \quad \begin{pmatrix} 4 & 8 \\ 0 & -5 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

Answer 31 (end of chapter) · Solution

Example Find the eigenvalues and eigenvectors of the given nonsingular matrix $\mathbf{A}$. Then without finding $\mathbf{A}^{-1}$, find its eigenvalues and corresponding eigenvectors

$$\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 5 \end{pmatrix}$$

Answer 32 (end of chapter) · Solution

Example True or False: If λ is an eigenvalue of an $n \times n$ matrix $\mathbf{A}$, then the matrix $\mathbf{A} - \lambda\mathbf{I}$ is singular. Justify your answer

Answer 33 (end of chapter) · Solution

Example Suppose λ is an eigenvalue with corresponding eigenvector $\mathbf{k}$ of an $n \times n$ matrix $\mathbf{A}$

1. If $\mathbf{A}^2 = \mathbf{AA}$, then show that $\mathbf{A}^2\mathbf{k} = \lambda^2\mathbf{k}$. Explain the meaning of the last equation

2. Verify the result obtained in part 1 for the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$$

3. Generalize the result in part 1

Answer 34 (end of chapter) · Solution

Example Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. The matrix $\mathbf{B}$ is said to be **similar** to the matrix $\mathbf{A}$ if there exists a nonsingular matrix $\mathbf{S}$ such that $\mathbf{B} = \mathbf{S}^{-1}\mathbf{AS}$. If $\mathbf{B}$ is similar to $\mathbf{A}$, then show that $\mathbf{A}$ is similar to $\mathbf{B}$

Answer 35 (end of chapter) · Solution

Example Suppose $\mathbf{A}$ and $\mathbf{B}$ are similar matrices. Show that $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues

Answer 36 (end of chapter) · Solution

Try it in Python: Eigenvalues and eigenvectors with `sympy` and `scipy.linalg`: Eigenvalue problems

6.10 Powers of Matrices

- **Cayley-Hamilton Theorem**

  If $(-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0$ is the characteristic equation of $n \times n$ matrix $\mathbf{A}$, then

  $$(-1)^n\mathbf{A}^n + c_{n-1}\mathbf{A}^{n-1} + \cdots + c_1\mathbf{A} + c_0\mathbf{I} = \mathbf{0}$$

  And we can write

  $$\mathbf{A}^m = a_{n-1}\mathbf{A}^{n-1} + a_{n-2}\mathbf{A}^{n-2} + \cdots + a_1\mathbf{A} + a_0\mathbf{I}$$

  and the equation for the eigenvalues

  $$\lambda^m = a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0$$

  hold for the same constants

Example Verify that the given matrix satisfies its own characteristic equation

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix}$$

Answer 37 (end of chapter) · Solution

Example Compute $\mathbf{A}^m$

$$\mathbf{A} = \begin{pmatrix} 8 & 5 \\ 4 & 0 \end{pmatrix};\ m = 5,\quad \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix};\ m = 10$$

Answer 38 (end of chapter) · Solution

Example Show that the given matrix has an eigenvalue λ_1 of multiplicity two. As a consequence, the equation $\lambda^m = c_0 + c_1\lambda$ does not yield enough independent equations to form a system for determining the coefficients c_i. Use the derivative (with respect to λ) of this equation evaluated at λ_1 as the extra needed equation to form a system. Compute $\mathbf{A}^m$ and use this result to compute the indicated power of the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 7 & 3 \\ -3 & 1 \end{pmatrix};\ m = 6$$

Answer 39 (end of chapter) · Solution

Example $\lambda = 0$ is an eigenvalue of each matrix. In this case, show that the coefficient c_0 in the characteristic equation

$$(-1)^n\mathbf{A}^n + c_{n-1}\mathbf{A}^{n-1} + \cdots + c_1\mathbf{A} + c_0\mathbf{I} = \mathbf{0}$$

is 0. Compute $\mathbf{A}^m$ in each case. In part (a), explain why we do not have to solve any system for the coefficients c_i in determining $\mathbf{A}^m$

$$(a)\ \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix},\quad (b)\ \mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 3 \end{pmatrix}$$

Answer 40 (end of chapter) · Solution

Example A non-zero $n \times n$ matrix $\mathbf{A}$ is said to be **nilpotent of index** m if m is the smallest positive integer for which $\mathbf{A}^m = \mathbf{0}$.

1. Explain why any nilpotent matrix $\mathbf{A}$ is singular

2. Show that all the eigenvalues of a nilpotent matrix $\mathbf{A}$ are 0

Answer 41 (end of chapter) · Solution

6.11 Orthogonal Matrices

- Let $\mathbf{A}$ be a *symmetric* matrix ($\mathbf{A} = \mathbf{A}^T$) with *real* entries. Then the eigenvalues of $\mathbf{A}$ are *real*
- Let $\mathbf{A}$ be a *symmetric* matrix. Then eigenvectors corresponding to distinct(different) eigenvalues are *orthogonal*
- An $n \times n$ matrix $\mathbf{A}$ is *orthogonal* ($\mathbf{A}^{-1} = \mathbf{A}^T$) if and only if its columns $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ form an orthonormal set

$$\mathbf{x}_i \cdot \mathbf{x}_j = 0, \ i \neq j \text{ and } \mathbf{x}_i \cdot \mathbf{x}_i = 1$$

- It may not be possible to find n linearly independent eigenvectors for an $n \times n$ matrix $\mathbf{A}$ when some of eigenvalues are repeated (defective matrix)
- But a symmetric matrix is an exception. It can be proved that a set of n linearly independent eigenvectors can always be found for an $n \times n$ symmetric matrix $\mathbf{A}$ even if there is some repetition of the eigenvalues
- However, this does not mean that all eigenvectors are mutually orthogonal for an $n \times n$ symmetric matrix $\mathbf{A}$. Eigenvectors corresponding to distinct eigenvalues are orthogonal; but different eigenvectors corresponding to a repeated eigenvalue may not be orthogonal
- But it is always possible to *find* or *construct* a set of n mutually orthogonal eigenvectors by using Gram-Schmidt orthogonalization. See Section 9.7

Example Construct an orthogonal matrix from the eigenvectors of

$$\mathbf{A} = \begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{pmatrix}$$

```
A = sympy.Matrix([[7, 4, -4], [4, -8, -1], [-4, -1, -8]])
A.eigenvects()
```

$$\left[\left(-9,\ 2,\ \left[\begin{bmatrix} -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix},\ \begin{bmatrix} \frac{1}{4} \\ 0 \\ 1 \end{bmatrix}\right]\right),\ \left(9,\ 1,\ \left[\begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}\right]\right)\right]$$

```
v1 = A.eigenvects()[0][2][0].T
v2 = A.eigenvects()[0][2][1].T
v3 = A.eigenvects()[1][2][0].T

B = sympy.Matrix(sympy.GramSchmidt([v1, v2, v3], orthonormal=True)).T; B
```

$$\begin{bmatrix} -\frac{\sqrt{17}}{17} & \frac{2\sqrt{34}}{51} & -\frac{2\sqrt{2}}{3} \\ \frac{4\sqrt{17}}{17} & \frac{\sqrt{34}}{102} & -\frac{\sqrt{2}}{6} \\ 0 & \frac{\sqrt{34}}{6} & \frac{\sqrt{2}}{6} \end{bmatrix}$$

```
B.T *B
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example Determine whether the given matrix is orthogonal

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ -\frac{12}{13} & \frac{5}{13} & 0 \\ \frac{5}{13} & \frac{12}{13} & 0 \end{pmatrix}$$

Answer 42 (end of chapter) · Solution

Example (a) Verify that the indicated column vectors are eigenvectors of the given symmetric matrix and (b) identify the corresponding eigenvalues. (c) Use Gram-Schmidt process to construct an orthogonal matrix $\mathbf{P}$ from the eigenvectors

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}; \quad \mathbf{k}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{k}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{k}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Answer 43 (end of chapter) · Solution

Example Answer the questions based on the supposition

- Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ orthogonal matrices. Then show that $\mathbf{AB}$ is orthogonal
- Suppose $\mathbf{A}$ is an orthogonal matrix. Is $\mathbf{A}^2$ orthogonal?
- Suppose $\mathbf{A}$ is an orthogonal matrix. Then show that $\mathbf{A}^{-1}$ is orthogonal
- Suppose $\mathbf{A}$ is an orthogonal matrix such that $\mathbf{A}^2 = \mathbf{I}$. Then show that $\mathbf{A}^T = \mathbf{A}$
- Show that the rotation matrix is orthogonal

Answer 44 (end of chapter) · Solution

6.12 Approximation of Eigenvalues

- Let $\lambda_1, \lambda_2, \cdots, \lambda_k, \cdots, \lambda_n$ denote the eigenvalues of an $n \times n$ matrix $\mathbf{A}$. The eigenvalue λ_k is said to be the **dominant eigenvalue** of $\mathbf{A}$ if $|\lambda_k| > |\lambda_i|, \;\; i = 1, 2, \cdots, n, \;\; \text{but} \;\; i \neq k$
- An eigenvector corresponding to λ_k is called the **dominant eigenvector** of $\mathbf{A}$
- **Power Method**

  Let us assume that the eigenvalues of $\mathbf{A}$ are such that $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$ and that the corresponding n eigenvectors $\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_n$ are linearly independent. Because of this last assumption, n eigenvectors can serve as a basis for $\mathbb{R}^n$. For any nonzero $n \times 1$ vector $\mathbf{x}_0$,

$$\mathbf{x}_0 = c_1\mathbf{k}_1 + c_2\mathbf{k}_2 + c_3\mathbf{k}_3 + \cdots + c_n\mathbf{k}_n$$

  We shall also assume $\mathbf{x}_0$ is chosen so that $c_1 \neq 0$. We do the following procedure

$$\begin{aligned}
\mathbf{A}\mathbf{x}_0 &= c_1\mathbf{A}\mathbf{k}_1 + c_2\mathbf{A}\mathbf{k}_2 + c_3\mathbf{A}\mathbf{k}_3 + \cdots + c_n\mathbf{A}\mathbf{k}_n \\
&\Downarrow \;\; \mathbf{x}_i = \mathbf{A}\mathbf{x}_{i-1}, \;\; \mathbf{A}\mathbf{k}_j = \lambda_j\mathbf{k}_j \\
\mathbf{x}_1 &= c_1\lambda_1\mathbf{k}_1 + c_2\lambda_2\mathbf{k}_2 + c_3\lambda_3\mathbf{k}_3 + \cdots + c_n\lambda_n\mathbf{k}_n \\
&\Downarrow \\
\mathbf{A}\mathbf{x}_1 &= c_1\lambda_1\mathbf{A}\mathbf{k}_1 + c_2\lambda_2\mathbf{A}\mathbf{k}_2 + c_3\lambda_3\mathbf{A}\mathbf{k}_3 + \cdots + c_n\lambda_n\mathbf{A}\mathbf{k}_n \\
&\Downarrow \\
\mathbf{x}_2 &= c_1\lambda_1^2\mathbf{k}_1 + c_2\lambda_2^2\mathbf{k}_2 + c_3\lambda_3^2\mathbf{k}_3 + \cdots + c_n\lambda_n^2\mathbf{k}_n \\
&\Downarrow \\
\mathbf{x}_m &= c_1\lambda_1^m\mathbf{k}_1 + c_2\lambda_2^m\mathbf{k}_2 + c_3\lambda_3^m\mathbf{k}_3 + \cdots + c_n\lambda_n^m\mathbf{k}_n \\
&= \lambda_1^m\left[c_1\mathbf{k}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^m\mathbf{k}_2 + c_3\left(\frac{\lambda_3}{\lambda_1}\right)^m\mathbf{k}_3 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^m\mathbf{k}_n\right] \\
&\Downarrow \;\; m \to \infty \\
\mathbf{x}_m &\simeq \lambda_1^m c_1\mathbf{k}_1
\end{aligned}$$

- For large values of m and under all the assumptions that were made, the $n \times 1$ vector $\mathbf{x}_m$ is an approximation to a dominant eigenvector associated with the dominant eigenvalue λ_1
- If $\mathbf{x}_m$ is an approximation to a dominant eigenvector, then the dominant eigenvalue λ_1 can be approximated by the **Rayleigh quotient**

$$\lambda_1 = \frac{\mathbf{A}\mathbf{x}_m \cdot \mathbf{x}_m}{\mathbf{x}_m \cdot \mathbf{x}_m}$$

- Iteration often results in vectors whose entries become very large. Large numbers can cause a problem in computation. One way around this difficulty is to use a **scaled-down** normalized vector

$$\mathbf{x}_m \leftarrow \frac{\mathbf{x}_m}{\|\mathbf{x}_m\|}$$

```python
import numpy as np
import pprint

def power_method(A, n_iteration):
    # Ideally choose a random vector
    # To decrease the chance that
    #   our vector is orthogonal to the eigenvector
    k = np.random.rand(A.shape[1])

    for _ in range(n_iteration):
        # calculate the matrix-by-vector product Ak
        k_1 = np.dot(A, k)

        # calculate the norm
        k_1_norm = np.linalg.norm(k_1)

        # re normalize the vector
        k = k_1 /k_1_norm

    return k

def Rayleigh_quotient(A, k):
    # calculate Rayleigh quotient
    lambda_ = np.dot(np.dot(A, k), k) /np.dot(k, k)

    return lambda_

A = np.array([[4, 2], [3, -1]])
k_1 = power_method(A, 10)
lambda_1 = Rayleigh_quotient(A, k_1)

print('A =')
pprint.pprint(A)
print('\nDominant Eigenvector:', np.around(k_1, 3))
print('Dominant Eigenvalue:', np.around(lambda_1, 3))
```

```
A =
array([[ 4,  2],
       [ 3, -1]])

Dominant Eigenvector: [0.894 0.447]
Dominant Eigenvalue: 5.0
```

- **Method of Deflation**
  - After we have found the dominant eigenvalue λ_1 of a matrix $\mathbf{A}$, it may still be necessary to find nondominant eigenvalues. We will limit the discussion to the case where $\mathbf{A}$ is a *symmetric* matrix
  - Suppose λ_1 and $\mathbf{k}_1$ are, respectively, the dominant eigenvalue and a corresponding *normalized* eigenvector of a symmetric matrix $\mathbf{A}$. Furthermore, suppose the eigenvalues of $\mathbf{A}$ are such that

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > \cdots > |\lambda_n|$$

In this case, the matrix

$$\mathbf{A}_1 = \mathbf{A} - \lambda_1 \mathbf{k}_1 \mathbf{k}_1^T$$

has eigenvalues 0, λ_2, λ_3, $\cdots$, λ_n and that eigenvectors of $\mathbf{A}_1$ are also eigenvectors of $\mathbf{A}$. Note that λ_2 is now the dominant eigenvalue of $\mathbf{A}_1$

```
def deflation_method(A, n_iteration):

    n = A.shape[1]

    K = np.zeros((n, n))
    Lambda = np.zeros(n)

    k0 = np.zeros(n)
    L0 = 0
    for i in range(n):
        A = A -L0 *np.outer(k0, k0)
        k0 = power_method(A, n_iteration)
        L0 = Rayleigh_quotient(A, k0)
        K[i, :] = k0
        Lambda[i] = L0

    return K, Lambda

# symmetric matrix
A = np.array([[1, 2, -1], [2, 1, 1], [-1, 1, 0]])

K, Lambda = deflation_method(A, 20)

print('A =')
pprint.pprint(A)

print('\nk_1 =', np.around(K[0, :], 3))
print('k_2 =', np.around(K[1, :], 3))
print('k_3 =', np.around(K[2, :], 3))

print('\nlambda_1 =', np.around(Lambda[0], 3))
print('lambda_2 =', np.around(Lambda[1], 3))
print('lambda_3 =', np.around(Lambda[2], 3))
```

```
A =
array([[ 1,  2, -1],
       [ 2,  1,  1],
       [-1,  1,  0]])

k_1 = [0.707 0.707 0.   ]
k_2 = [ 0.577 -0.577  0.577]
```

```
k_3 = [-0.408  0.408  0.817]

lambda_1 = 3.0
lambda_2 = -2.0
lambda_3 = 1.0
```

- **Inverse Power Method**

  If we want to find the smallest eigenvalue instead of the largest one, then we perform power iteration for $\mathbf{A}^{-1}$ (since the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{A}$). Of course, we do not want to compute $\mathbf{A}^{-1}$

```python
def inverse_power_method(A, n_iteration):
    # Ideally choose a random vector
    # To decrease the chance that
    #   our vector is orthogonal to the eigenvector
    k = np.random.rand(A.shape[1])

    for _ in range(n_iteration):
        # calculate the matrix-by-vector product Ak
        w = np.linalg.solve(A, k)

        # calculate the norm
        w_norm = np.linalg.norm(w)

        # re normalize the vector
        k = w /w_norm

    return k

A = np.array([[4, 2], [3, -1]])
k_2 = inverse_power_method(A, 10)
lambda_2 = Rayleigh_quotient(A, k_2)

print('A =')
pprint.pprint(A)
print('\nEigenvector =', np.around(k_2, 3))
print('Eigenvalue of Least Magnitude =', np.around(lambda_2, 3))
```

```
A =
array([[ 4,  2],
       [ 3, -1]])

Eigenvector = [-0.316  0.949]
Eigenvalue of Least Magnitude = -2.0
```

6.13 Diagonalization

- For an $n \times n$ matrix $\mathbf{A}$, can we find an $n \times n$ nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ is a diagonal matrix?
- An $n \times n$ matrix $\mathbf{A}$ is diagonalizable if and only if $\mathbf{A}$ has n linearly independent eigenvectors

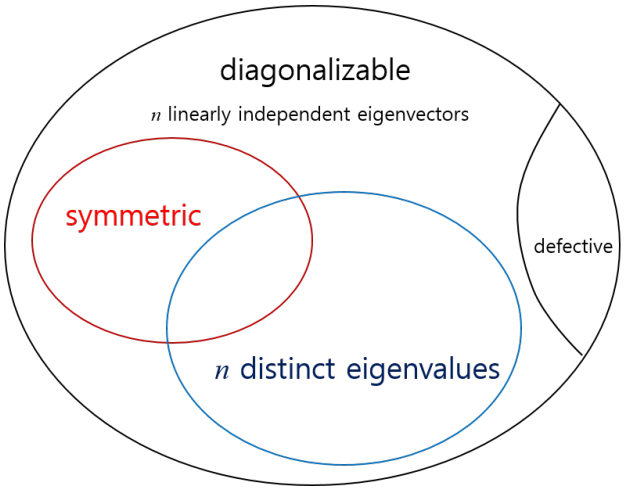
- Let $\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_n$ be linearly independent eigenvectors corresponding to eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Next form the matrix $\mathbf{P}$ with column vectors $\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_n$

$$\mathbf{P} = \begin{pmatrix} \mathbf{k}_1 & \mathbf{k}_2 & \cdots & \mathbf{k}_n \end{pmatrix}$$

We can write the product $\mathbf{AP}$ as

$$\begin{aligned} \mathbf{AP} &= \begin{pmatrix} \mathbf{Ak}_1 & \mathbf{Ak}_2 & \cdots & \mathbf{Ak}_n \end{pmatrix} = \begin{pmatrix} \lambda_1\mathbf{k}_1 & \lambda_2\mathbf{k}_2 & \cdots & \lambda_n\mathbf{k}_n \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{k}_1 & \mathbf{k}_2 & \cdots & \mathbf{k}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = \mathbf{PD} \end{aligned}$$

Multiplying by $\mathbf{P}^{-1}$ on the left then gives $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$



- If an $n \times n$ matrix $\mathbf{A}$ has n distinct eigenvalues, it is diagonalizable
- An $n \times n$ matrix $\mathbf{A}$ can be *orthogonally* diagonalized if and only if $\mathbf{A}$ is symmetric

Example Diagonalize

$$\mathbf{A} = \begin{pmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{pmatrix}$$

- The eigenvalues and corresponding orthogonal eigenvectors are $\lambda_1 = 11$, $\lambda_2 = \lambda_3 = 8$

$$\mathbf{k}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{k}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{k}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

- Multiplying these vectors, in turn, by the reciprocals of the norms $\|\mathbf{k}_1\| = \sqrt{3},\ \|\mathbf{k}_2\| = \sqrt{6}$ and $\|\mathbf{k}_3\| = \sqrt{2},$ we obtain an orthonormal set

$$\mathbf{k}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix},\ \mathbf{k}_2 = \begin{pmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \text{ and } \mathbf{k}_3 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

  We then use these vectors as columns to construct an orthogonal matrix that diagonalizes $\mathbf{A}$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

- This transforms $\mathbf{A}$ to $\mathbf{D}$

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} 11 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} = \mathbf{D}$$

  The entries in $\mathbf{D}$ are the eigenvalues of $\mathbf{A}$ and the order in which these numbers appear on the diagonal corresponds to the order in which the eigenvectors are used as columns in the matrix $\mathbf{P}$

Example - Quadratic Forms: Identify the conic section whose equation $2x^2 + 4xy - y^2 = 1$

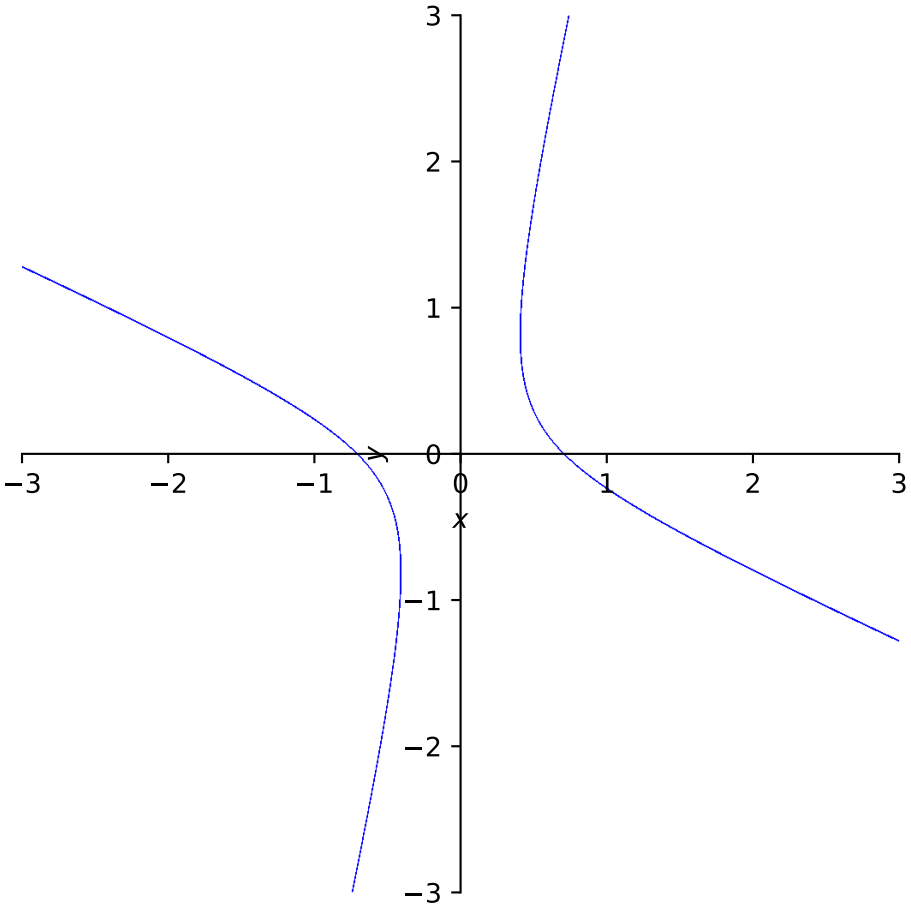
- We can write the given equation as

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \text{ or } \mathbf{x}^T\mathbf{A}\mathbf{x} = 1$$

- The eigenvalues and corresponding eigenvectors of $\mathbf{A}$ are found to be

$$\lambda_1 = -2,\ \lambda_2 = 3,\ \mathbf{k}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix},\ \mathbf{k}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

```
from sympy import symbols, Eq, plot_implicit

x, y = symbols('x y')

p1 = plot_implicit(Eq(2*x**2 +4*x*y -y**2, 1),
                   (x, -3, 3), (y, -3, 3),
                   aspect_ratio=(1, 1), size=(5, 5))
```



- Observe that $\mathbf{k}_1$ and $\mathbf{k}_2$ are orthogonal. Moreover, $\|\mathbf{k}_1\| = \|\mathbf{k}_2\| = \sqrt{5}$, and so the vectors

$$\mathbf{k}_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \text{ and } \mathbf{k}_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

  are orthonormal. Hence, the matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

  is orthogonal

- If we define the change of variables $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$ where $\bar{\mathbf{x}} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$, then the quadratic form can be written

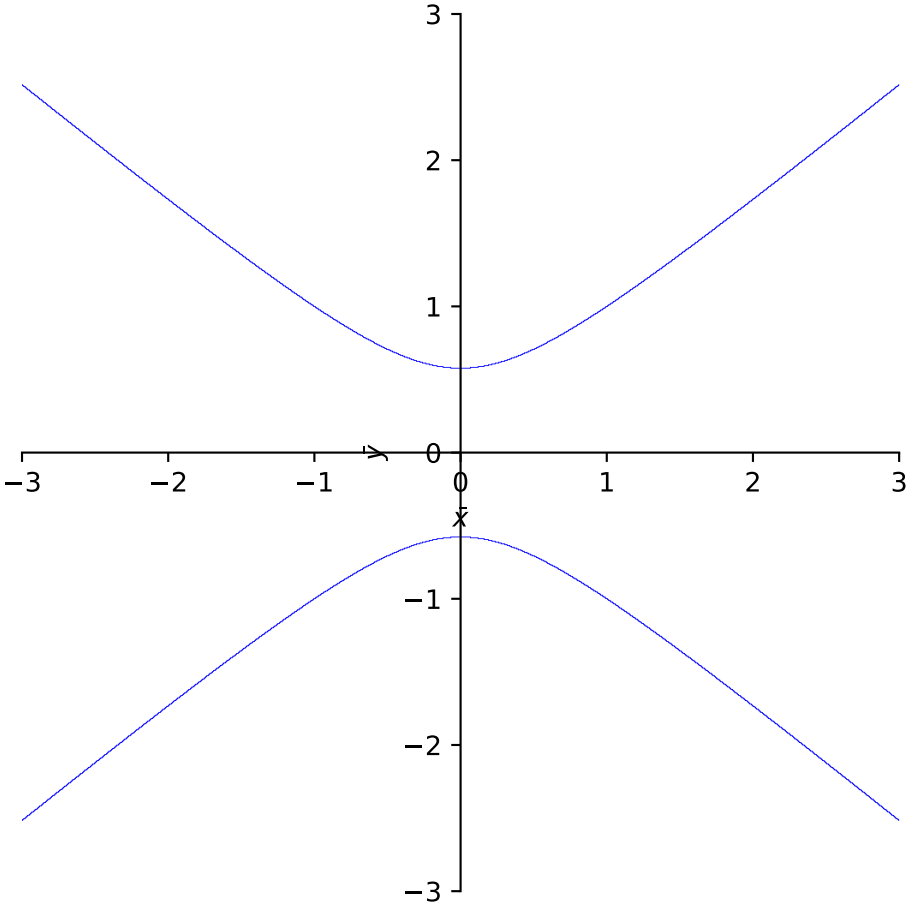
$$\begin{aligned} \mathbf{x}^T\mathbf{A}\mathbf{x} &= \bar{\mathbf{x}}^T\mathbf{P}^T\mathbf{A}\mathbf{P}\bar{\mathbf{x}} = \bar{\mathbf{x}}^T\mathbf{D}\bar{\mathbf{x}} \\ &= \begin{pmatrix} \bar{x} & \bar{y} \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = 1 \end{aligned}$$

  or

$$-2\bar{x}^2 + 3\bar{y}^2 = 1$$

  This last equation is recognized as the standard form of a hyperbola

```
bar_x, bar_y = symbols('bar_x bar_y')

p1 = plot_implicit(Eq(-2*bar_x**2 +3*bar_y**2, 1),
                   (bar_x, -3, 3), (bar_y, -3, 3),
                   aspect_ratio=(1, 1), size=(5, 5),
                   xlabel=r'$\bar{x}$', ylabel=r'$\bar{y}$')
```



Example Determine whether the given matrix $\mathbf{A}$ is diagonalizable. If so, find the matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$ and the diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$

$$\begin{pmatrix} -9 & 13 \\ -2 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Answer 45 (end of chapter) · Solution

Example The given matrix $\mathbf{A}$ is symmetric. Find an orthogonal matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$ and the diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^{T}\mathbf{AP}$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix}$$

Answer 46 (end of chapter) · Solution

Example Identify the given conic section

$$5x^2 - 2xy + 5y^2 = 24$$

$$-3x^2 + 8xy + 3y^2 = 20$$

Answer 47 (end of chapter) · Solution

Example Find 3×3 symmetric matrix that has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = 5$ and corresponding eigenvectors

$$\mathbf{k}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{k}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{k}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Answer 48 (end of chapter) · Solution

Example If $\mathbf{A}$ is an $n \times n$ diagonalizable matrix, then $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, where $\mathbf{D}$ is a diagonal matrix. Show that if m is a positive integer, then $\mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1}$

Answer 49 (end of chapter) · Solution

Example Find the indicated power of the given matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \quad \mathbf{A}^5$$

$$\mathbf{A} = \begin{pmatrix} 6 & -10 \\ 3 & -5 \end{pmatrix}, \quad \mathbf{A}^{10}$$

Answer 50 (end of chapter) · Solution

Example Suppose $\mathbf{A}$ is a nonsingular diagonalizable matrix. Then show that $\mathbf{A}^{-1}$ is diagonalizable

Answer 51 (end of chapter) · Solution

Example Suppose $\mathbf{A}$ is a diagonalizable matrix. Is the matrix $\mathbf{P}$ unique?

Answer 52 (end of chapter) · Solution

6.14 LU Factorization

- Let $\mathbf{A}$ be a square matrix. An *LU factorization* refers to the factorization of $\mathbf{A}$ into two factors – a lower triangular matrix $\mathbf{L}$ and an upper triangular matrix $\mathbf{U}$:

$$\mathbf{A} = \mathbf{LU}$$

- Without a proper ordering or permutations in the matrix, the factorization may fail to materialize. This is a procedural problem. It can be removed by simply reordering the rows of $\mathbf{A}$. It turns out that a proper permutation in rows (or columns) is sufficient for LU factorization. LU factorization with partial pivoting (LUP) refers often to LU factorization with row permutations only

$$\mathbf{PA} = \mathbf{LU}$$

  where $\mathbf{P}$ is a permutation matrix, which, when left-multiplied to $\mathbf{A}$, reorders the rows of $\mathbf{A}$. It turns out that all square matrices can be factorized in this form

- If $\mathbf{A}$ is invertible, then it admits an LU factorization if and only if all its leading principal minors are nonzero. If $\mathbf{A}$ is a singular matrix of rank k, then it admits an LU factorization if the first k leading principal minors are nonzero
- LU decomposition is basically a modified form of Gaussian elimination. We transform the matrix $\mathbf{A}$ into an upper triangular matrix $\mathbf{U}$ by eliminating the entries below the main diagonal. The **Doolittle algorithm** does the column-by-column elimination, starting from the left, by multiplying $\mathbf{A}$ to the left with atomic lower triangular matrices. It results in a *unit* lower triangular matrix and an upper triangular matrix

- **Doolittle Algorithm**
  - We define

$$\mathbf{A}^{(0)} = \mathbf{A}$$

  - We eliminate the matrix elements below the main diagonal in the k-th column of $\mathbf{A}^{(k-1)}$ by subtracting from the i-th row of this matrix the k-th row multiplied by

$$l_{i,k} = \frac{a_{i,k}^{(k-1)}}{a_{k,k}^{(k-1)}} \quad \text{for} \quad i = k+1, \cdots, n$$

    This can be done by multiplying $\mathbf{A}^{(k-1)}$ to the left with the lower triangular matrix

$$\mathbf{L}_k = \begin{pmatrix} 1 & 0 & & \cdots & 0 \\ 0 & \ddots & \ddots & & \\ & & 1 & & \\ \vdots & & -l_{k+1,k} & & \vdots \\ & & \vdots & \ddots & 0 \\ 0 & & -l_{n,k} & 0 & 1 \end{pmatrix}$$

– We set

$$\mathbf{A}^{(k)} = \mathbf{L}_k \mathbf{A}^{(k-1)},\ k = 1, \cdots, n-1$$

After $n-1$ steps, we eliminated all the matrix elements below the main diagonal, so we obtain an upper triangular matrix $\mathbf{A}^{(n-1)}$. We find the decomposition

$$\begin{aligned} \mathbf{A} &= \mathbf{L}_1^{-1}\mathbf{L}_1\mathbf{A}^{(0)} = \mathbf{L}_1^{-1}\mathbf{A}^{(1)} = \mathbf{L}_1^{-1}\mathbf{L}_2^{-1}\mathbf{L}_2\mathbf{A}^{(1)} = \mathbf{L}_1^{-1}\mathbf{L}_2^{-1}\mathbf{A}^{(2)} \\ &\vdots \\ &= \mathbf{L}_1^{-1} \cdots \mathbf{L}_{n-1}^{-1}\mathbf{A}^{(n-1)} \end{aligned}$$

Denote the upper triangular matrix $\mathbf{A}^{(n-1)}$ by $\mathbf{U}$, and $\mathbf{L} = \mathbf{L}_1^{-1} \cdots \mathbf{L}_{n-1}^{-1}$

– Because the inverse of a lower triangular matrix $\mathbf{L}_k$ is again a lower triangular matrix, and the multiplication of two lower triangular matrices is again a lower triangular matrix, it follows that $\mathbf{L}$ is a lower triangular matrix:

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & & \cdots & & 0 \\ l_{2,1} & \ddots & \ddots & & & \\ & & 1 & & & \\ \vdots & & l_{k+1,k} & & & \vdots \\ & & \vdots & & 1 & 0 \\ l_{n,1} & \cdots & l_{n,k} & \cdots & l_{n,n-1} & 1 \end{pmatrix}$$

We obtain $\mathbf{A} = \mathbf{LU}$

NOTE It is clear that in order for this algorithm to work, one needs to have $a_{k,k}^{(k-1)} \neq 0$ at each step (see the definition of $l_{i,k}$). If this assumption fails at some point, one needs to interchange k-th row with another row below it before continuing. This is why an LU decomposition in general looks like $\mathbf{PA} = \mathbf{LU}$

```
def lu_factor(A):
    """
        LU factorization with partial pivoting

        PA = LU
        P(permutation),
        L(unit Lower triangular) and
        U(upper triangular)

        Return P, L, U
    """
```

```
    n = A.shape[0]
    U = A.copy()
    P = np.identity(n)
    L = np.identity(n)

    for k in range(n -1):

        # Partial Pivoting
        max_row_index = np.argmax(abs(U[k:n,k])) +k
        P[[k,max_row_index]] = P[[max_row_index,k]]
        U[[k,max_row_index]] = U[[max_row_index,k]]

        # LU
        L[k+1:,k] = U[k+1:,k] /U[k,k]
        U[k+1:,k] = 0.0
        U[k+1:,k+1:] -= np.tensordot(L[k+1:,k],
                            U[k,k+1:], axes=0)

    return P, L, U

A = np.array([[7, 3, -1, 2], [3, 8, 1, -4], [-1, 1, 4, -1],
              [2, -4, -1, 6]], dtype='float64')
P, L, U = lu_factor(A)

print('A ='); pprint.pprint(A)
print('\nP ='); pprint.pprint(P)
print('\nL ='); pprint.pprint(L)
print('\nU ='); pprint.pprint(U)
```

```
A =
array([[ 7.,  3., -1.,  2.],
       [ 3.,  8.,  1., -4.],
       [-1.,  1.,  4., -1.],
       [ 2., -4., -1.,  6.]])

P =
array([[1., 0., 0., 0.],
       [0., 1., 0., 0.],
       [0., 0., 1., 0.],
       [0., 0., 0., 1.]])

L =
array([[ 1.        ,  0.        ,  0.        ,  0.        ],
       [ 0.42857143,  1.        ,  0.        ,  0.        ],
       [-0.14285714,  0.21276596,  1.        ,  0.        ],
       [ 0.28571429, -0.72340426,  0.08982036,  1.        ]])

U =
array([[ 7.        ,  3.        , -1.        ,  2.        ],
       [ 0.        ,  6.71428571,  1.42857143, -4.85714286],
       [ 0.        ,  0.        ,  3.55319149,  0.31914894],
       [ 0.        ,  0.        ,  0.        ,  1.88622754]])
```

```
from scipy.linalg import lu

A = np.array([[7, 3, -1, 2], [3, 8, 1, -4], [-1, 1, 4, -1],
              [2, -4, -1, 6]], dtype='float64')
P, L, U = lu(A)

print('P =')
pprint.pprint(P)
print('\nL =')
pprint.pprint(L)
print('\nU =')
pprint.pprint(U)
```

```
P =
array([[1., 0., 0., 0.],
       [0., 1., 0., 0.],
       [0., 0., 1., 0.],
       [0., 0., 0., 1.]])

L =
array([[ 1.        ,  0.        ,  0.        ,  0.        ],
       [ 0.42857143,  1.        ,  0.        ,  0.        ],
       [-0.14285714,  0.21276596,  1.        ,  0.        ],
       [ 0.28571429, -0.72340426,  0.08982036,  1.        ]])

U =
array([[ 7.        ,  3.        , -1.        ,  2.        ],
       [ 0.        ,  6.71428571,  1.42857143, -4.85714286],
       [ 0.        ,  0.        ,  3.55319149,  0.31914894],
       [ 0.        ,  0.        ,  0.        ,  1.88622754]])
```

```
A_ = sympy.Matrix([[7, 3, -1, 2], [3, 8, 1, -4], [-1, 1, 4, -1],
              [2, -4, -1, 6]])
L_, U_, _ = A_.LUdecomposition()
L_, U_
```

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{7} & 1 & 0 & 0 \\ -\frac{1}{7} & \frac{10}{47} & 1 & 0 \\ \frac{2}{7} & -\frac{34}{47} & \frac{15}{167} & 1 \end{bmatrix}, \begin{bmatrix} 7 & 3 & -1 & 2 \\ 0 & \frac{47}{7} & \frac{10}{7} & -\frac{34}{7} \\ 0 & 0 & \frac{167}{47} & \frac{15}{47} \\ 0 & 0 & 0 & \frac{315}{167} \end{bmatrix} \right)$$

- **Solving Linear Equations**

  Given a system of linear equations in matrix form

  $$\mathbf{Ax} = \mathbf{b}$$

  Suppose we have already obtained the LUP decomposition of $\mathbf{A}$ such that $\mathbf{PA} = \mathbf{LU}$, so

$$\mathbf{LUx} = \mathbf{Pb}$$

In this case the solution is done in two logical steps:

- First, we solve the equation $\mathbf{Ly} = \mathbf{Pb}$ for $\mathbf{y}$
- Second, we solve the equation $\mathbf{Ux} = \mathbf{y}$ for $\mathbf{x}$
- Note that in both cases we are dealing with triangular matrices $\mathbf{L}$ and $\mathbf{U}$, which can be solved directly by forward and backward substitution without using the Gaussian elimination process (however we do need this process or equivalent to compute the LU decomposition itself)
- The cost of solving a system of linear equations is approximately $\frac{2}{3}n^3$ floating-point operations
- The above procedure can be repeatedly applied to solve the equation multiple times for different $\mathbf{b}$. In this case it is faster (and more convenient) to do an LU decomposition of the matrix $\mathbf{A}$ once and then solve the triangular matrices for the different $\mathbf{b}$, rather than using Gaussian elimination each time
- The matrices $\mathbf{L}$ and $\mathbf{U}$ could be thought to have *encoded* the Gaussian elimination process

```
def ufsub(L, b):
    """ Unit row oriented forward substitution """
    y = b.copy()
    for i in range(1, L.shape[0]):
        y[i] -= np.dot(L[i,:i], y[:i])
    return y

def bsub(U, y):
    """ Row oriented backward substitution """
    x = y.copy()
    x[-1] /= U[-1,-1]
    for i in range(U.shape[0] -2, -1, -1):
        x[i] -= np.dot(U[i,i+1:], x[i+1:])
        x[i] /= U[i,i]
    return x

A = np.array([[7, 3, -1, 2], [3, 8, 1, -4],
              [-1, 1, 4, -1], [2, -4, -1, 6]], dtype='float64')
b = np.array([1, 2, 3, 4], dtype='float64')

P, L, U = lu_factor(A)
Pb = np.matmul(P, b)

y = ufsub(L, Pb)
x = bsub(U, y)
```

```
print('A ='); pprint.pprint(A)
print('\nb =', end=' '); pprint.pprint(b)
print('\nx =', end=' '); pprint.pprint(x)
```

```
A =
array([[ 7.,  3., -1.,  2.],
       [ 3.,  8.,  1., -4.],
       [-1.,  1.,  4., -1.],
       [ 2., -4., -1.,  6.]])

b = array([1., 2., 3., 4.])

x = array([-1.27619048,  1.87619048,  0.57142857,  2.43809524])
```

```
A_ = sympy.Matrix([[7, 3, -1, 2], [3, 8, 1, -4],
                   [-1, 1, 4, -1], [2, -4, -1, 6]])
b_ = sympy.Matrix([1, 2, 3, 4])

A_.LUsolve(b_)
```

$$\begin{bmatrix} -\frac{134}{105} \\ \frac{197}{105} \\ \frac{4}{7} \\ \frac{256}{105} \end{bmatrix}$$

- **Inverting a Matrix**

  In matrix inversion, instead of vector $\mathbf{b}$, we have matrix $\mathbf{I}_n$ so that we are trying to find a matrix $\mathbf{X}$

$$\mathbf{LUX} = \mathbf{I}_n$$

  We can use the same algorithm presented earlier to solve for each column of matrix $\mathbf{X}$

```
A_.inv(method='LU')
```

$$\begin{bmatrix} \frac{122}{315} & -\frac{101}{315} & \frac{2}{21} & -\frac{103}{315} \\ -\frac{101}{315} & \frac{143}{315} & -\frac{2}{21} & \frac{124}{315} \\ \frac{2}{21} & -\frac{2}{21} & \frac{2}{7} & -\frac{1}{21} \\ -\frac{103}{315} & \frac{124}{315} & -\frac{1}{21} & \frac{167}{315} \end{bmatrix}$$

- **Computing the Determinant**

  Given the LUP decomposition $\mathbf{A} = \mathbf{P}^{-1}\mathbf{LU}$ of a square matrix $\mathbf{A}$, the determinant of $\mathbf{A}$ can be computed straightforwardly as

$$\det \mathbf{A} = \det \mathbf{P}^{-1} \det \mathbf{L} \det \mathbf{U} = (-1)^s \prod_{i=1}^{n} l_{ii} \prod_{i=1}^{n} u_{ii}$$

where s is the number of row exchanges in the permutation matrix

Example Find the LU factorization of the given matrix

$$\begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} -1 & 4 \\ 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 4 & -2 & 1 \\ -4 & 1 & 2 \\ 12 & 1 & 3 \end{pmatrix}$$

Answer 53 (end of chapter) · Solution

Example Use the LU factorization to solve the given linear system of equations

$$\begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 & 1 \\ -4 & 1 & 2 \\ 12 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 28 \end{pmatrix}$$

Answer 54 (end of chapter) · Solution

Try it in Python: LU decomposition with `sympy` and `scipy.linalg.lu`: System of Linear equations

6.15 Applications

6.15.1 Cryptography

Cryptography is the study of making *secret writings* or *codes*. We will consider a system of encoding and decoding messages that requires both the sender and the receiver to know:

- A specified rule of correspondence between a set of symbols and a set of integers; and
- A specified *nonsingular* matrix **A**

Example A correspondence between the twenty-seven integers and the letters of the alphabet and a blank space is given by

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	2
space	j	k	l	n	m	s	t	u	w	x	g	h	i	o	p	q	r	v	y	z	a	b	c	d	

The numerical equivalent of the message **DR JOHN IS A DOUBLE SPY** is

24	17	0	1	14	12	4	0	13	6	0	21	0	24	14	
8	22	3	25	0	6	15	19								

- The sender will **encode** the message by means of the nonsingular matrix $\mathbf{A}$ and the receiver will **decode** the encoded message by means of the matrix $\mathbf{A}^{-1}$. We choose to write the numerical message as the 3×8 matrix

$$\mathbf{M} = \begin{pmatrix} 24 & 17 & 0 & 1 & 14 & 12 & 4 & 0 \\ 13 & 6 & 0 & 21 & 0 & 24 & 14 & 8 \\ 22 & 3 & 25 & 0 & 6 & 15 & 19 & 0 \end{pmatrix}$$

  Note that the last entry m_{38} has been simply padded with a space (the number 0)

- A 3×8 matrix allows us to encode the message by means of a 3×3 matrix. The encoding matrix $\mathbf{A}$ is constructed, so that
  - $\mathbf{A}$ is nonsingular
  - $\mathbf{A}$ has only integer entries, and
  - $\mathbf{A}^{-1}$ has only integer entries

- To accomplish the last criterion, we need only select the integer entries of $\mathbf{A}$ in such a manner that $\det \mathbf{A} = \pm 1$. We choose

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & -1 \\ 2 & 3 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

  and

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 4 & -3 \\ 2 & 3 & -2 \\ -2 & -4 & 3 \end{pmatrix}$$

  You should verify that $\det \mathbf{A} = -1$

- The original message is encoded as following:

$$\mathbf{B} = \mathbf{AM} = \begin{pmatrix} -46 & -20 & -25 & -1 & -20 & -27 & -23 & 0 \\ 175 & 64 & 100 & 65 & 52 & 156 & 126 & 24 \\ 210 & 73 & 125 & 86 & 58 & 195 & 159 & 32 \end{pmatrix}$$

  To send the encoded message as letters of the alphabet rather than as numbers, we rewrite $\mathbf{B}$ as $\mathbf{B}'$ using integers modulo 27:

$$\mathbf{B}' = \begin{pmatrix} 8 & 7 & 2 & 26 & 7 & 0 & 4 & 0 \\ 13 & 10 & 19 & 11 & 25 & 21 & 18 & 24 \\ 21 & 19 & 17 & 5 & 4 & 6 & 24 & 5 \end{pmatrix}$$

  The encoded message to be sent in letters is

  UTKFT N IXYGEAVDAYRMNSDM

```
import numpy as np

#-- Data for Encoding and Decoding-----------------------------------

cp = { 0:' ',  1:'j',  2:'k',  3:'l',  4:'n',  5:'m',  6:'s',  7:'t',
       8:'u',  9:'w', 10:'x', 11:'g', 12:'h', 13:'i', 14:'o', 15:'p',
      16:'q', 17:'r', 18:'v', 19:'y', 20:'z', 21:'a', 22:'b', 23:'c',
      24:'d', 25:'e', 26:'f' }

A = np.array([[-1, 0, -1], [2, 3, 4], [2, 4, 5]])

inv_cp =  {v: k for k, v in cp.items()}
inv_A = np.rint(np.linalg.inv(A)).astype('int32')

print('inv_A = ')
print(inv_A)
print('det_A =', np.linalg.det(A))
```

```
inv_A =
[[ 1  4 -3]
 [ 2  3 -2]
 [-2 -4  3]]
det_A = -1.0
```

```
def convert_to_numeric(message):
    return np.array([inv_cp[m] for m in message])

def convert_to_letter(numeric_message):
    return ''.join([cp[m] for m in numeric_message])

def encoding_message(message):

    numeric_message = convert_to_numeric(message)

    n_app = (3 -len(numeric_message) % 3) % 3
    M = np.append(numeric_message, [0]*n_app).reshape(3, -1)
    B = (A @ M) % 27

    numeric_message_encoded = B.flatten()

    return convert_to_letter(numeric_message_encoded)
```

```
message = 'dr john is a double spy'
message_encoded = encoding_message(message)

print('Message =', message.upper())
print('Encoded message =', message_encoded.upper())
```

```
Message = DR JOHN IS A DOUBLE SPY
Encoded message = UTKFT N IXYGEAVDAYRMNSDM
```

- You should try to imagine the difficulty of decoding the encoded message without prior knowledge. Using the original correspondence and $\mathbf{A}$, the decoding is the straightforward computation

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{B}'$$

```
def decoding_message(message_encoded):

    B_ = convert_to_numeric(message_encoded).reshape(3, -1)
    M_ = (inv_A @ B_) % 27
    numeric_message_decoded = M_.flatten()

    return convert_to_letter(numeric_message_decoded)


message_decoded = decoding_message(message_encoded)

print('Decoding message =', message_decoded.upper())
```

```
Decoding message = DR JOHN IS A DOUBLE SPY
```

6.15.2 An Error-Correcting Code

We are going to examine briefly the concept of digital communication between a satellite and a computer. As a result, we will deal only with matrices whose entries are binary digits, namely 0s and 1s

- When adding and multiplying such matrices, we will use arithmetic modulo 2. This arithmetic is defined by the addition and multiplication tables

+	0	1
0	0	1
1	1	0

x	0	1
0	0	0
1	0	1

- In digital communication the messages or words are binary n-tuples. An n-bit word is also said to be a binary string of length n. By **encoding** a message, we mean a process whereby we transform a word $\mathbf{W}$ of length n into another word $\mathbf{C}$ of length $n + m$ by augmenting $\mathbf{W}$ with m additional bits, called **parity check bits**. An encoding/decoding scheme is called a **code**

- The **Hamming (7, 4) code** is an encoding/decoding scheme that can detect the presence of a single error in a received message and can tell which bit must be corrected. In $(7, 4)$ code the encoding process consists of transforming a 4-bit word

$$\mathbf{W} = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix}$$

into a 7-bit code word

$$\mathbf{C} = \begin{bmatrix} c_1 & c_2 & w_1 & c_3 & w_2 & w_3 & w_4 \end{bmatrix}$$

where c_1, c_2, and c_3 denote the parity check bits and are defined in terms of the information bits w_1, w_2, w_3, and w_4

$$\begin{aligned} c_1 &= w_1 + w_2 + w_4 \\ c_2 &= w_1 + w_3 + w_4 \\ c_3 &= w_2 + w_3 + w_4 \end{aligned} \tag{C1}$$

We first observe that in modulo 2 arithmetic there are no negative numbers; the additive inverse of 1 is 1, not -1. With this in mind, we can write the system (C1) in the equivalent form

$$\begin{aligned} c_3 + w_2 + w_3 + w_4 &= 0 \\ c_2 + w_1 + w_3 + w_4 &= 0 \\ c_1 + w_1 + w_2 + w_4 &= 0 \end{aligned} \tag{C2}$$

These are called **parity check equations**

- As a matrix product, (C2) can be written

$$\mathbf{HC}^T = \mathbf{0}$$

where

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

is called the **parity check matrix**. A closer inspection of $\mathbf{H}$ shows a surprising fact: The columns of $\mathbf{H}$, left to right, are the numbers 1 through 7 written in binary

- Let $\mathbf{R}$ be a 1×7 matrix representing the received message. The product

$$\mathbf{S} = \mathbf{HR}^T$$

is called the **syndrome** of $\mathbf{R}$. If $\mathbf{S} = \mathbf{0}$, it is assumed that the transmission is correct and that $\mathbf{R}$ is the same as the original encoded message $\mathbf{C}$. The decoding of the message is accomplished by simply dropping the three check bits in $\mathbf{R}$

- Let

$$\mathbf{E} = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{bmatrix}$$

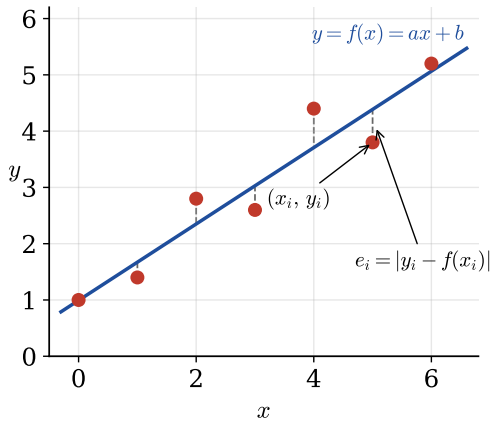
be a *single-error noise* word added to $\mathbf{C}$ during its transmission

- If noise changes the i-th bit, $e_i = 1$. The received message is then $\mathbf{R} = \mathbf{C} + \mathbf{E}$. We see that

$$\begin{aligned} \mathbf{S} &= \mathbf{H}\mathbf{R}^T = \mathbf{H}(\mathbf{C}^T + \mathbf{E}^T) = \mathbf{H}\mathbf{E}^T \\ &= e_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + e_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + e_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + e_4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e_5 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + e_6 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + e_7 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

Hence, if $\mathbf{S} \neq \mathbf{0}$, then $\mathbf{S}$ must be one of the columns of $\mathbf{H}$. If $\mathbf{R}$ contains a single error, we see that the syndrome itself indicates which bit is in error

6.15.3 Method of Least Squares



When performing experiments we often tabulate data in the form of ordered pairs (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n), with each x_i distinct. Given the data, it is then often desirable to predict y from x by finding a mathematical model, that is, a function $f(x)$ that approximates or fits the data

- We shall confine our attention to the problem of finding a linear polynomial $f(x) = ax + b$ that best fits the data (x_i, y_i), $i = 1, \cdots, n$. The procedure for finding this linear function is known as **the method of least squares**
- One way to determine how well the linear function $f(x) = ax + b$ fits the data is to measure the vertical distances between the data points y_i and the graphs $f(x_i)$

$$e_i = |y_i - f(x_i)|, \quad i = 1, \cdots, n$$

- An actual approach is to find a linear function f so that the sum of the *squares* of all the e_i values is a minimum

$$\min_{a,\,b} E = \min_{a,\,b} \sum_{i=1}^{n} [y_i - ax_i - b]^2$$

Then to find the minimum value of E, we set the first partial derivatives with respect to a and b to zero:

$$\frac{\partial E}{\partial a} = 0 \quad \text{and} \quad \frac{\partial E}{\partial b} = 0$$

- The last two conditions yield, in turn,

$$-2\sum_{i=1}^{n} x_i[y_i - ax_i - b] = 0$$
$$-2\sum_{i=1}^{n} [y_i - ax_i - b] = 0$$

Expanding the sums and rearranging, we find the above system is the same as

$$\begin{pmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{pmatrix}$$

and, in terms of matrices, is equivalent to

$$\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$$

where

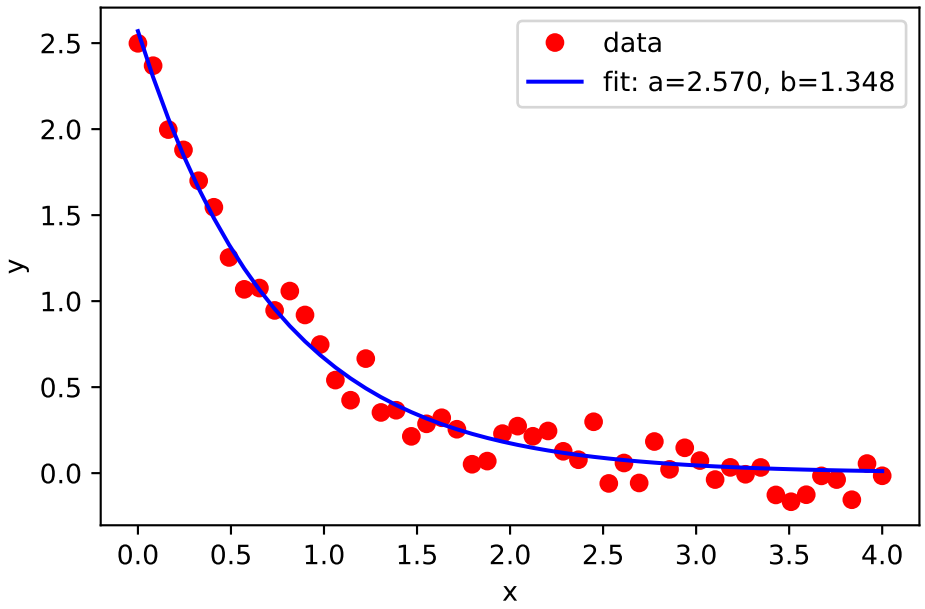
$$\mathbf{A} = \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{C3}$$

Unless the data points all lie on the same vertical line, the matrix $\mathbf{A}^T\mathbf{A}$ is nonsingular. Thus (C3) has the unique solution

$$\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\,\mathbf{A}^T\mathbf{b}$$

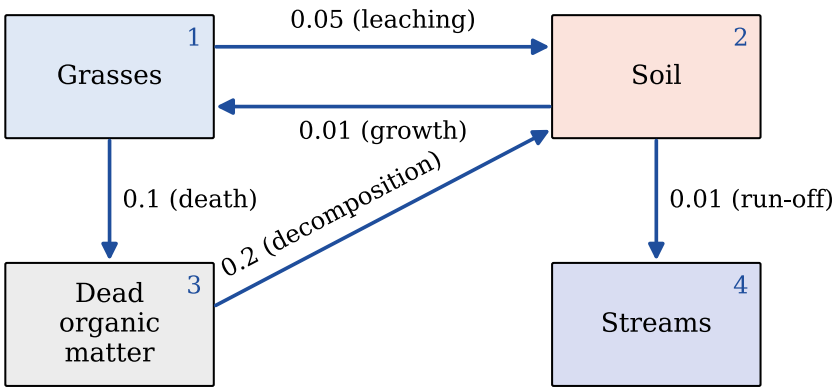
Example Nonlinear Least Squares: $f(x) = 2.5e^{-1.3x}$

```
from scipy.optimize import curve_fit
import matplotlib.pyplot as plt

np.random.seed(1970)

def func(x, a, b):
    return a *np.exp(-b*x)

# Define the data to be fit with some noise
xdata = np.linspace(0, 4, 50)
ydata = func(xdata, 2.5, 1.3) +0.1 *np.random.normal(size=xdata.size)
plt.plot(xdata, ydata, 'ro', label='data')

# Fit for the parameters a and b of the function func:
popt, pcov = curve_fit(func, xdata, ydata)
plt.plot(xdata, func(xdata, *popt),
         'b-', label='fit: a=%5.3f, b=%5.3f' % tuple(popt))
plt.xlabel('x'), plt.ylabel('y'), plt.legend()
plt.show()
```



6.15.4 Discrete Compartmental Models

- Strontium 90 is deposited into pastureland by rainfall. To study how this material is cycled through the ecosystem, we divide the system into the compartments:



Suppose that $\Delta t = 1$ month and the transfer coefficients shown in the figure are measured in fraction/month

- Suppose that rainfall has deposited the strontium 90 into the compartments so that

$$\mathbf{x}_0 = \begin{pmatrix} 20 & 60 & 15 & 20 \end{pmatrix}^T$$

Units might be grams per hectare. Compute the states of the ecosystem over the next 12 months

- From the data in figure, we see that transfer matrix $\mathbf{T}$ is

$$\mathbf{T} = \begin{pmatrix} 0.85 & 0.01 & 0 & 0 \\ 0.05 & 0.98 & 0.2 & 0 \\ 0.1 & 0 & 0.8 & 0 \\ 0 & 0.01 & 0 & 1 \end{pmatrix}$$

We must compute $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{12}$ by using the recursion formula $\mathbf{x}_{n+1} = \mathbf{T}\mathbf{x}_n$

```
T = np.array([[0.85, 0.01, 0.00, 0.00],
              [0.05, 0.98, 0.20, 0.00],
              [0.10, 0.00, 0.80, 0.00],
              [0.00, 0.01, 0.00, 1.00]])
x = np.array([20, 60, 15, 20]).T

print('-' *47)
print('Month     Grasses   Soil      Dead_OM   Streams')
print('-' *47)
for month in range(13):
    print(f'{month:5d} {x[0]:9.2f} {x[1]:9.2f} {x[2]:9.2f} {x[3]:9.2f}')
    x = T @ x
```

```
-----------------------------------------------
Month     Grasses   Soil      Dead_OM   Streams
-----------------------------------------------
    0     20.00     60.00     15.00     20.00
    1     17.60     62.80     14.00     20.60
```

2	15.59	65.22	12.96	21.23
3	13.90	67.29	11.93	21.88
4	12.49	69.03	10.93	22.55
5	11.31	70.46	9.99	23.24
6	10.32	71.61	9.13	23.95
7	9.48	72.52	8.33	24.66
8	8.79	73.21	7.61	25.39
9	8.20	73.71	6.97	26.12
10	7.71	74.04	6.40	26.86
11	7.29	74.22	5.89	27.60
12	6.94	74.28	5.44	28.34

Summary

Topic	Key result
Elimination	$(\mathbf{A}\|\mathbf{b}) \to$ row echelon form by elementary row operations, then back substitution
Rank	$\mathbf{Ax} = \mathbf{b}$ is consistent iff $\operatorname{rank}\mathbf{A} = \operatorname{rank}(\mathbf{A}\|\mathbf{b})$; the solution then has $n - \operatorname{rank}\mathbf{A}$ parameters
Determinant	$\det\mathbf{A} = \sum_j a_{ij}C_{ij}$, $C_{ij} = (-1)^{i+j}M_{ij}$; $\det(\mathbf{AB}) = \det\mathbf{A}\det\mathbf{B}$, $\det\mathbf{A}^T = \det\mathbf{A}$
Inverse	exists iff $\det\mathbf{A} \neq 0$: $\mathbf{A}^{-1} = \dfrac{\operatorname{adj}\mathbf{A}}{\det\mathbf{A}}$, or $(\mathbf{A}\|\mathbf{I}) \to (\mathbf{I}\|\mathbf{A}^{-1})$
Cramer's rule	$x_k = \dfrac{\det\mathbf{A}_k}{\det\mathbf{A}}$ ($\mathbf{A}_k$: column k replaced by $\mathbf{b}$)
Eigenvalues	$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, $\quad(\mathbf{A} - \lambda\mathbf{I})\mathbf{k} = \mathbf{0}$
Cayley-Hamilton	$\mathbf{A}$ satisfies its characteristic equation; $\mathbf{A}^m = c_0\mathbf{I} + c_1\mathbf{A} + \cdots + c_{n-1}\mathbf{A}^{n-1}$ with $\lambda^m = c_0 + c_1\lambda + \cdots$
Orthogonal and symmetric	orthogonal: $\mathbf{A}^{-1} = \mathbf{A}^T$; symmetric: real eigenvalues, orthogonal eigenvectors
Power method	$\mathbf{x}_i = \mathbf{Ax}_{i-1}$, $\quad\lambda_1 \approx \dfrac{\mathbf{Ax}_m \cdot \mathbf{x}_m}{\mathbf{x}_m \cdot \mathbf{x}_m}$ (Rayleigh quotient)
Diagonalization	$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ (n independent eigenvectors); symmetric: $\mathbf{D} = \mathbf{P}^T\mathbf{AP}$; $\quad\mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1}$
LU factorization	$\mathbf{A} = \mathbf{LU}$: solve $\mathbf{Ly} = \mathbf{b}$, then $\mathbf{Ux} = \mathbf{y}$

Worked Exercises

1. Use the inverse of the matrix $\mathbf{A}$ to solve the system $\mathbf{Ax} = \mathbf{b}_i$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

and the vectors $\mathbf{b}_i$, $i = 1, 2$ are given by (a) $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, (b) $\mathbf{b}_2 = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

Solution

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \cdot \text{adj}(\mathbf{A}) = \frac{1}{4} \begin{pmatrix} 6 & -1 & -9 \\ -4 & 2 & 6 \\ 2 & -1 & -1 \end{pmatrix}$$

For $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\mathbf{x}_1 = \mathbf{A}^{-1}\mathbf{b}_1 = \frac{1}{4} \begin{pmatrix} 6 & -1 & -9 \\ -4 & 2 & 6 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

For $\mathbf{b}_2 = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

$$\mathbf{x}_2 = \frac{1}{4} \begin{pmatrix} 6 & -1 & -9 \\ -4 & 2 & 6 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -10 \\ 7 \\ -2 \end{pmatrix}$$

2. Use the given LU-factorization

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

to solve the linear system $\mathbf{Ax} = \mathbf{b_i}$ for the given column matrix $\mathbf{b}_i$, $i = 1, 2, 3, 4$

(a) $\mathbf{b}_1 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$, (b) $\mathbf{b}_2 = \begin{pmatrix} -4 \\ 7 \\ 10 \end{pmatrix}$, (c) $\mathbf{b}_3 = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \\ -\frac{1}{2} \end{pmatrix}$, (d) $\mathbf{b}_4 = \begin{pmatrix} 30 \\ -42 \\ -18 \end{pmatrix}$

Solution

To solve $\mathbf{Ax} = \mathbf{b}$, we use the factorization:

$$\mathbf{Ax} = \mathbf{LUx} = \mathbf{b}$$

- Step 1: Solve $\mathbf{Ly} = \mathbf{b}$ (forward substitution)
- Step 2: Solve $\mathbf{Ux} = \mathbf{y}$ (backward substitution)

We will apply this for each given $\mathbf{b}_i$

(a) $\mathbf{b}_1 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$

Step 1: Solve $\mathbf{Ly} = \mathbf{b}_1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

- $y_1 = 2$
- $y_1 + y_2 = 4 \Rightarrow y_2 = 2$
- $y_1 + y_2 + y_3 = 1 \Rightarrow y_3 = -3$

So:

$$\mathbf{y} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

Step 2: Solve $\mathbf{Ux} = \mathbf{y}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

Back substitution:

- $x_3 = -3$
- $x_2 + x_3 = 2 \Rightarrow x_2 = 5$
- $x_1 + x_2 + x_3 = 2 \Rightarrow x_1 = 0$

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 5 \\ -3 \end{pmatrix}$$

(b) $\mathbf{b}_2 = \begin{pmatrix} -4 \\ 7 \\ 10 \end{pmatrix}$

Step 1: $\mathbf{Ly} = \mathbf{b}_2$

- $y_1 = -4$
- $y_1 + y_2 = 7 \Rightarrow y_2 = 11$
- $y_1 + y_2 + y_3 = 10 \Rightarrow y_3 = 3$

$$\mathbf{y} = \begin{pmatrix} -4 \\ 11 \\ 3 \end{pmatrix}$$

Step 2: $\mathbf{Ux} = \mathbf{y}$

- $x_3 = 3$
- $x_2 + x_3 = 11 \Rightarrow x_2 = 8$
- $x_1 + x_2 + x_3 = -4 \Rightarrow x_1 = -15$

$$\mathbf{x}_2 = \begin{pmatrix} -15 \\ 8 \\ 3 \end{pmatrix}$$

(c) $\mathbf{b}_3 = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \\ -\frac{1}{2} \end{pmatrix}$

Step 1:

- $y_1 = \frac{1}{2}$
- $y_1 + y_2 = \frac{3}{4} \Rightarrow y_2 = \frac{1}{4}$
- $y_1 + y_2 + y_3 = -\frac{1}{2} \Rightarrow y_3 = -\frac{5}{4}$

$$\mathbf{y} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ -\frac{5}{4} \end{pmatrix}$$

Step 2:

- $x_3 = -\frac{5}{4}$
- $x_2 + x_3 = \frac{1}{4} \Rightarrow x_2 = \frac{3}{2}$
- $x_1 + x_2 + x_3 = \frac{1}{2} \Rightarrow x_1 = \frac{1}{4}$

$$\mathbf{x}_3 = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{2} \\ -\frac{5}{4} \end{pmatrix}$$

(d) $\mathbf{b}_4 = \begin{pmatrix} 30 \\ -42 \\ -18 \end{pmatrix}$

Step 1:

- $y_1 = 30$
- $y_1 + y_2 = -42 \Rightarrow y_2 = -72$
- $y_1 + y_2 + y_3 = -18 \Rightarrow y_3 = 24$

$$\mathbf{y} = \begin{pmatrix} 30 \\ -72 \\ 24 \end{pmatrix}$$

Step 2:

- $x_3 = 24$
- $x_2 + x_3 = -72 \Rightarrow x_2 = -96$
- $x_1 + x_2 + x_3 = 30 \Rightarrow x_1 = 102$

$$\mathbf{x}_4 = \begin{pmatrix} 102 \\ -96 \\ 24 \end{pmatrix}$$

3. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(a) Find an orthogonal matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$ and the diagonal matrix $\mathbf{D}$

(b) Calculate the eigenvalues and eigenvectors of the matrix

$$3\mathbf{A}^2 + 4\mathbf{A}^{-1} - 7\mathbf{I}$$

Solution (a)

Compute $\det(\mathbf{A} - \lambda\mathbf{I})$:

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 3-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix}$$

Use cofactor expansion along the 2nd row:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)(\lambda^2 - 4\lambda + 2) = 0$$

So the eigenvalues are:

$$\lambda_1 = 1, \quad \lambda_2 = 2 + \sqrt{2}, \quad \lambda_3 = 2 - \sqrt{2}$$

Let's find eigenvectors for each eigenvalue. We'll normalize them at the end.

- For $\lambda = 1$,

  Solve $(\mathbf{A} - \mathbf{I})\mathbf{v}_1 = \mathbf{0}$:

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- For $\lambda = 2 + \sqrt{2}$,

  Solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{0}$:

  This yields:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 + \sqrt{2} \end{pmatrix}$$

  Normalize:

$$\|\mathbf{v}_2\| = \sqrt{1^2 + (-1 + \sqrt{2})^2} = \sqrt{4 - 2\sqrt{2}}$$

  Let's denote the normalized vector as $\mathbf{u}_2 = \mathbf{v}_2/\|\mathbf{v_2}\|$

- For $\lambda = 2 - \sqrt{2}$,

  Similarly:

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 - \sqrt{2} \end{pmatrix}$$

$$\|\mathbf{v}_3\| = \sqrt{1^2 + (1 + \sqrt{2})^2} = \sqrt{4 + 2\sqrt{2}}$$

  Normalize to get $\mathbf{u}_3 = \mathbf{v}_3/\|\mathbf{v_3}\|$

Let:

$$\mathbf{P} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 + \sqrt{2} & 0 \\ 0 & 0 & 2 - \sqrt{2} \end{pmatrix}$$

Solution (b)

Let's use the fact that if $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ, then:

$$\mathbf{A}^2\mathbf{v} = \lambda^2\mathbf{v}$$
$$\mathbf{A}^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$$
$$\mathbf{I}\mathbf{v} = \mathbf{v}$$

So:

$$\left(3\mathbf{A}^2 + 4\mathbf{A}^{-1} - 7\mathbf{I}\right)\mathbf{v} = \left(3\lambda^2 + 4\lambda^{-1} - 7\right)\mathbf{v}$$

So the eigenvalues of this new matrix are:

$$\mu_i = 3\lambda_i^2 + \frac{4}{\lambda_i} - 7$$

This matrix shares the same set of eigenvectors with matrix $\mathbf{A}$

Now compute for each λ_i:

$$\mu_1 = 3(1)^2 + \frac{4}{1} - 7 = 3 + 4 - 7 = 0$$
$$\mu_2 = 3(6 + 4\sqrt{2}) + 4 \cdot \frac{2 - \sqrt{2}}{2} - 7 = 15 + 10\sqrt{2}$$
$$\mu_3 = 3(6 - 4\sqrt{2}) + 4 \cdot \frac{2 + \sqrt{2}}{2} - 7 = 15 - 10\sqrt{2}$$

4. Simplify the following equation of conic section in two-dimensional space, and give the corresponding coordinate tranformation:

$$16x^2 + 24xy + 9y^2 - 3x + 4y = 0$$

and plot the transformed conic section

Solution

We diagonalized the quadratic part:

$$\mathbf{A} = \begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix}$$

The eigenvalues of this matrix give the coefficients of the conic in the rotated (u, v) coordinate system, and the eigenvectors provide the rotation matrix

Eigenvalues:

$$\lambda_1 = 0, \quad \lambda_2 = 25$$

These are the new coefficients of the quadratic terms in the rotated coordinate system. That is, the conic becomes:

$$0 \cdot u^2 + 25v^2 + (\text{linear terms in } u, v) = 0$$

Eigenvectors (columns of rotation matrix):

$$\mathbf{P} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{pmatrix}$$

This gives the coordinate transformation from (u, v) to (x, y):

$$\mathbf{x} = \mathbf{P}\mathbf{y}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} u \\ v \end{pmatrix}$$

The linear terms come from substituting $x = \frac{3}{5}u - \frac{4}{5}v$, $y = -\frac{4}{5}u - \frac{3}{5}v$ into the original equation's linear part:

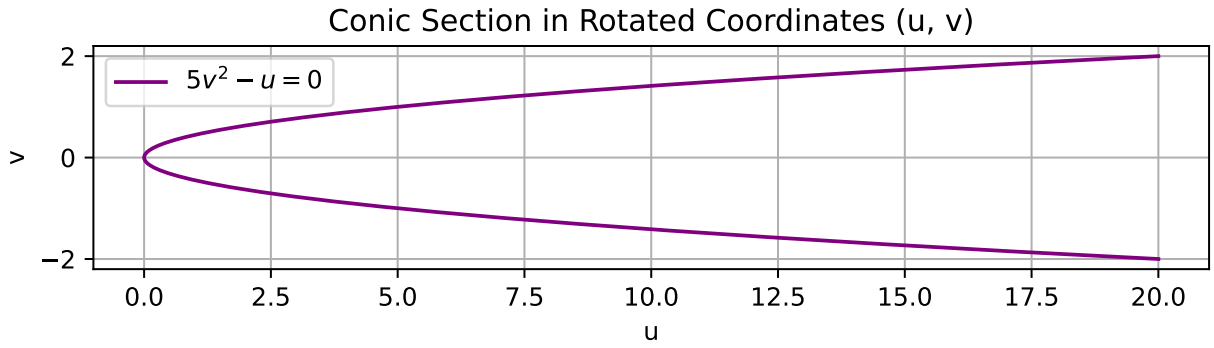
$$-3x + 4y = -3(\frac{3}{5}u - \frac{4}{5}v) + 4(-\frac{4}{5}u - \frac{3}{5}v) = -5u$$

So, the full rotated equation becomes:

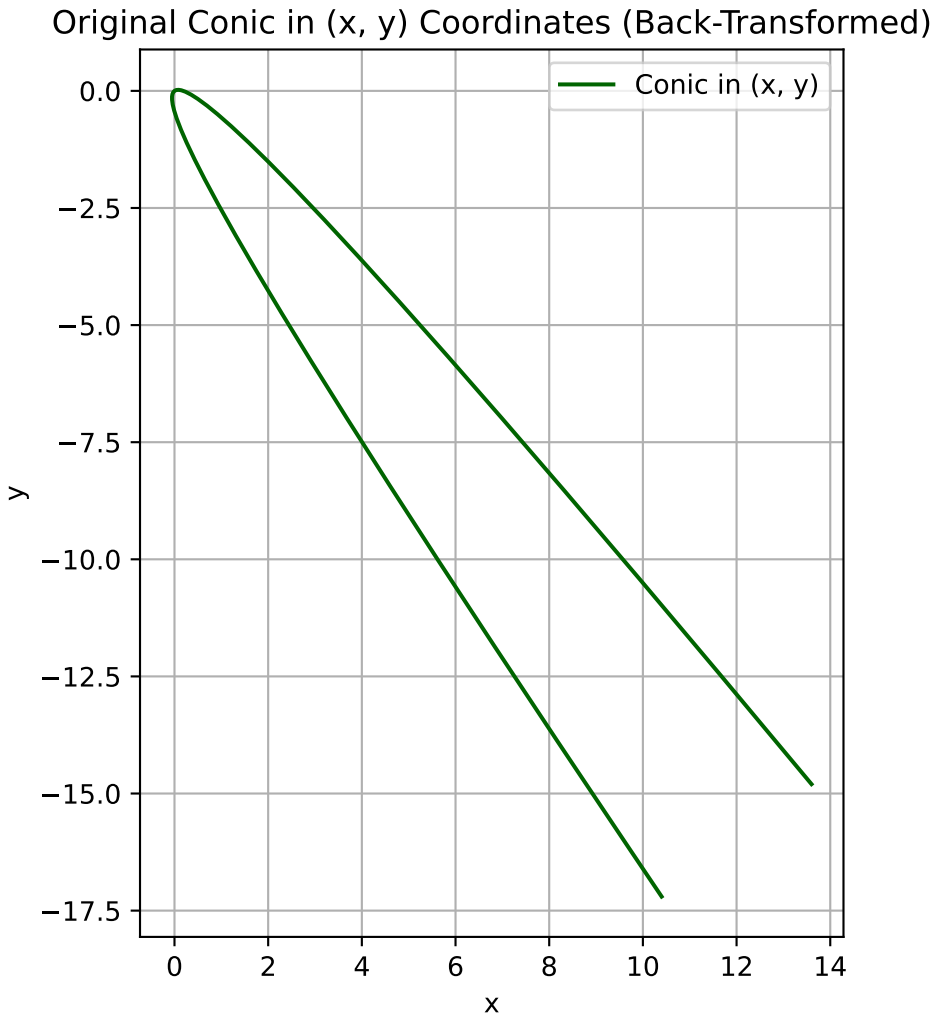
$$5v^2 - u = 0$$

This is a parabola that opens along the u-axis with its vertex at the origin in the (u, v) coordinate system

```
import numpy as np
import matplotlib.pyplot as plt

# Define v values
v_vals = np.linspace(-2, 2, 400)

# Compute corresponding u values from u = 5 *v^2
u_vals = 5 *v_vals**2

# Plot the parabola in (u, v) coordinates
plt.figure(figsize=(8, 6))

plt.plot(u_vals, v_vals, label=r'$5v^2 -u = 0$', color='purple')
plt.gca().set_aspect('equal')

plt.title("Conic Section in Rotated Coordinates (u, v)")
plt.xlabel("u")
plt.ylabel("v")
plt.grid(True)
plt.legend()
plt.show()
```



```
# Step 1: Define the (u, v) parabola
v_vals = np.linspace(-2, 2, 400)
u_vals = 5 *v_vals**2  # From u = 5v^2

# Step 2: Rotation matrix (from previous eigenvectors)
P = np.array([[ 0.6, -0.8],
              [-0.8, -0.6]])

# Step 3: Stack u and v into coordinate array
UV = np.vstack((u_vals, v_vals))  # Shape (2, N)

# Step 4: Transform back to original (x, y)
# using inverse rotation: [x, y]^T = P @ [u, v]^T
XY = P @ UV
x_vals, y_vals = XY[0, :], XY[1, :]

# Step 5: Plot the result in original (x, y) coordinates
plt.figure(figsize=(8, 6))

plt.plot(x_vals, y_vals, label="Conic in (x, y)", color='darkgreen')
plt.gca().set_aspect('equal')

plt.title("Original Conic in (x, y) Coordinates (Back-Transformed)")
plt.xlabel("x")
plt.ylabel("y")
plt.grid(True)
plt.legend()
plt.show()
```



5. Find the inverse of the given matrix or show that no inverse exists

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & 2 & 1 \\ 2 & 1 & -3 & 0 \\ 1 & 1 & 2 & 1 \end{pmatrix}$$

Solution

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & 2 & 1 \\ 2 & 1 & -3 & 0 \\ 1 & 1 & 2 & 1 \end{pmatrix}$$

$$\Downarrow$$

$$(\mathbf{A} \mid \mathbf{I}) \overset{\text{row operations}}{\longrightarrow} (\mathbf{I} \mid \mathbf{A}^{-1})$$

$$\Downarrow$$

$$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{6} & \frac{7}{6} \\ 1 & \frac{1}{3} & \frac{1}{3} & -\frac{4}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

6. Use Cramer's rule to determine the solution of the system

$$\begin{array}{rcrcl} (2-k)x_1 & + & kx_2 & = & 4 \\ kx_1 & + & (3-k)x_2 & = & 3 \end{array}$$

For what value(s) of k is the system inconsistent?

Solution

$$\mathbf{A} = \begin{pmatrix} 2-k & k \\ k & 3-k \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\Downarrow$$

$$|\mathbf{A}| = \begin{vmatrix} 2-k & k \\ k & 3-k \end{vmatrix} = -5k + 6$$

$$|\mathbf{A}_1| = \begin{vmatrix} 4 & k \\ 3 & 3-k \end{vmatrix} = -7k + 12, \quad |\mathbf{A}_2| = \begin{vmatrix} 2-k & 4 \\ k & 3 \end{vmatrix} = -7k + 6$$

$$\Downarrow$$

$$x_1 = \frac{12-7k}{6-5k}, \quad x_2 = \frac{6-7k}{6-5k} \quad \text{when } k \neq \frac{6}{5}$$

7. Simplify the following equations of quadric surfaces in three-dimensional space, and give the corresponding coordinate tranformations:

$$x^2 - 2y^2 + z^2 + 6xy - 2yz + 8x + 2y - 12z - 3 = 0$$

Solution

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix},\ \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Downarrow$$

$$\mathbf{x}^T\mathbf{A}\mathbf{x} + \begin{pmatrix} 8 & 2 & -12 \end{pmatrix}\mathbf{x} = 3$$

$$\Downarrow$$

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \ \Rightarrow\ \lambda_1 = -4,\ \lambda_2 = 1,\ \lambda_3 = 3$$

$$\lambda_1 = -4 \ \rightarrow\ \mathbf{k}_1 = \frac{1}{\sqrt{35}}\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1 \ \rightarrow\ \mathbf{k}_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$\lambda_3 = 3 \ \rightarrow\ \mathbf{k}_3 = \frac{1}{\sqrt{14}}\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$$

$$\Downarrow$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{k}_1 & \mathbf{k}_2 & \mathbf{k}_3 \end{pmatrix} \ \Rightarrow\ \mathbf{x} = \mathbf{P}\mathbf{y}$$

$$\Downarrow$$

$$\mathbf{y}^T\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{y} + \begin{pmatrix} 8 & 2 & -12 \end{pmatrix}\mathbf{P}\mathbf{y} = 3$$

$$\Downarrow$$

$$\mathbf{y}^T\begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\mathbf{y} - \begin{pmatrix} \frac{26}{\sqrt{35}} & \frac{28}{\sqrt{10}} & \frac{40}{\sqrt{14}} \end{pmatrix}\mathbf{y} = 3$$

$$\Downarrow$$

$$-4\left(y_1 + \frac{13}{4\sqrt{35}}\right)^2 + \left(y_2 - \frac{14}{\sqrt{10}}\right)^2 + 3\left(y_3 - \frac{20}{3\sqrt{14}}\right)^2 = \frac{2597}{84}$$

$$\Downarrow$$

$$-\frac{(y_1 + 0.5494)^2}{2.7801^2} + \frac{(y_2 - 4.4272)^2}{5.5603^2} + \frac{(y_3 - 1.7817)^2}{3.2102^2} \approx 1$$

$$\Downarrow$$

$$\mathbf{z} = \mathbf{y} + \begin{pmatrix} 0.5494 \\ -4.4272 \\ -1.7817 \end{pmatrix}$$

$$\Downarrow$$

$$-\frac{z_1^2}{2.7801^2} + \frac{z_2^2}{5.5603^2} + \frac{z_3^2}{3.2102^2} \approx 1$$

$$\Downarrow$$

$$\mathbf{x} = \mathbf{P}\left[\mathbf{z} - \begin{pmatrix} 0.5494 \\ -4.4272 \\ -1.7817 \end{pmatrix}\right]$$

8. Simplify the following equations of quadric surfaces in three-dimensional space, and give the corresponding coordinate tranformations:

$$5x^2 + 6y^2 + 7z^2 - 4xy + 4yz - 10x + 8y + 14z - 6 = 0$$

Solution

$$\mathbf{A} = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{pmatrix},\ \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Downarrow$$

$$\mathbf{x}^T\mathbf{A}\mathbf{x} + \begin{pmatrix} -10 & 8 & 14 \end{pmatrix}\mathbf{x} = 6$$

$$\Downarrow$$

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \ \Rightarrow\ \lambda_1 = 3,\ \lambda_2 = 6,\ \lambda_3 = 9$$

$$\lambda_1 = 3 \ \rightarrow\ \mathbf{k}_1 = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 6 \ \rightarrow\ \mathbf{k}_2 = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$$\lambda_3 = 9 \ \rightarrow\ \mathbf{k}_3 = \frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$\Downarrow$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{k}_1 & \mathbf{k}_2 & \mathbf{k}_3 \end{pmatrix} \ \Rightarrow\ \mathbf{x} = \mathbf{P}\mathbf{y}$$

$$\Downarrow$$

$$\mathbf{y}^T \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}\mathbf{y} + \begin{pmatrix} 6 & 0 & 18 \end{pmatrix}\mathbf{y} = 6$$

$$\Downarrow$$

$$3(y_1^2 + 2y_1 + 1) + 6y_2^2 + 9(y_3^2 + 2y_3 + 1) = 18$$

$$\Downarrow$$

$$\frac{(y_1 + 1)^2}{6} + \frac{y_2^2}{3} + \frac{(y_3 + 1)^2}{2} = 1$$

$$\Downarrow$$

$$\mathbf{z} = \mathbf{y} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Downarrow$$

$$\frac{z_1^2}{6} + \frac{z_2^2}{3} + \frac{z_3^2}{2} = 1$$

$$\Downarrow$$

$$\mathbf{x} = \mathbf{P}\left[\mathbf{z} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right]$$

Answers

Answer 1 (Section 6.2)

$\mathbf{AB} = \begin{pmatrix} -11 & 6 \\ 17 & -22 \end{pmatrix}$, $\mathbf{BA} = \begin{pmatrix} -32 & 27 \\ -4 & -1 \end{pmatrix}$, $\mathbf{A}^2 = \begin{pmatrix} 19 & -18 \\ -30 & 31 \end{pmatrix}$, $\mathbf{B}^2 = \begin{pmatrix} 19 & 6 \\ 3 & 22 \end{pmatrix}$

Answer 2 (Section 6.2)

$(\mathbf{A}\mathbf{A}^T)^T = (\mathbf{A}^T)^T\mathbf{A}^T = \mathbf{A}\mathbf{A}^T$

Answer 3 (Section 6.2)

For example $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, or $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Answer 4 (Section 6.2)

$(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$, and $\mathbf{AB} + \mathbf{BA} = 2\mathbf{AB}$ only if $\mathbf{AB} = \mathbf{BA}$, which is not true in general

Answer 5 (Section 6.2)

$\mathbf{b} = \langle -1, 1 \rangle$

Answer 6 (Section 6.2)

$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} ax + \frac{b}{2}y \\ \frac{b}{2}x + cy \end{pmatrix} = ax^2 + bxy + cy^2$

Answer 7 (Section 6.3)

(1) $x_1 = 4,\ x_2 = -7$

(2) $x_1 = -\frac{2}{3},\ x_2 = \frac{1}{3}$

(3) $x_1 = 0,\ x_2 = 4,\ x_3 = -1$

(4) $(x_1, x_2, x_3) = (-t,\ t,\ 0)$ (infinitely many solutions)

(5) No solution (inconsistent)

Answer 8 (Section 6.3)

(1) $\mathrm{C_5H_8} + 7\,\mathrm{O_2} \rightarrow 5\,\mathrm{CO_2} + 4\,\mathrm{H_2O}$

(2) $3\,\mathrm{Cu} + 8\,\mathrm{HNO_3} \rightarrow 3\,\mathrm{Cu(NO_3)_2} + 4\,\mathrm{H_2O} + 2\,\mathrm{NO}$

Answer 9 (Section 6.3)

$\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ ca_{21} + a_{31} & ca_{22} + a_{32} & ca_{23} + a_{33} \end{pmatrix}$ (rows 1 and 2 swapped, then $c\times$ row 2 added to row 3)

Answer 10 (Section 6.4)

(1) rank $= 1$ **(2)** rank $= 3$ **(3)** rank $= 3$

Answer 11 (Section 6.4)

(1) linearly independent (rank 3) **(2)** linearly independent (rank 3)

Answer 12 (Section 6.4)

$8 - 3 = 5$ parameters

Answer 13 (Section 6.4)

1. 4 **2.** inconsistent if $\operatorname{rank}(\mathbf{A}) = 1$ (consistent if $\operatorname{rank}(\mathbf{A}) = 2$) **3.** $6 - 3 = 3$ parameters

Answer 14 (Section 6.4)

$\operatorname{rank}(\mathbf{A}) = 2$ (so $\det \mathbf{A} = 0$)

Answer 15 (Section 6.5)

$M_{12} = 9, \quad M_{32} = 0, \quad C_{13} = 1, \quad C_{22} = 18$

Answer 16 (Section 6.5)

$17, \quad \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$

Answer 17 (Section 6.5)

(1) -85 **(2)** $-x + 2y - z$

Answer 18 (Section 6.6)

(1) interchanging two rows changes the sign

(2) adding a multiple of one column to another leaves the determinant unchanged

(3) common factors 2 (row 2) and 3 (column 3) are taken out: $2 \cdot 3 = 6$

Answer 19 (Section 6.6)

(1) -5 **(2)** 30 **(3)** 10

Answer 20 (Section 6.6)

Adding row 2 to row 3 makes row 3 equal to $(x + y + z)\langle 1, 1, 1\rangle$, a multiple of row 1, so $\det \mathbf{A} = 0$

Answer 21 (Section 6.6)

$(b - a)(c - a)(d - a)(c - b)(d - b)(d - c)$

Answer 22 (Section 6.7)

$\mathbf{AB} = \mathbf{BA} = \mathbf{I}$

Answer 23 (Section 6.7)

(1) nonsingular ($\det = -36$): $\mathbf{A}^{-1} = \operatorname{diag}\left(\frac{1}{3}, \frac{1}{6}, -\frac{1}{2}\right)$

(2) nonsingular ($\det = 27$): $\mathbf{A}^{-1} = \frac{1}{27}\begin{pmatrix} 6 & 21 & -9 & -36 \\ -1 & 1 & 6 & -3 \\ 10 & 17 & -6 & -51 \\ 4 & -4 & 3 & 12 \end{pmatrix}$

(3) nonsingular ($\det = 1$): $\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix}$

Answer 24 (Section 6.7)

(1) no inverse ($\det = 0$) **(2)** $\begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & 0 \end{pmatrix}$

Answer 25 (Section 6.7)

$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T = \begin{pmatrix} 5 & -1 \\ -2 & \frac{1}{2} \end{pmatrix}$

Answer 26 (Section 6.7)

$\mathbf{M}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \mathbf{M}(-\theta) = \mathbf{M}^T$: $\mathbf{a}$ is $\mathbf{b}$ rotated back through $-\theta$

Answer 27 (Section 6.7)

In both cases $\mathbf{A}^T\mathbf{A} = \mathbf{I}$ (the columns are orthonormal)

Answer 28 (Section 6.7)

Use $\det(\mathbf{AB}) = \det\mathbf{A}\det\mathbf{B}$ and "nonsingular $\iff$ $\det \neq 0$":

$(\det\mathbf{A})^2 = \det(\mathbf{A}^T\mathbf{A}) = 1$; $\det(\mathbf{AB}) \neq 0$; $\det(\mathbf{AB}) = 0$; $\det\mathbf{A}\det\mathbf{A}^{-1} = 1$; if $\mathbf{A}$ is nonsingular, multiply $\mathbf{A}^2 = \mathbf{A}$ by $\mathbf{A}^{-1}$

$\mathbf{B} = \mathbf{A}^{-1}(\mathbf{AB}) = \mathbf{0}$; $\mathbf{B} = \mathbf{A}^{-1}(\mathbf{AC}) = \mathbf{C}$; $\mathbf{A} + \mathbf{B}$ need not be nonsingular ($\mathbf{B} = -\mathbf{A}$); $\det\mathbf{A}^T = \det\mathbf{A} \neq 0$; if either were nonsingular, the other would be $\mathbf{0}$

Answer 29 (Section 6.8)

(1) $x_1 = -\frac{3}{5},\ x_2 = \frac{6}{5}$ **(2)** $x_1 = 0.1,\ x_2 = -0.3$ **(3)** $x = 4,\ y = -7$

Answer 30 (Section 6.8)

1. $x_1 = \dfrac{\epsilon - 2}{\epsilon - 1} = 1 - \dfrac{1}{\epsilon - 1}, \quad x_2 = \dfrac{1}{\epsilon - 1}$

2. $\epsilon = 1.01$: $(x_1, x_2) = (-99, 100)$; $\epsilon = 0.99$: $(101, -100)$. A 2% change in ϵ flips the solution, because the lines are almost parallel ($\det\mathbf{A} = \epsilon - 1 \approx 0$)

Answer 31 (Section 6.9)

(1) $\lambda = 1$: $\langle 1, 1\rangle$; $\lambda = 6$: $\langle 2, 7\rangle$; nonsingular

(2) $\lambda = 4$: $\langle 1, 0\rangle$; $\lambda = -5$: $\langle -8, 9\rangle$; nonsingular

(3) $\lambda = -1$: $\langle 1, -1, 1\rangle$; $\lambda = \pm i$: $\langle \pm i, 1, 1\rangle$; nonsingular

Answer 32 (Section 6.9)

(1) $\lambda = 4$: $\langle -1, 1\rangle$; $\lambda = 6$: $\langle 1, 1\rangle$; $\mathbf{A}^{-1}$: $\frac{1}{4}, \frac{1}{6}$ with the same eigenvectors

(2) $\lambda = 3$: $\langle -2, 1, 0\rangle$; $\lambda = 4$: $\langle 1, 0, 0\rangle$; $\lambda = 5$: $\langle -3, -1, 1\rangle$; $\mathbf{A}^{-1}$: $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ with the same eigenvectors

Answer 33 (Section 6.9)

True: $(\mathbf{A} - \lambda\mathbf{I})\mathbf{k} = \mathbf{0}$ has a nonzero solution $\mathbf{k}$, so $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

Answer 34 (Section 6.9)

1. $\mathbf{A}^2\mathbf{k} = \mathbf{A}(\lambda\mathbf{k}) = \lambda^2\mathbf{k}$: λ^2 is an eigenvalue of $\mathbf{A}^2$ with the same eigenvector

2. $\lambda = -1$: $\langle -1, 1\rangle$, $\lambda = 7$: $\langle 3, 5\rangle$; $\mathbf{A}^2\langle 3, 5\rangle^T = 49\langle 3, 5\rangle^T$

3. $\mathbf{A}^n\mathbf{k} = \lambda^n\mathbf{k}$, and $p(\mathbf{A})\mathbf{k} = p(\lambda)\mathbf{k}$ for any polynomial p

Answer 35 (Section 6.9)

$\mathbf{A} = \mathbf{SBS}^{-1} = \mathbf{T}^{-1}\mathbf{BT}$ with $\mathbf{T} = \mathbf{S}^{-1}$

Answer 36 (Section 6.9)

$\det(\mathbf{B} - \lambda\mathbf{I}) = \det\mathbf{S}^{-1}\det(\mathbf{A} - \lambda\mathbf{I})\det\mathbf{S} = \det(\mathbf{A} - \lambda\mathbf{I})$

Answer 37 (Section 6.10)

$\lambda^2 - 6\lambda + 13 = 0$ and $\mathbf{A}^2 - 6\mathbf{A} + 13\mathbf{I} = \mathbf{0}$

Answer 38 (Section 6.10)

(1) $\mathbf{A}^5 = 16640\,\mathbf{I} + 8336\,\mathbf{A} = \begin{pmatrix} 83328 & 41680 \\ 33344 & 16640 \end{pmatrix}$

(2) $\mathbf{A}^{10} = -340\,\mathbf{I} + 341\,\mathbf{A}^2 = \begin{pmatrix} 1 & 1023 & 1023 \\ 0 & 683 & 682 \\ 0 & 341 & 342 \end{pmatrix}$

Answer 39 (Section 6.10)

$\mathbf{A}^m = 4^{m-1}\begin{pmatrix} 4+3m & 3m \\ -3m & 4-3m \end{pmatrix}, \qquad \mathbf{A}^6 = \begin{pmatrix} 22528 & 18432 \\ -18432 & -14336 \end{pmatrix}$

Answer 40 (Section 6.10)

(a) $\mathbf{A}^m = 4^{m-1}\mathbf{A}$ (only one unknown c_1 remains, so no system needs to be solved)

(b) $\mathbf{A}^m = (3 \cdot 2^{m-1} - 2 \cdot 3^{m-1})\mathbf{A} + (3^{m-1} - 2^{m-1})\mathbf{A}^2$

Answer 41 (Section 6.10)

1. $(\det \mathbf{A})^m = \det \mathbf{A}^m = 0$, so $\det \mathbf{A} = 0$

2. $\mathbf{0} = \mathbf{A}^m\mathbf{k} = \lambda^m\mathbf{k}$ with $\mathbf{k} \neq \mathbf{0}$, so $\lambda = 0$

Answer 42 (Section 6.11)

(1) orthogonal **(2)** orthogonal

Answer 43 (Section 6.11)

(a), **(b)** $\mathbf{A}\mathbf{k}_1 = -2\mathbf{k}_1$, $\mathbf{A}\mathbf{k}_2 = -2\mathbf{k}_2$, $\mathbf{A}\mathbf{k}_3 = 4\mathbf{k}_3$

(c) $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad \mathbf{P}^T\mathbf{A}\mathbf{P} = \operatorname{diag}(-2, -2, 4)$

Answer 44 (Section 6.11)

$(\mathbf{AB})^T(\mathbf{AB}) = \mathbf{B}^T\mathbf{A}^T\mathbf{AB} = \mathbf{I}$; so $\mathbf{A}^2$ is orthogonal too

$(\mathbf{A}^{-1})^T\mathbf{A}^{-1} = \mathbf{A}\mathbf{A}^T = \mathbf{I}$; if $\mathbf{A}^2 = \mathbf{I}$ then $\mathbf{A}^T = \mathbf{A}^{-1} = \mathbf{A}$; for the rotation matrix $\mathbf{M}^T\mathbf{M} = \mathbf{I}$

Answer 45 (Section 6.13)

(1) $\mathbf{P} = \begin{pmatrix} 13 & 1 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{D} = \operatorname{diag}(-7, 4)$

(2) $\mathbf{P} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{D} = \operatorname{diag}(0, 1, 2)$

(3) $\mathbf{P} = \begin{pmatrix} 0 & 1-\sqrt{5} & 1+\sqrt{5} \\ 0 & 2 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{D} = \operatorname{diag}(1, -\sqrt{5}, \sqrt{5})$

Answer 46 (Section 6.13)

(1) $\mathbf{P} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{D} = \operatorname{diag}(0, 2)$

(2) $\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{D} = \operatorname{diag}(-1, 1, 1)$

(3) $\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \mathbf{D} = \operatorname{diag}(-6, 1, 8)$

Answer 47 (Section 6.13)

(1) ellipse: $\dfrac{X^2}{6} + \dfrac{Y^2}{4} = 1$ **(2)** hyperbola: $\dfrac{Y^2}{4} - \dfrac{X^2}{4} = 1$

Answer 48 (Section 6.13)

$\mathbf{A} = \dfrac{1}{3}\begin{pmatrix} 8 & 4 & -1 \\ 4 & 11 & 4 \\ -1 & 4 & 8 \end{pmatrix}$

Answer 49 (Section 6.13)

$\mathbf{A}^m = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})\cdots(\mathbf{PDP}^{-1}) = \mathbf{PD}^m\mathbf{P}^{-1}$

Answer 50 (Section 6.13)

(1) $\mathbf{A}^5 = \begin{pmatrix} 21 & 11 \\ 22 & 10 \end{pmatrix}$ **(2)** $\mathbf{A}^{10} = \mathbf{A} = \begin{pmatrix} 6 & -10 \\ 3 & -5 \end{pmatrix}$

Answer 51 (Section 6.13)

$\mathbf{D}^{-1} = \mathbf{P}^{-1}\mathbf{A}^{-1}\mathbf{P}$ is diagonal, so the same $\mathbf{P}$ diagonalizes $\mathbf{A}^{-1}$

Answer 52 (Section 6.13)

No: the columns of $\mathbf{P}$ can be rescaled or reordered, and for a repeated eigenvalue other eigenvectors can be chosen. Only $\mathbf{D}$ (up to order) is unique.

Answer 53 (Section 6.14)

(1) $\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix}$ **(2)** $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 0 & 10 \end{pmatrix}$

(3) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -7 & 1 \end{pmatrix}\begin{pmatrix} 4 & -2 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 21 \end{pmatrix}$

Answer 54 (Section 6.14)

(1) $x_1 = -\frac{1}{3},\ x_2 = -\frac{5}{6}$ **(2)** $x_1 = 1,\ x_2 = 1,\ x_3 = 5$

7 Systems of Linear Differential Equations

Learning objectives After this chapter you should be able to

- write coupled or higher-order linear equations as a system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$, and use fundamental sets and the fundamental matrix
- solve homogeneous systems with distinct, repeated and complex eigenvalues
- solve nonhomogeneous systems by undetermined coefficients, variation of parameters and diagonalization
- compute the matrix exponential by its series or by the Laplace transform

7.1 Theory of Linear Systems

- Here we confine our study to systems of first-order DEs that are special cases of systems that have the normal form

$$\begin{aligned} \frac{dx_1}{dt} &= g_1(t, x_1, x_2, \cdots, x_n) \\ \frac{dx_2}{dt} &= g_2(t, x_1, x_2, \cdots, x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= g_n(t, x_1, x_2, \cdots, x_n) \end{aligned}$$

  This system is called a **first-order system**

- When each of the functions g_i, $i = 1, \cdots, n$, is linear in the dependent variables x_i, $i = 1, \cdots, n$, we get the normal form of a first-order system of linear equations

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t) \end{aligned} \tag{LS}$$

  We refer to (LS) simply as a **linear system**. We assume that the coefficients $a_{ij}(t)$ as well as the functions $f_i(t)$ are continuous on a common interval I. When $f_i(t) = 0$, $i = 1, \cdots, n$, the linear system is said to be **homogeneous**; otherwise it is **nonhomogeneous**

- If $\mathbf{x}$, $\mathbf{A}$, and $\mathbf{f}$ denote the respective matrices

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \ \mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

  then the system of linear first-order DEs can be written as

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{f} \tag{LSM}$$

- A solution vector of (LSM) can be interpreted geometrically as a set of parametric equations of a space curve. In the case $n = 2$ and $n = 3$, the equations $x_1 = \phi_1(t)$, $x_2 = \phi_2(t)$, and $x_1 = \phi_1(t)$, $x_2 = \phi_2(t)$, $x_3 = \phi_3(t)$ represent curves in 2-space and 3-space, respectively. It is common practice to call such a solution curve a **trajectory**. The plane is also called the **phase plane**

- **Initial-Value Problem**

  Let t_0 denote a point on an interval I and

$$\mathbf{x}_0 = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}$$

  where γ_i, $i = 1, \cdots, n$ are given constants. Then the problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

  is an **initial-value problem** on the interval. Here there exists a unique solution of the initial-value problem if the entries of $\mathbf{A}$ and $\mathbf{f}$ are continuous functions on a common interval I

- **Superposition Principle**

  Let $\mathbf{x}_1$, $\mathbf{x}_2$, $\cdots$, $\mathbf{x}_k$ be a set of solution vectors of the homogeneous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}$$

  on an interval I. Then the linear combination

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$$

  where c_i, $i = 1, 2, \cdots, k$, are arbitrary constants, is also a solution on the interval

- We can introduce the concept of the **Wronskian** determinant as a test for linear independence. Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ be n solution vectors of the homogeneous system on I. Then the set of solution vectors is linearly independent on I if and only if the Wronskian

$$W(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} \neq 0$$

  for every t in the interval.

- Either $W(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) \neq 0$ for every t in I, or $W(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) = 0$ for every t in I. Thus if we can show that $W \neq 0$ for some t_0 in I, then $W \neq 0$ for every t, hence the set of solutions is linearly independent on the interval

- Any set of n linearly independent solution vectors of the homogeneous system on an interval I is said to be a **fundamental set of solutions**. Then the **general solution** of the homogeneous system is

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

  where c_i, $i = 1, 2, \cdots, n$ are arbitrary constants

- **General Solution of the Nonhomogeneous System**

  For nonhomogeneous systems, a **particular solution** $\mathbf{x}_p$ on an interval I is any vector, free of arbitrary parameters, whose entries are functions that satisfy system (LSM). Let

$$\mathbf{x}_h = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

  denote the general solution of the associated homogeneous system. Then the general solution of the nonhomogeneous system is

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$$

7.2 Homogeneous Linear Systems

- For the general homogeneous linear first-order system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \tag{HE}$$

  we are prompted to ask whether we can always find a solution of the form

$$\mathbf{x} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = \mathbf{k}e^{\lambda t} \tag{SL}$$

- If (SL) is to be a solution vector of the system, then

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{A}\mathbf{x} \\ \Downarrow \quad & \mathbf{x} = \mathbf{k}e^{\lambda t} \\ \lambda \mathbf{k} e^{\lambda t} &= \mathbf{A}\mathbf{k}e^{\lambda t} \\ \Downarrow \quad & \div e^{\lambda t} \\ (\mathbf{A} - \lambda \mathbf{I})\mathbf{k} &= \mathbf{0} \end{aligned} \tag{CE}$$

  In order for (CE) to have solutions other than the obvious trivial solution $\mathbf{k} = \mathbf{0}$, we must have

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

  This polynomial equation of λ is called the **characteristic equation** of $\mathbf{A}$; its solutions are the **eigenvalues** of $\mathbf{A}$. A solution $\mathbf{k} \neq \mathbf{0}$ of (CE) corresponding to an eigenvalue λ is called an **eigenvector** of $\mathbf{A}$

- **Distinct Real Eigenvalues**

  When $\mathbf{A}$ possesses n distinct real eigenvalues λ_i, $i = 1, 2, \cdots, n$, then a set of n linearly independent eigenvectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\cdots$, $\mathbf{k}_n$ can always be found and

$$\mathbf{x}_1 = \mathbf{k}_1 e^{\lambda_1 t}, \mathbf{x}_2 = \mathbf{k}_2 e^{\lambda_2 t}, \cdots, \mathbf{x}_n = \mathbf{k}_n e^{\lambda_n t}$$

  is a fundamental set of solutions of the homogeneous system (HE) on $(-\infty, \infty)$

Example Solve

$$\begin{aligned} \frac{dx}{dt} &= 2x + 3y \\ \frac{dy}{dt} &= 2x + y \end{aligned} \quad \Rightarrow \quad \frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}, \ \mathbf{A} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

- From the characteristic equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 2 - \lambda & 3 \\ 2 & 1 - \lambda \end{vmatrix} = (\lambda + 1)(\lambda - 4) = 0$$

- Now for $\lambda_1 = -1$,

$$\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  Thus $k_1 = -k_2$. When $k_2 = -1$, the related eigenvector is

$$\mathbf{k}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- For $\lambda_2 = 4$,

$$\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  so that $k_1 = \frac{3}{2}k_2$, and therefore with $k_2 = 2$, the corresponding eigenvector is

$$\mathbf{k}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

- Since we have found two linearly independent solutions

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$$

  the general solution of the system is

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

```
import numpy as np
import matplotlib.pyplot as plt

w = 6
xl = np.arange(-w, w, 0.1)
yl = np.arange(-w, w, 0.1)
x, y = np.meshgrid(xl, yl)

xdot = 2*x +3*y
ydot = 2*x +y

y_1 = -xl
y_2 = 2/3 *xl
```

```
fig, ax = plt.subplots(figsize=(5, 5))

ax.streamplot(x, y, xdot, ydot, color='blue')
ax.plot(xl, y_1, 'r:', xl, y_2, 'r:')
ax.set_title('Phase Portrait')
ax.axis((-w, w, -w, w))
ax.set_xlabel('x')
ax.set_ylabel('y')
ax.grid()
ax.tick_params(axis='both', direction='in')
```

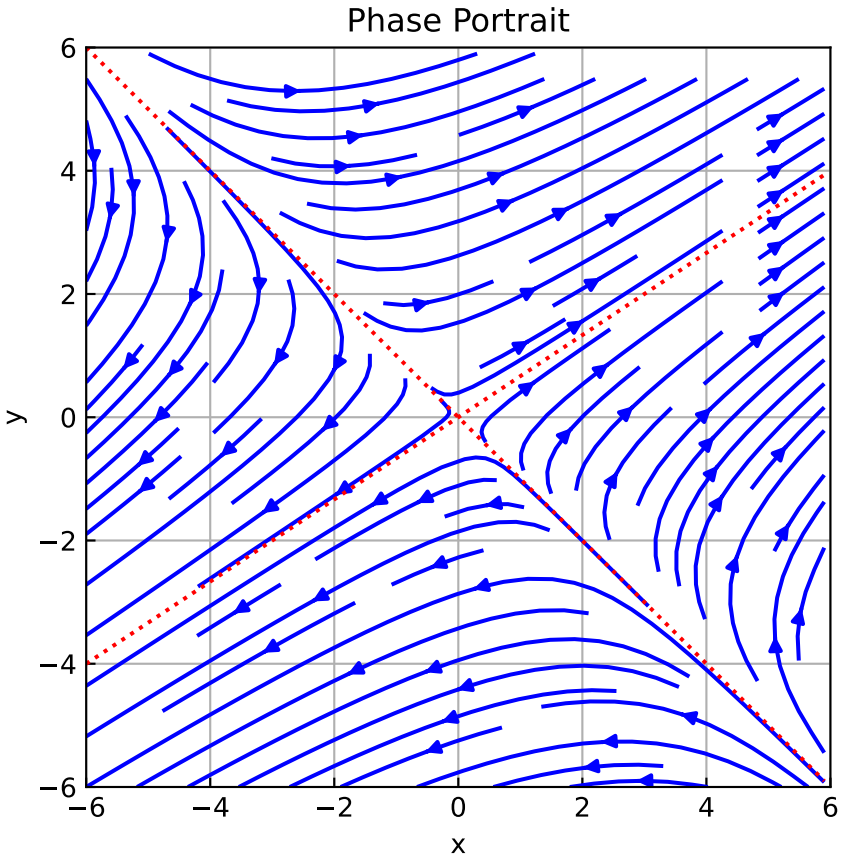


Figure 7.1: Phase Portrait of Saddle-Point System

```
import pprint

A = np.array([[2, 3], [2, 1]])
ld, k = np.linalg.eig(A)

print('Eigenvalues =', end = ' '); pprint.pprint(ld)
print('Eigenvectors ='); pprint.pprint(k)
```

```
Eigenvalues = array([ 4., -1.])
Eigenvectors =
array([[ 0.83205029, -0.70710678],
       [ 0.5547002 ,  0.70710678]])
```

```
import sympy
sympy.init_printing()

A_ = sympy.Matrix(A)
A_.eigenvects()
```

$$\left[\left(-1,\ 1,\ \left[\begin{bmatrix}-1\\1\end{bmatrix}\right]\right),\ \left(4,\ 1,\ \left[\begin{bmatrix}\frac{3}{2}\\1\end{bmatrix}\right]\right)\right]$$

Example Use python to obtain the phase portraits of the systems

Attractor

$$\frac{dx}{dt} = -\frac{5}{2}x + 2y$$
$$\frac{dy}{dt} = \frac{3}{4}x - 2y$$

Repeller

$$\frac{dx}{dt} = 2x + 2y$$
$$\frac{dy}{dt} = x + 3y$$

Answer 1 (end of chapter) · Solution

- **Repeated Eigenvalues**

  For an $n \times n$ matrix $\mathbf{A}$, if m is a positive integer and $(\lambda - \lambda_1)^m$ is a factor of the characteristic equation while $(\lambda - \lambda_1)^{m+1}$ is not a factor, then λ_1 is said to be an **eigenvalue of multiplicity** m

  – It may be possible to find m *linearly independent eigenvectors* $\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_m$ corresponding to an eigenvalue λ_1 of multiplicity $m \leq n$. In this case, the general solution contains the linear combination

$$c_1\mathbf{k}_1 e^{\lambda_1 t} + c_2\mathbf{k}_2 e^{\lambda_1 t} + \cdots + c_m\mathbf{k}_m e^{\lambda_1 t}$$

- If there is *only one eigenvector* corresponding to the eigenvalue λ_1 of multiplicity m, then m linearly independent solutions of the form

$$\begin{aligned}
\mathbf{x}_1 &= \mathbf{k}_{11} e^{\lambda_1 t} \\
\mathbf{x}_2 &= \mathbf{k}_{11} t e^{\lambda_1 t} + \mathbf{k}_{12} e^{\lambda_1 t} \\
&\vdots \\
\mathbf{x}_m &= \mathbf{k}_{11} \frac{t^{m-1}}{(m-1)!} e^{\lambda_1 t} + \mathbf{k}_{12} \frac{t^{m-2}}{(m-2)!} e^{\lambda_1 t} + \cdots + \mathbf{k}_{1m} e^{\lambda_1 t}
\end{aligned}$$

  can always be found

Example Solve

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{x}$$

- The characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0$$

  yields $-(\lambda+1)^2(\lambda-5) = 0$. Then $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 5$

- For $\lambda_1 = \lambda_2 = -1$, we have

$$(\mathbf{A} + \mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

  The first row of the last matrix means $k_1 - k_2 + k_3 = 0$. By choosing $k_2 = 1$, $k_3 = 0$ and $k_2 = 1$, $k_3 = 1$, two linearly independent eigenvectors corresponding to λ_1

$$\mathbf{k}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{k}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

  are obtained

- Last, for $\lambda_3 = 5$, the Gauss elimination

$$(\mathbf{A} - 5\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} -4 & -2 & 2 & 0 \\ -2 & -4 & -2 & 0 \\ 2 & -2 & -4 & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

  implies $k_1 = k_3$ and $k_2 = -k_3$. Picking $k_3 = 1$ gives $k_1 = 1$, $k_2 = -1$, and thus a third eigenvector is

$$\mathbf{k}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

- Thus we conclude that the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t}$$

Example Solve

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}$$

- The characteristic equation $-(\lambda - 2)^3 = 0$ shows that $\lambda_1 = 2$ is an eigenvalue of multiplicity three. We have

$$(\mathbf{A} - 2\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

  Therefore $0 \cdot k_1 + k_2 + 6k_3 = 0$ and $k_3 = 0 \;\rightarrow\; k_2 = k_3 = 0$. The choice $k_1 = 1$ gives the single eigenvector

$$\mathbf{k}_{11} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- A second solution can be found of the form

$$\mathbf{x}_2 = \mathbf{k}_{11} t e^{2t} + \mathbf{k}_{12} e^{2t} \tag{S1}$$

  To see this, we substitute (S1) into the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$

$$\underbrace{(\mathbf{A} - 2\mathbf{I})\mathbf{k}_{11}}_{\mathbf{0}} t e^{2t} + (\mathbf{A}\mathbf{k}_{12} - 2\mathbf{k}_{12} - \mathbf{k}_{11}) e^{2t} = \mathbf{0}$$

  Since this last equation is to hold for all values of t, we must have

$$(\mathbf{A} - 2\mathbf{I})\mathbf{k}_{12} = \mathbf{k}_{11}$$

- To find $\mathbf{x}_2$, we need only solve the additional system for the generalized eigenvector $\mathbf{k}_{12}$

$$(\mathbf{A} - 2\mathbf{I}|\mathbf{k}_{11}) = \left(\begin{array}{ccc|c} 0 & 1 & 6 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

  Then $0 \cdot k_1 + k_2 + 6k_3 = 1$ and $k_3 = 0 \;\rightarrow\; k_2 = 1, k_3 = 0$. The choice $k_1 = 0$ gives

$$\mathbf{k}_{12} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- A third solution can be found of the form

$$\mathbf{x}_3 = \mathbf{k}_{11}\frac{t^2}{2}e^{2t} + \mathbf{k}_{12}te^{2t} + \mathbf{k}_{13}e^{2t} \tag{S2}$$

To see this, we substitute (S2) into the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$

$$\underbrace{(\mathbf{A} - 2\mathbf{I})\mathbf{k}_{11}}_{\mathbf{0}}\frac{t^2}{2}e^{2t} + \underbrace{(\mathbf{A}\mathbf{k}_{12} - 2\mathbf{k}_{12} - \mathbf{k}_{11})}_{\mathbf{0}}te^{2t} + (\mathbf{A}\mathbf{k}_{13} - 2\mathbf{k}_{13} - \mathbf{k}_{12})e^{2t} = \mathbf{0}$$

- Then, to find $\mathbf{k}_{13}$

$$(\mathbf{A} - 2\mathbf{I}|\mathbf{k}_{12}) = \left(\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Then $0 \cdot k_1 + k_2 + 6k_3 = 0$ and $5k_3 = 1 \;\rightarrow\; k_2 = -\frac{6}{5}$, $k_3 = \frac{1}{5}$. The choice $k_1 = 0$ gives

$$\mathbf{k}_{13} = \begin{pmatrix} 0 \\ -\frac{6}{5} \\ \frac{1}{5} \end{pmatrix}$$

- Thus we see that the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}e^{2t}\right] + c_3\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\frac{t^2}{2}e^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ -\frac{6}{5} \\ \frac{1}{5} \end{pmatrix}e^{2t}\right]$$

Example Use python to obtain the phase portrait of the system

$$\frac{dx}{dt} = 3x - 18y$$
$$\frac{dy}{dt} = 2x - 9y$$

```
A = np.array([[3, -18], [2, -9]])
ld, k = np.linalg.eig(A)

print('Eigenvalues =', end = ' '); pprint.pprint(ld)
print('Eigenvectors ='); pprint.pprint(np.around(k, decimals=4))
```

```
Eigenvalues = array([-3., -3.])
Eigenvectors =
array([[0.9487, 0.9487],
       [0.3162, 0.3162]])
```

```
A_ = sympy.Matrix(A)
A_.eigenvects()
```

$$\left[\left(-3,\ 2,\ \left[\begin{bmatrix}3\\1\end{bmatrix}\right]\right)\right]$$

```
w = 6
xl = np.arange(-w, w, 0.1)
yl = np.arange(-w, w, 0.1)
x, y = np.meshgrid(xl, yl)

xdot = 3*x -18*y
ydot = 2*x -9*y

y_1 = 1/3 *xl

fig, ax = plt.subplots(figsize=(5, 5))

ax.streamplot(x, y, xdot, ydot, color='blue')
ax.plot(xl, y_1, 'r:')
ax.set_title('Phase Portrait')
ax.axis((-w, w, -w, w))
ax.set_xlabel('x')
ax.set_ylabel('y')
ax.grid()
ax.tick_params(axis='both', direction='in')
```

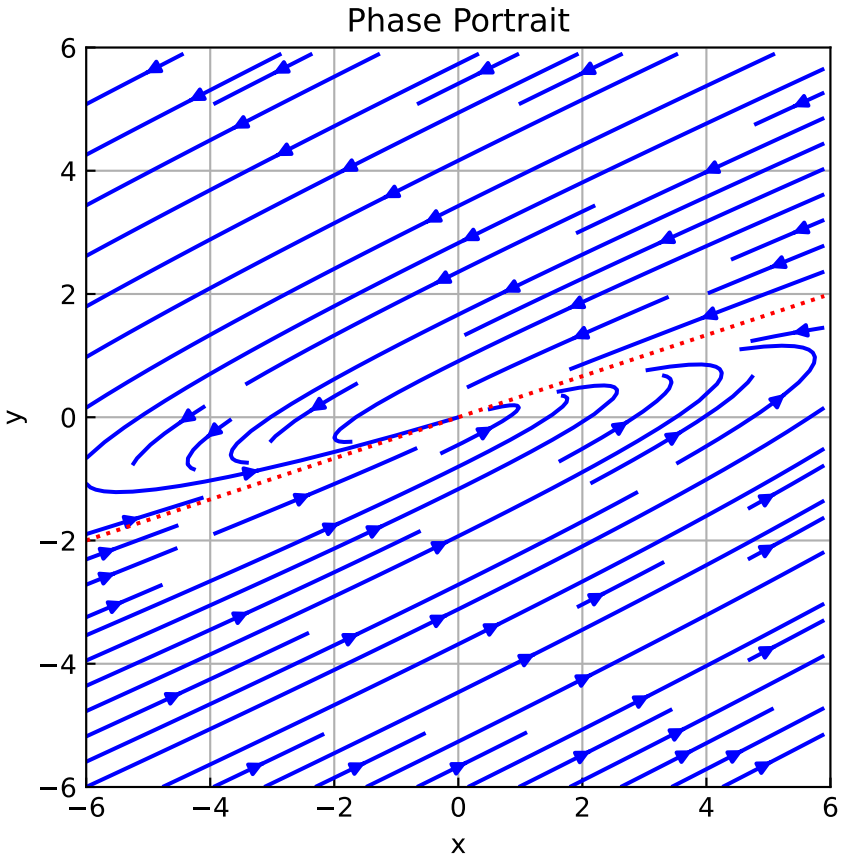


Figure 7.2: Phase Portrait of Defective System

Example The 5×5 matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

has an eigenvalue $\lambda_1 = 2$ of multiplicity 5. Show that three linearly independent eigenvectors corresponding to λ_1 can be found. And construct the generalized eigenvectors

- The system

$$(\mathbf{A} - 2\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

implies $k_2 = 0$ and $k_5 = 0$. Then we can pick $(k_1 = 1, k_3 = k_4 = 0)$, $(k_3 = 1, k_1 = k_4 = 0)$ and $(k_4 = 1, k_1 = k_3 = 0)$ in turn to build three linearly independent eigenvectors

$$\mathbf{k}_{11} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{k}_{21} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{k}_{31} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

- The following systems

$$(\mathbf{A} - 2\mathbf{I}|\mathbf{k}_{11}) = \left(\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad (\mathbf{A} - 2\mathbf{I}|\mathbf{k}_{31}) = \left(\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

  are solved to construct two generalized eigenvectors: The first system gives $k_2 = 1$ and $k_5 = 0$, and then, for $\mathbf{k}_{12}$, we can pick $k_1 = k_3 = k_4 = 0$. The second system yields $k_2 = 0$ and $k_5 = 1$, we can also pick $k_1 = k_3 = k_4 = 0$ for $\mathbf{k}_{32}$

$$\mathbf{k}_{12} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{k}_{32} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- **Complex Eigenvalues**

  Let $\mathbf{A}$ be the coefficient matrix having real entries of the homogeneous system, and let $\mathbf{k}_1$ be an eigenvector corresponding to the complex eigenvalue $\lambda_1 = \alpha + i\beta$, α and β real. Then

$$\mathbf{k}_1 e^{\lambda_1 t} \text{ and } \bar{\mathbf{k}}_1 e^{\bar{\lambda}_1 t}$$

  are linearly independent solutions of the homogeneous system. By defining

$$\mathbf{b}_1 = \text{Re}(\mathbf{k}_1) \text{ and } \mathbf{b}_2 = \text{Im}(\mathbf{k}_1)$$

  the following vectors are also linearly independent solutions

$$\begin{aligned} \mathbf{x}_1 &= [\mathbf{b}_1 \cos \beta t - \mathbf{b}_2 \sin \beta t]\, e^{\alpha t} \\ \mathbf{x}_2 &= [\mathbf{b}_2 \cos \beta t + \mathbf{b}_1 \sin \beta t]\, e^{\alpha t} \end{aligned}$$

Example Solve the initial value problem

$$\begin{aligned} \frac{dx}{dt} &= 2x + 8y \\ \frac{dy}{dt} &= -x - 2y \end{aligned}, \quad x(0) = 2, \ \ y(0) = -1$$

- First we obtain the eigenvalues from

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

The eigenvalues are $\lambda_1 = 2i$ and $\lambda_2 = \bar{\lambda}_1$

```
A = np.array([[2, 8], [-1, -2]])

A_ = sympy.Matrix(A)
A_.eigenvects()
```

$$\left[\left(-2i,\ 1,\ \left[\begin{bmatrix}-2+2i\\1\end{bmatrix}\right]\right),\ \left(2i,\ 1,\ \left[\begin{bmatrix}-2-2i\\1\end{bmatrix}\right]\right)\right]$$

- For λ_1, the system

$$(\mathbf{A}-2i\mathbf{I}|\mathbf{0}) = \left(\begin{array}{cc|c} 2-2i & 8 & 0 \\ -1 & -2-2i & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{cc|c} 1 & 2+2i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

gives $k_1 = -(2+2i)k_2$. By choosing $k_2 = -1$, we get

$$\mathbf{k}_1 = \begin{pmatrix} 2+2i \\ -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{\mathbf{b}_1} + i \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{\mathbf{b}_2}$$

- Now we form the general solution

$$\mathbf{x} = c_1 \left[\begin{pmatrix} 2 \\ -1 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right] + c_2 \left[\begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \sin 2t \right]$$

The initial condition yields $2c_1 + 2c_2 = 2$, $-c_1 = -1$ whose solution is $c_1 = 1$, $c_2 = 0$. Thus solution is

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t$$

```
from scipy.integrate import solve_ivp

def myODE(t, y):
    return [2*y[0] +8*y[1], -y[0] -2*y[1]]

tf = 20.0

sol = solve_ivp(myODE, [0, tf], [2, -1],
```

```
                t_eval = np.linspace(0, tf, 200))

fig, ax = plt.subplots(figsize=(6, 4))

ax.plot(sol.t, sol.y[0], 'r-', label = 'x')
ax.plot(sol.t, sol.y[1], 'b-', label = 'y')
ax.axis((0, tf, -4, 4))
ax.tick_params(axis='both', direction='in', pad=8)
ax.legend()
ax.set_xlabel('t')
ax.set_ylabel('x, y');
```

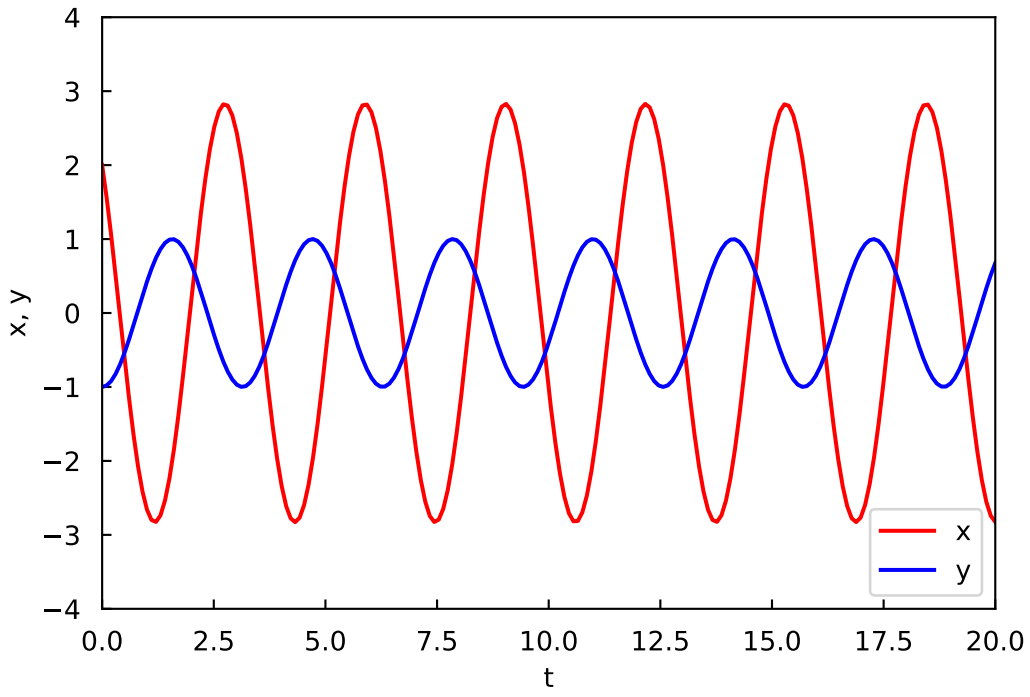


Figure 7.3: Time Series of System with Complex Eigenvalues

```
w = 6

xl = np.arange(-w, w, 0.1)
yl = np.arange(-w, w, 0.1)
x, y = np.meshgrid(xl, yl)

xdot = 2*x +8*y
ydot = -x -2*y

fig, ax = plt.subplots(figsize=(5, 5))

ax.streamplot(x, y, xdot, ydot, color='blue')
ax.plot(sol.y[0], sol.y[1], 'r:')

ax.set_title('Phase Portrait')
ax.axis((-w, w, -w, w))
```

```
ax.set_xlabel('x')
ax.set_ylabel('y')
ax.grid()
ax.tick_params(axis='both', direction='in')
```

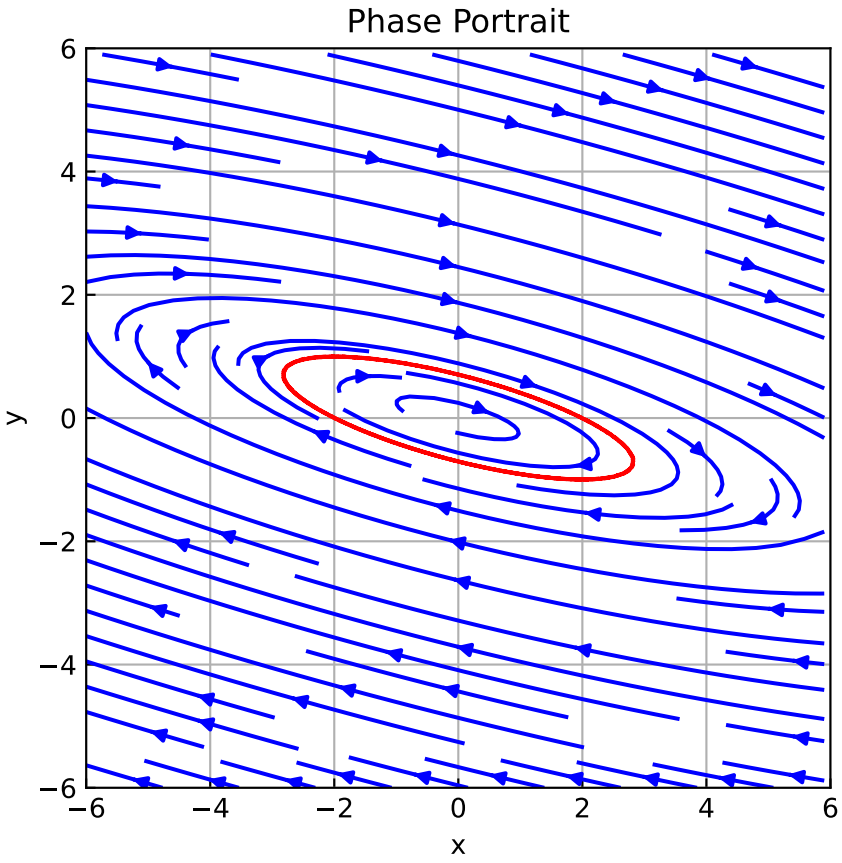


Figure 7.4: Phase Portrait of System with Complex Eigenvalues

Try it in Python: Eigenvalues and eigenvectors numerically: Eigenvalue problems; integrating linear systems with `solve_ivp`: Numerical integration of ODEs using 'scipy'

7.3 Solution by Diagonalization

- A homogeneous system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

  is said to be **coupled** if each $\dot{x}_i$ is expressed as a linear combination of $x_1, x_2, \cdots, x_n$. If $\mathbf{A}$ is diagonalizable, the system can be **decoupled** so each $\dot{x}_i$ is expressed solely in terms of x_i

- If $\mathbf{A}$ has n linearly independent eigenvectors, we can find a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ where $\mathbf{D}$ is a diagonal matrix

- Making the substitution $\mathbf{x} = \mathbf{P}\mathbf{y}$ in $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} \\ \Downarrow\ \mathbf{x} &= \mathbf{P}\mathbf{y} \\ \mathbf{P}\dot{\mathbf{y}} &= \mathbf{A}\mathbf{P}\mathbf{y} \\ \dot{\mathbf{y}} &= \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{y} = \mathbf{D}\mathbf{y} \end{aligned}$$

  The last equation is the same as

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

- The general solution of $\dot{\mathbf{y}} = \mathbf{D}\mathbf{y}$ can be written as

$$\mathbf{y} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

  Then the general solution of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is obtained from $\mathbf{x} = \mathbf{P}\mathbf{y}$

Example Solve

$$\dot{\mathbf{x}} = \begin{pmatrix} -2 & -1 & 8 \\ 0 & -3 & 8 \\ 0 & -4 & 9 \end{pmatrix} \mathbf{x}$$

- We begin by finding the eigenvalues and corresponding eigenvectors of the coefficient matrix. From $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, we get $\lambda_1 = -2$, $\lambda_2 = 1$, and $\lambda_3 = 5$. Since the eigenvalues are distinct, the eigenvectors are linearly independent. Solving $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{k} = \mathbf{0}$ for $i = 1, 2$, and 3 gives, respectively,

$$\mathbf{k}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{k}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{k}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- Thus a matrix that diagonalizes $\mathbf{A}$ is

$$\mathbf{P} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

  The substitution $\mathbf{x} = \mathbf{P}\mathbf{y}$ in $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ gives the decoupled system $\dot{\mathbf{y}} = \mathbf{D}\mathbf{y}$. The entries on the main diagonal of $\mathbf{D}$ are the eigenvalues of $\mathbf{A}$ corresponding to the order in which the eigenvectors appear in $\mathbf{P}$. The general solution of the decoupled system is

$$\mathbf{y} = \begin{pmatrix} c_1 e^{-2t} \\ c_2 e^{t} \\ c_3 e^{5t} \end{pmatrix}$$

- Hence the solution of the given system is

$$\begin{aligned} \mathbf{x} = \mathbf{P}\mathbf{y} &= \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-2t} \\ c_2 e^{t} \\ c_3 e^{5t} \end{pmatrix} \\ &= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{5t} \end{aligned}$$

7.4 Nonhomogeneous Linear Systems

- The general solution of a nonhomogeneous linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$ is $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$ where $\mathbf{x}_h$ is the complementary function of the associated homogeneous linear system and $\mathbf{x}_p$ is any particular solution of the nonhomogeneous system

7.4.1 Method of undetermined coefficients

- The **method of undetermined coefficients** consists of making an educated guess about the form of a particular solution vector $\mathbf{x}_p$. The matrix version of undetermined coefficients is only applicable when the entries of $\mathbf{A}$ are constants and the entries of $\mathbf{f}(t)$ are constants, polynomials, exponential functions, sines and cosines, or finite sums and products of these functions

Example Solve

$$\dot{\mathbf{x}} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -8 \\ 3 \end{pmatrix}$$

- We first solve the associated homogeneous system

$$\dot{\mathbf{x}} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \mathbf{x}$$

The characteristic equation of $\mathbf{A}$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -1-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

yields the complex eigenvalues $\lambda_1 = i$ and $\lambda_2 = \bar{\lambda}_1 = -i$

- For $\lambda_1 = i$,

$$(\mathbf{A} - i\mathbf{I}|\mathbf{0}) = \left(\begin{array}{cc|c} -1-i & 2 & 0 \\ -1 & 1-i & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{cc|c} 1 & -1+i & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

gives an eigenvector

$$\mathbf{k}_1 = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$$

Now we build the general solution of the associated homogeneous system

$$\mathbf{x}_h = c_1 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin t \right] + c_2 \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin t \right]$$

- Since $\mathbf{f}$ is a constant vector, we assume a constant particular solution vector

$$\mathbf{x}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

$$\mathbf{0} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} -8 \\ 3 \end{pmatrix}$$

Solving this algebraic system gives

$$\mathbf{x}_p = \begin{pmatrix} 14 \\ 11 \end{pmatrix}$$

Example Solve

$$\dot{\mathbf{x}} = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 6t \\ -10t + 4 \end{pmatrix}$$

- The eigenvalues and corresponding eigenvectors of the associated homogeneous system are found to be $\lambda_1 = 2$, $\lambda_2 = 7$,

$$\mathbf{k}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \text{ and } \mathbf{k}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hence the complementary function is

$$\mathbf{x}_h = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}$$

- Now we shall try to find the particular solution of the form

$$\mathbf{x}_p = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

Substituting this assumption into the given system and solving the resultant algebraic equations yields

$$\mathbf{x}_p = \begin{pmatrix} -2 \\ 6 \end{pmatrix} t + \frac{1}{7} \begin{pmatrix} -4 \\ 10 \end{pmatrix}$$

Example Determine the form of a particular solution vector $\mathbf{x}_p$, for the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2e^{-t} + 1 \\ e^{-t} - 5t + 7 \end{pmatrix}$$

- Because $\mathbf{f}$ can be written in matrix terms as

$$\mathbf{f} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ -5 \end{pmatrix} t + \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

a natural assumption for a particular solution would be

$$\mathbf{x}_p = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} e^{-t} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

- The method of undetermined coefficients for linear systems is not as straightforward as the last three examples would seem to indicate. But there are further difficulties
- For example, in 2×2 system, if $\mathbf{f}(t)$ is a constant vector and $\lambda = 0$ is an eigenvalue of multiplicity one, then the complementary function $\mathbf{x}_h$ contains a constant vector. We would ordinarily try a particular solution of the form

$$\mathbf{x}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} t$$

This is not the proper assumption for linear systems; it should be

$$\mathbf{x}_p = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

- Similarly, in the last Example, if we replace e^{-t} in $\mathbf{f}(t)$ by e^{2t} ($\lambda = 2$ is an eigenvalue), then the correct form of the particular solution vector is

$$\mathbf{x}_p = \begin{pmatrix} a_4 \\ b_4 \end{pmatrix} te^{2t} + \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} e^{2t} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

7.4.2 Method of Variation of Parameters

- **A Fundamental Matrix**

  If $\mathbf{x}_1$, $\mathbf{x}_2$, $\cdots$, $\mathbf{x}_n$ is a fundamental set of solutions of the homogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, then its general solution is the linear combination $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$ or

$$\mathbf{x} = c_1 \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} + c_2 \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} + \cdots + c_n \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

- In other words, the general solution can be written as the product

$$\mathbf{x} = \Phi(t)\mathbf{c}$$

  where $\mathbf{c}$ is the column vector of constants and Φ is the **fundamental matrix**

- We need to know two properties of a fundamental matrix:
  - A fundamental matrix $\Phi(t)$ is nonsingular
  - If $\Phi(t)$ is a fundamental matrix of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, then

$$\dot{\Phi}(t) = \mathbf{A}\Phi(t)$$

- **Variation of Parameters**

  We ask whether it is possible to replace $\mathbf{c}$ by

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix} \quad \text{so that} \quad \mathbf{x}_p = \Phi(t)\mathbf{u}(t)$$

  is a particular solution of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{f}(t) \\ &\Downarrow \ \mathbf{x}_p = \Phi(t)\mathbf{u}(t) \\ \dot{\Phi}(t)\mathbf{u}(t) + \Phi(t)\dot{\mathbf{u}}(t) &= \mathbf{A}\Phi(t)\mathbf{u}(t) + \mathbf{f}(t) \\ &\Downarrow \ \dot{\Phi}(t) = \mathbf{A}\Phi(t) \\ \Phi(t)\dot{\mathbf{u}}(t) &= \mathbf{f}(t) \\ &\Downarrow \\ \mathbf{u}(t) &= \int \Phi^{-1}(t)\mathbf{f}(t)\, dt + \mathbf{c}_0 \\ &\Downarrow \\ \mathbf{x}_p &= \Phi(t) \int \Phi^{-1}(t)\mathbf{f}(t)\, dt + \Phi(t)\mathbf{c}_0 \end{aligned}$$

- Thus the general solution is

$$\mathbf{x} = \Phi(t)\bar{\mathbf{c}} + \Phi(t)\int \Phi^{-1}(t)\mathbf{f}(t)\,dt$$

  where $\bar{\mathbf{c}} = \mathbf{c} + \mathbf{c}_0$

- If we set $\mathbf{x}_0 = \Phi(t_0)\bar{\mathbf{c}}$ with an initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, the solution of the initial value problem is

$$\mathbf{x} = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)\,ds$$

Example Find the general solution of the nonhomogeneous system

$$\dot{\mathbf{x}} = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}$$

- The solution vectors of the homogeneous system are then

$$\mathbf{x}_1 = \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix} \text{ and } \mathbf{x}_2 = \begin{pmatrix} e^{-5t} \\ -2e^{-5t} \end{pmatrix}$$

- Hence

$$\Phi(t) = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix} \text{ and } \Phi^{-1}(t) = \frac{1}{3}\begin{pmatrix} 2e^{2t} & e^{2t} \\ e^{5t} & -e^{5t} \end{pmatrix}$$

  Then we obtain

$$\begin{aligned} \mathbf{x}_p &= \Phi(t)\int \Phi^{-1}(t)\mathbf{f}(t)\,dt \\ &= \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\int \frac{1}{3}\begin{pmatrix} 2e^{2t} & e^{2t} \\ e^{5t} & -e^{5t} \end{pmatrix}\begin{pmatrix} 3t \\ e^{-t} \end{pmatrix} dt \\ &= \begin{pmatrix} \frac{6}{5}t - \frac{27}{50} + \frac{1}{4}e^{-t} \\ \frac{3}{5}t - \frac{21}{50} + \frac{1}{2}e^{-t} \end{pmatrix} \end{aligned}$$

7.4.3 Diagonalization

- Suppose $\mathbf{P}$ is the matrix such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix. Substituting $\mathbf{x} = \mathbf{Py}$ into the nonhomogeneous system gives

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{f} \\ &\Downarrow \mathbf{x} = \mathbf{Py} \\ \mathbf{P}\dot{\mathbf{y}} &= \mathbf{APy} + \mathbf{f} \\ \dot{\mathbf{y}} &= \mathbf{P}^{-1}\mathbf{APy} + \mathbf{P}^{-1}\mathbf{f} \\ \dot{\mathbf{y}} &= \mathbf{Dy} + \mathbf{P}^{-1}\mathbf{f} \\ &\Downarrow \mathbf{g} = \mathbf{P}^{-1}\mathbf{f} \\ \dot{\mathbf{y}} &= \mathbf{Dy} + \mathbf{g} \end{aligned}$$

In the last equation, $\mathbf{g}$ is a column vector. So each differential equation has the form of $\dot{y}_i = \lambda_i y_i + g_i(t)$, $i = 1, \cdots, n$. Unlike the procedure for solving a homogeneous system, we now are required to compute the inverse of $\mathbf{P}$

7.5 Matrix Exponential

- We can define a **matrix exponential** $e^{\mathbf{A}t}$ so that $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ has a solution $\mathbf{x} = e^{\mathbf{A}t}\mathbf{c}$

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^k\frac{t^k}{k!} + \cdots = \sum_{k=0}^{\infty} \mathbf{A}^k\frac{t^k}{k!}$$

The derivative of $e^{\mathbf{A}t}$ is obtained by differentiating term by term

$$\begin{aligned}\frac{d}{dt}e^{\mathbf{A}t} &= \frac{d}{dt}\left[\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^k\frac{t^k}{k!} + \cdots\right] \\ &= \mathbf{A} + \mathbf{A}^2 t + \mathbf{A}^3\frac{t^2}{2!} + \cdots + \mathbf{A}^{k+1}\frac{t^k}{k!} + \cdots \\ &= \mathbf{A}\left[\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^k\frac{t^k}{k!} + \cdots\right] = \mathbf{A}e^{\mathbf{A}t}\end{aligned}$$

- Now we can prove that $\mathbf{x} = e^{\mathbf{A}t}\mathbf{c}$ is the solution of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ for every constant vector $\mathbf{c}$

$$\dot{\mathbf{x}} = \frac{d}{dt}e^{\mathbf{A}t}\mathbf{c} = \mathbf{A}e^{\mathbf{A}t}\mathbf{c} = \mathbf{A}\left(e^{\mathbf{A}t}\mathbf{c}\right) = \mathbf{A}\mathbf{x}$$

- If we denote $e^{\mathbf{A}t}$ by the symbol $\Psi(t)$, then $\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}$ is equivalent to $\dot{\Psi}(t) = \mathbf{A}\Psi(t)$. In addition, $\Psi(0) = e^{\mathbf{A}0} = \mathbf{I}$ and so $\det\Psi(0) \neq 0$. These two properties are sufficient to conclude that $\Psi(t)$ is a **fundamental matrix** of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Then it is always nonsingular and $\left(e^{\mathbf{A}t}\right)^{-1} = e^{-\mathbf{A}t}$

- For a nonhomogeneous system of linear first-order DEs, the general solution is

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p = e^{\mathbf{A}t}\mathbf{c} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{f}(s)\,ds$$

- **Computation of $e^{\mathbf{A}t}$:** Using the Laplace Transform

  $\mathbf{x} = e^{\mathbf{A}t}$ is a solution of the initial value problem

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{I}$$

If $\mathbf{x}(s) = \mathcal{L}\{e^{\mathbf{A}t}\}$, then the Laplace transform of the above initial value problem is

$$\begin{aligned} s\mathbf{x}(s) - \mathbf{I} &= \mathbf{A}\mathbf{x}(s) \\ (s\mathbf{I} - \mathbf{A})\mathbf{x}(s) &= \mathbf{I} \\ \mathbf{x}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\end{aligned}$$

In other words,

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}$$

- **Computation of $e^{\mathbf{A}t}$:** Using Powers $\mathbf{A}^m$

  For computing an arbitrary power $\mathbf{A}^k$, k a nonnegative integer, we can write

$$\mathbf{A}^k = \sum_{j=0}^{n-1} c_j(k)\mathbf{A}^j \text{ and } \lambda^k = \sum_{j=0}^{n-1} c_j(k)\lambda^j$$

  where the last expression is valid for the eigenvalues λ_1, λ_2, $\cdots$, λ_n. If the eigenvalues are distinct, by setting $\lambda = \lambda_1, \lambda_2, \cdots, \lambda_n$ in the second expression, we were able to find $c_j(k)$ in the first expression by solving n equations in n unknowns. From the definition of $e^{\mathbf{A}t}$, we have

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!} \text{ and } e^{\lambda t} = \sum_{k=0}^{\infty} \lambda^k \frac{t^k}{k!}$$

  We next use these to replace $\mathbf{A}^k$ and λ^k as finite sums followed by an interchange of the order of summations

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\sum_{j=0}^{n-1} c_j(k)\mathbf{A}^j \right) = \sum_{j=0}^{n-1} \mathbf{A}^j \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} c_j(k) \right) = \sum_{j=0}^{n-1} \mathbf{A}^j b_j$$
$$e^{\lambda t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\sum_{j=0}^{n-1} c_j(k)\lambda^j \right) = \sum_{j=0}^{n-1} \lambda^j \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} c_j(k) \right) = \sum_{j=0}^{n-1} \lambda^j b_j$$

  Analogous to how we used the eigenvalues of $\mathbf{A}$ to determine the c_j, we again use the eigenvalues to solve a system of equations to determine the b_j

Example Compute $e^{\mathbf{A}t}$ for

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$$

- First we compute the matrix $s\mathbf{I} - \mathbf{A}$ and then find its inverse

$$s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s-1 & 1 \\ -2 & s+2 \end{pmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} \frac{s+2}{s(s+1)} & \frac{-1}{s(s+1)} \\ \frac{2}{s(s+1)} & \frac{s-1}{s(s+1)} \end{pmatrix}$$

- Then we decompose the entries of the last matrix into partial fractions

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} \frac{2}{s} - \frac{1}{s+1} & -\frac{1}{s} + \frac{1}{s+1} \\ \frac{2}{s} - \frac{2}{s+1} & -\frac{1}{s} + \frac{2}{s+1} \end{pmatrix}$$

  Taking the inverse Laplace transform gives the desired result

$$e^{\mathbf{A}t} = \begin{pmatrix} 2 - e^{-t} & -1 + e^{-t} \\ 2 - 2e^{-t} & -1 + 2e^{-t} \end{pmatrix}$$

```
A = sympy.Matrix([[1, -1], [2, -2]])
s = sympy.symbols('s')
t = sympy.symbols('t', real=True, positive=True)

lp_sol = (s*sympy.eye(2) -A).inv()
sympy.inverse_laplace_transform(lp_sol, s, t)
```

$$\begin{bmatrix} 2 - e^{-t} & -1 + e^{-t} \\ 2 - 2e^{-t} & -1 + 2e^{-t} \end{bmatrix}$$

Example Compute $e^{\mathbf{A}t}$ for

$$\mathbf{A} = \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}$$

- The eigenvalues of $\mathbf{A}$ are $\lambda_1 = -1$ and $\lambda_2 = 2$. Now since $\mathbf{A}$ is a 2×2 matrix, we have

  $$e^{\mathbf{A}t} = b_0\mathbf{I} + b_1\mathbf{A} \quad \text{and} \quad e^{\lambda t} = b_0 + b_1\lambda$$

  Setting $\lambda = -1$ and $\lambda = 2$ in the second equation gives two equations in the two unknowns b_0 and b_1. Solving the system yields

  $$b_0 = \frac{1}{3}\left[e^{2t} + 2e^{-t}\right], \; b_1 = \frac{1}{3}\left[e^{2t} - e^{-t}\right]$$

  Substituting these values in the first equation and simplifying the entries yields

  $$e^{\mathbf{A}t} = \begin{pmatrix} -\frac{1}{3}e^{2t} + \frac{4}{3}e^{-t} & \frac{4}{3}e^{2t} - \frac{4}{3}e^{-t} \\ -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t} & \frac{4}{3}e^{2t} - \frac{1}{3}e^{-t} \end{pmatrix}$$

```
A = sympy.Matrix([[-2, 4], [-1, 3]])
t = sympy.symbols('t', real = True, positive = True)
sympy.exp(A*t)
```

$$\begin{bmatrix} -\frac{e^{2t}}{3} + \frac{4e^{-t}}{3} & \frac{4e^{2t}}{3} - \frac{4e^{-t}}{3} \\ -\frac{e^{2t}}{3} + \frac{e^{-t}}{3} & \frac{4e^{2t}}{3} - \frac{e^{-t}}{3} \end{bmatrix}$$

Summary

Topic	Result
General solution	$\mathbf{x} = \Phi(t)\mathbf{c} + \mathbf{x}_p$, with the fundamental matrix Φ ($\dot{\Phi} = \mathbf{A}\Phi$)
Distinct eigenvalues	$\mathbf{x}_i = \mathbf{k}_i e^{\lambda_i t}$
Repeated eigenvalue	$\mathbf{x}_2 = \mathbf{k}_{11} t e^{\lambda t} + \mathbf{k}_{12} e^{\lambda t}$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{k}_{12} = \mathbf{k}_{11}$
Complex eigenvalues $\alpha \pm i\beta$	$\mathbf{b}_1 = \operatorname{Re}\mathbf{k}_1$, $\mathbf{b}_2 = \operatorname{Im}\mathbf{k}_1$: $\mathbf{x}_1 = [\mathbf{b}_1 \cos\beta t - \mathbf{b}_2 \sin\beta t]e^{\alpha t}$, $\mathbf{x}_2 = [\mathbf{b}_2 \cos\beta t + \mathbf{b}_1 \sin\beta t]e^{\alpha t}$
Variation of parameters	$\mathbf{x}_p = \Phi(t) \int \Phi^{-1}(t)\mathbf{f}(t)\, dt$
Initial-value problem	$\mathbf{x} = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0 + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)\, ds$
Diagonalization	$\mathbf{x} = \mathbf{P}\mathbf{y}$ uncouples the system: $\dot{\mathbf{y}} = \mathbf{D}\mathbf{y} + \mathbf{P}^{-1}\mathbf{f}$
Matrix exponential	$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \dfrac{\mathbf{A}^2 t^2}{2!} + \cdots = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}$, $\mathbf{x} = e^{\mathbf{A}t}\mathbf{c}$

Worked Exercises

1. Compute $e^{\mathbf{A}t}$ for the matrix

$$\begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

and then use the matrix exponential to solve the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

Solution A

Step 1: Characteristic Polynomial

$$\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda^3 = 0$$

So the eigenvalue is $\lambda = 0$ with algebraic multiplicity 3. This means $\mathbf{A}$ is nilpotent (some power of $\mathbf{A}$ equals zero), and not diagonalizable

Step 2: Matrix Exponential

Since $\mathbf{A}$ is nilpotent of index 3 (i.e., $\mathbf{A}^3 = \mathbf{0}$), the matrix exponential can be computed using the truncated power series:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2$$

We compute:

$$\mathbf{A}^2 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Then:

$$e^{\mathbf{A}t} = \begin{pmatrix} 1 - t - \frac{t^2}{2} & t & t + \frac{t^2}{2} \\ -t & 1 & t \\ -t - \frac{t^2}{2} & t & 1 + t + \frac{t^2}{2} \end{pmatrix}$$

Step 3: Final Solution

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 = \begin{pmatrix} 1 - t - \frac{t^2}{2} & t & t + \frac{t^2}{2} \\ -t & 1 & t \\ -t - \frac{t^2}{2} & t & 1 + t + \frac{t^2}{2} \end{pmatrix} \mathbf{x}_0$$

Solution B

Step 1: Compute $s\mathbf{I} - \mathbf{A}$

$$s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s+1 & -1 & -1 \\ 1 & s & -1 \\ 1 & -1 & s-1 \end{pmatrix}$$

Step 2: Inverse of $s\mathbf{I} - \mathbf{A}$

After computing the inverse, we obtain:

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^3} \begin{pmatrix} s^2 - s - 1 & s & s+1 \\ -s & s^2 & s \\ -s - 1 & s & s^2 + s + 1 \end{pmatrix}$$

Step 3: Inverse Laplace Transform

So the time-domain solution becomes:

$$e^{\mathbf{A}t} = \begin{pmatrix} 1 - t - \frac{t^2}{2} & t & t + \frac{t^2}{2} \\ -t & 1 & t \\ -t - \frac{t^2}{2} & t & 1 + t + \frac{t^2}{2} \end{pmatrix}$$

Step 3: Final Solution

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 = \begin{pmatrix} 1 - t - \frac{t^2}{2} & t & t + \frac{t^2}{2} \\ -t & 1 & t \\ t - \frac{t^2}{2} & t & 1 + t + \frac{t^2}{2} \end{pmatrix} \mathbf{x}_0$$

2. Solve the given system

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ e^{2t} \\ te^{3t} \end{pmatrix}$$

Solution

We are given the nonhomogeneous system of differential equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(t), \quad \text{where} \quad \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} e^t \\ e^{2t} \\ te^{3t} \end{pmatrix}$$

Step 1: Solve the Homogeneous System

We first solve:

$$\dot{\mathbf{x}}_h = \mathbf{A}\mathbf{x}_h$$

Let's find the general solution to the homogeneous system using eigenvalues and eigenvectors of $\mathbf{A}$

- Characteristic Polynomial

  We compute the determinant of this matrix:

  $$\det(\mathbf{A} - \lambda\mathbf{I}) = (3-\lambda) \cdot \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda(\lambda-2)(\lambda-3) = 0$$

  Thus, the eigenvalues are:

  $$\lambda_1 = 0,\ \lambda_2 = 2,\ \lambda_3 = 3$$

- Eigenvectors

  For $\lambda_1 = 0$: Solve $(\mathbf{A} - 0\mathbf{I})\mathbf{v}_1 = \mathbf{0}$:

  $$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{v}_1 = \mathbf{0} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

  For $\lambda_2 = 2$: Solve $(\mathbf{A} - 2\mathbf{I})\mathbf{v}_2 = \mathbf{0}$:

  $$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{v}_2 = \mathbf{0} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

  For $\lambda = 3$:

  Solve $(\mathbf{A} - 3\mathbf{I})\mathbf{v}_3 = \mathbf{0}$:

  $$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{v}_3 = \mathbf{0} \Rightarrow \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So, the homogeneous solution is:

$$\mathbf{x}_h(t) = c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{3t}$$

Step 2: Find a Particular Solution

So the fundamental matrix is:

$$\Phi(t) = \begin{pmatrix} -1 & e^{2t} & 0 \\ 1 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$$

Then

$$\Phi(t)^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix}$$

We compute:

$$\mathbf{x}_p(t) = \Phi(t) \int \Phi(t)^{-1}\mathbf{f}(t)\, dt$$

Compute $\Phi(t)^{-1}\mathbf{f}(t)$:

$$\Phi(t)^{-1}\mathbf{f}(t) = \begin{pmatrix} -\frac{1}{2}e^{t} + \frac{1}{2}e^{2t} \\ \frac{1}{2}e^{-t} + \frac{1}{2} \\ t \end{pmatrix}$$

Now integrate:

$$\int \Phi(t)^{-1}\mathbf{f}(t)\, dt = \begin{pmatrix} -\frac{1}{2}e^{t} + \frac{1}{4}e^{2t} \\ \frac{1}{2}t - \frac{1}{2}e^{-t} \\ \frac{1}{2}t^2 \end{pmatrix}$$

Let's simplify and write the particular solution in general form:

$$\mathbf{x}_p(t) = \Phi(t) \int \Phi(t)^{-1}\mathbf{f}(t)\, dt = \begin{pmatrix} -\frac{1}{4}e^{2t} + \frac{1}{2}te^{2t} \\ -e^{t} + \frac{1}{4}e^{2t} + \frac{1}{2}te^{2t} \\ \frac{1}{2}t^2e^{3t} \end{pmatrix}$$

Step 3: Final General Solution

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$$

3. Use diagonalization to solve the given system

$$\mathbf{x}' = \begin{pmatrix} 1 & \frac{1}{4} \\ 1 & 1 \end{pmatrix} \mathbf{x}$$

Solution

Step 1: Find eigenvalues

Solve $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$:

$$\det \begin{pmatrix} 1-\lambda & \frac{1}{4} \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - \frac{1}{4} = 0$$

$$(1-\lambda)^2 = \frac{1}{4} \Rightarrow 1-\lambda = \pm\frac{1}{2} \Rightarrow \lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{3}{2}$$

Step 2: Find eigenvectors

For $\lambda_1 = \frac{1}{2}$:

Solve $(A - \frac{1}{2}I)\mathbf{v} = 0$ and so one eigenvector is:

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \quad \text{(or any scalar multiple)}$$

For $\lambda_2 = \frac{3}{2}$:

So one eigenvector is:

$$\mathbf{v}_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

Step 3: Form matrix $\mathbf{P}$ and diagonal $\mathbf{D}$

Let:

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

Then:

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

Step 4: Solve system using diagonalization

Let $\mathbf{x} = \mathbf{P}\mathbf{y}$, then:

$$\mathbf{x}\text{'} = \mathbf{A}\mathbf{x} \Rightarrow \mathbf{P}\mathbf{y}\text{'} = \mathbf{A}\mathbf{P}\mathbf{y} \Rightarrow \mathbf{y}\text{'} = \mathbf{D}\mathbf{y}$$

So the system becomes:

$$\mathbf{y}' = \mathbf{D}\mathbf{y} \Rightarrow \begin{cases} y_1' = \frac{1}{2}y_1 \\ y_2' = \frac{3}{2}y_2 \end{cases} \Rightarrow \begin{cases} y_1(t) = c_1 e^{\frac{1}{2}t} \\ y_2(t) = c_2 e^{\frac{3}{2}t} \end{cases}$$

Then:

$$\mathbf{x}(t) = \mathbf{P}\mathbf{y}(t) = c_1 e^{\frac{1}{2}t} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} + c_2 e^{\frac{3}{2}t} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

4. Use diagonalization to solve the given system

$$\mathbf{x}' = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \mathbf{x}$$

Solution

Step 1: Find eigenvalues of $\mathbf{A}$

We solve the characteristic polynomial:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

The determinant is:

$$\begin{vmatrix} -\lambda & 2 & 0 \\ 2 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} = -\lambda \cdot \begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 2 \\ 0 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 4) + 4\lambda = -\lambda^3 + 8\lambda$$

Set equal to 0:

$$-\lambda^3 + 8\lambda = 0 \Rightarrow \lambda(-\lambda^2 + 8) = 0 \Rightarrow \lambda = 0,\ \pm 2\sqrt{2}$$

Step 2: Find eigenvectors

- For $\lambda = 0$:

  From the system:

  - $2y = 0 \Rightarrow y = 0$
  - Then $2x + 2z = 0 \Rightarrow x + z = 0 \Rightarrow z = -x$

  So eigenvector:

$$\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

- For $\lambda = 2\sqrt{2}$:

  Solve $(\mathbf{A} - 2\sqrt{2}\mathbf{I})\mathbf{v} = \mathbf{0}$

  So:

$$\mathbf{v}_+ = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

- For $\lambda = -2\sqrt{2}$:

$$\mathbf{v}_{-} = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Step 3: Construct the matrices **P** and **D**

We construct matrix P using the eigenvectors as columns:

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ \sqrt{2} & -\sqrt{2} & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

Then the diagonal matrix is:

$$\mathbf{D} = \begin{pmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & -2\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, we have:

$$\mathbf{A} = \mathbf{PDP}^{-1}$$

Step 4: Solve system using diagonalization

Let $\mathbf{x} = \mathbf{Py}$, then:

$$\mathbf{x}\text{'} = \mathbf{Ax} \Rightarrow \mathbf{Py}\text{'} = \mathbf{APy} \Rightarrow \mathbf{y}\text{'} = \mathbf{Dy}$$

So the system becomes:

$$\begin{cases} y_1\text{'}(t) = 2\sqrt{2}y_1(t) \\ y_2\text{'}(t) = -2\sqrt{2}y_2(t) \\ y_3\text{'}(t) = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} y_1(t) = c_1 e^{2\sqrt{2}t} \\ y_2(t) = c_2 e^{-2\sqrt{2}t} \\ y_3(t) = c_3 \end{cases}$$

Step 5: Final general solution

Now substitute back:

$$\mathbf{x}(t) = P\mathbf{y}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 e^{2\sqrt{2}t} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} + c_3 e^{-2\sqrt{2}t} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

5. Compute $e^{\mathbf{A}t}$ for the coefficient matrix and find the general solution of the given system

$$\mathbf{x}' = \begin{pmatrix} 2 & 8 \\ 0 & 4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 16t \end{pmatrix}$$

Solution

Step 1: Solve the homogeneous system

We begin by solving the homogeneous system:

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

We need to compute $e^{\mathbf{A}t}$

Step 1.1: Diagonalize $\mathbf{A}$

$A = \begin{pmatrix} 2 & 8 \\ 0 & 4 \end{pmatrix}$ is upper triangular, so eigenvalues are on the diagonal:

$$\lambda_1 = 2,\ \lambda_2 = 4$$

Eigenvector for $\lambda = 2$:

$$(\mathbf{A} - 2\mathbf{I}) = \begin{pmatrix} 0 & 8 \\ 0 & 2 \end{pmatrix} \Rightarrow v^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Eigenvector for $\lambda = 4$:

$$(\mathbf{A} - 4\mathbf{I}) = \begin{pmatrix} -2 & 8 \\ 0 & 0 \end{pmatrix} \Rightarrow -2x + 8y = 0 \Rightarrow v^{(2)} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Construct $\mathbf{P}$ and $\mathbf{D}$:

$$\mathbf{P} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \quad \mathbf{P}^{-1} = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$$

Step 1.2: Compute the matrix exponential

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{pmatrix} \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$$

Multiply:

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{2t} & 4e^{2t}(e^{2t} - 1) \\ 0 & e^{4t} \end{pmatrix}$$

Step 2: Particular solution by variation of parameters

The general solution is:

$$\mathbf{x}(t) = e^{At}\mathbf{c} + \mathbf{x}_p(t)$$

where:

$$\mathbf{x}_p(t) = e^{At} \int_0^t e^{-As}\mathbf{f}(s)\, ds$$

Step 2.1: Compute $e^{-\mathbf{A}s}\mathbf{f}(s)$

$$e^{-\mathbf{A}s}\mathbf{f}(s) = \begin{pmatrix} 2e^{-2s} - 64s(e^{2s}-1)e^{-4s} \\ 16se^{-4s} \end{pmatrix}$$

Now integrate each component:

Step 2.2: Compute the integrals

From direct integration:

$$\int_0^t (2e^{-2s} - 64s(e^{2s}-1)e^{-4s})\, ds = (32t+15)e^{-2t} - (16t+4)e^{-4t} - 11$$

$$\int_0^t 16se^{-4s} ds = (-4t-1)e^{-4t} + 1$$

Step 2.3: Multiply by $e^{\mathbf{A}t}$

Now compute:

$$\mathbf{x}_p(t) = e^{\mathbf{A}t} \begin{pmatrix} (32t+15)e^{-2t} - (16t+4)e^{-4t} - 11 \\ (-4t-1)e^{-4t} + 1 \end{pmatrix}$$

After simplification, the final particular solution is:

$$\mathbf{x}_p(t) = \begin{pmatrix} 16t + 11 - 15e^{2t} + 4e^{4t} \\ e^{4t} - 4t - 1 \end{pmatrix}$$

Final General Solution

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{c} + \begin{pmatrix} 16t + 11 - 15e^{2t} + 4e^{4t} \\ e^{4t} - 4t - 1 \end{pmatrix}$$

where:

$$e^{\mathbf{A}t}\mathbf{c} = \begin{pmatrix} e^{2t} & 4e^{2t}(e^{2t}-1) \\ 0 & e^{4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

6. Compute $e^{\mathbf{A}t}$ for the coefficient matrix and find the general solution of the given system

$$\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t}$$

Solution

Step 1: Solve the homogeneous system

Step 1.1: Find eigenvalues of $\mathbf{A}$

Compute the characteristic polynomial:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) + 1 = (\lambda - 2)^2$$

So, $\lambda = 2$ is a repeated eigenvalue

Step 1.2: Find generalized eigenvectors

Eigenvector for $\lambda = 2$:

$$(\mathbf{A} - 2\mathbf{I}) = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Generalized eigenvector:

We solve $(\mathbf{A} - 2\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Step 1.3: Matrix exponential for defective matrix

If $\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}$ with Jordan form $\mathbf{J}$, then:

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{J}t}\mathbf{P}^{-1}$$

We use:

$$\mathbf{J} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad e^{\mathbf{J}t} = e^{2t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

We form:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Now compute:

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{J}t}\mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = e^{2t} \begin{pmatrix} 1+t & t \\ -t & -t+1 \end{pmatrix}$$

The inverse of the matrix $e^{\mathbf{A}t}$ is:

$$e^{-2t} \begin{pmatrix} 1-t & -t \\ t & t+1 \end{pmatrix}$$

Step 2: Variation of parameters

The general solution is:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{c} + \mathbf{x}_p(t), \quad \mathbf{x}_p(t) = e^{\mathbf{A}t} \int_0^t e^{-\mathbf{A}s}\mathbf{f}(s)\, ds$$

The particular solution is:

$$\mathbf{x}_p(t) = e^{2t} \begin{pmatrix} -\dfrac{t(t+4)}{2} \\ \dfrac{t(t+2)}{2} \end{pmatrix}$$

Answers

Answer 1 (Section 7.2)

Attractor: $\mathbf{x} = c_1 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-7t/2}$

Repeller: $\mathbf{x} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

8 Systems of Nonlinear Differential Equations

Learning objectives After this chapter you should be able to

- write a higher-order equation as a plane autonomous system and find its critical points
- classify the critical point of a linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ from $\tau = \operatorname{tr}\mathbf{A}$ and $\Delta = \det\mathbf{A}$
- determine local stability by linearization, and use the phase-plane method when linearization is not decisive
- analyze models such as the pendulum, the bead on a wire, predator-prey and competition
- rule out periodic solutions (Bendixson, Dulac) or show that they exist (Poincaré-Bendixson)

8.1 Autonomous Systems

A system of first-order differential equations is called **autonomous** when the system can be written in the form

$$\begin{aligned}\frac{dx_1}{dt} &= g_1(x_1, x_2, \cdots, x_n)\\ \frac{dx_2}{dt} &= g_2(x_1, x_2, \cdots, x_n)\\ &\vdots\\ \frac{dx_n}{dt} &= g_n(x_1, x_2, \cdots, x_n)\end{aligned}$$

Notice that the independent variable t does not appear explicitly on the right-hand side of each differential equation

- **Second-Order DE as a System**

  Any second-order differential equation $x'' = g(x, x')$ can be written as an autonomous system. If we let $y = x'$, the second-order differential equation becomes the system of two first-order equations

$$\begin{aligned}x' &= y\\ y' &= g(x, y)\end{aligned}$$

Example The displacement angle θ for a pendulum satisfies the nonlinear second-order differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

If we let $x = \theta$ and $y = \dot{\theta}$, this second-order differential equation can be written as the autonomous system

$$\dot{x} = y, \quad \dot{y} = -\frac{g}{l}\sin x$$

- **Plane Autonomous System**

  When $n = 2$, the system is called a **plane autonomous system**, and we write the system as

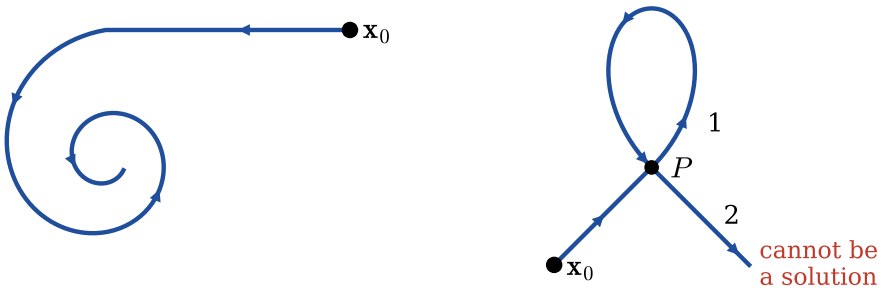
$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y)$$

  If P, Q, and the first-order partial derivatives $\partial P/\partial x$, $\partial P/\partial y$, $\partial Q/\partial x$, and $\partial Q/\partial y$ are continuous in a region R of the plane, then a solution to the plane autonomous system that satisfies $\mathbf{x}(0) = \mathbf{x}_0$ is unique and one of three basic types:

  – A **constant solution**, $\mathbf{x}(t) = \mathbf{x}_0$ for all t. A constant solution is called a **critical** or **stationary point**. Since $\mathbf{x}' = \mathbf{0}$, a critical point is a solution of the system of algebraic equations

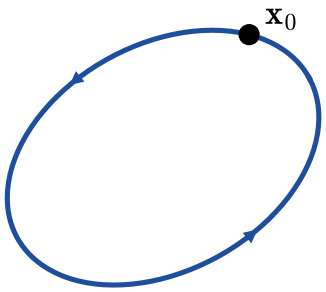
$$P(x, y) = 0$$
$$Q(x, y) = 0$$

  – An **arc**, $\mathbf{x}(t)$ - a plane curve that does not cross itself



(a) an arc (b) a curve that crosses itself

  – A **periodic solution** or **cycle**, $\mathbf{x}(t + p) = \mathbf{x}(t)$



Example Find all critical points of the following plane autonomous system

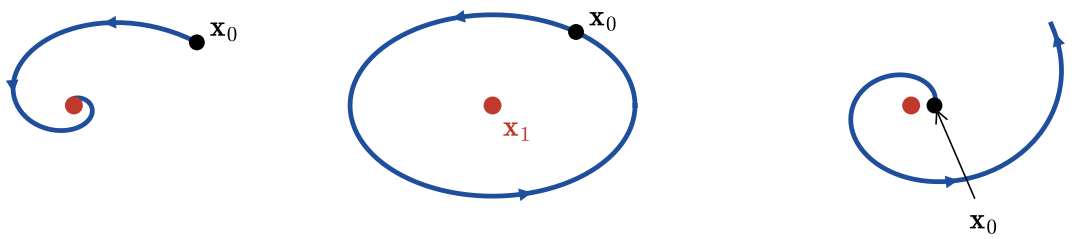
$$x' = x^2 + y^2 - 6$$
$$y' = x^2 - y$$

Answer 1 (end of chapter) · Solution

Try it in Python: Finding critical points of nonlinear systems with `scipy.optimize.fsolve`: System of nonlinear equations

8.2 Stability of Linear Systems

If $\mathbf{x}_1$ is a critical point of a plane autonomous system and $\mathbf{x} = \mathbf{x}(t)$ is a solution satisfying $\mathbf{x}(0) = \mathbf{x}_0$, when $\mathbf{x}_0$ is placed near $\mathbf{x}_1$



(a) returns to $\mathbf{x}_1$ (b) stays close to $\mathbf{x}_1$ (c) moves away from $\mathbf{x}_1$

(1) It may return to the critical point
(2) It may remain close to the critical point without returning
(3) It may move away from the critical point

- **Stability Analysis**
  - A careful geometric analysis of the solutions to the *linear* plane autonomous system

$$x' = ax + by$$
$$y' = cx + dy$$

    in terms of the eigenvalues and eigenvectors of the coefficient matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

    drives the stability analysis

  - To ensure that $\mathbf{x}_0 = (0, 0)$ is the only critical point, we will assume that the determinant $\Delta = ad - bc \neq 0$. If $\tau = a + d$ is the trace of matrix $\mathbf{A}$, then the characteristic equation, $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, can be rewritten as

$$\lambda^2 - \tau\lambda + \Delta = 0$$

Therefore the eigenvalues of $\mathbf{A}$ are

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

and the usual three cases for these roots occur according to whether $\tau^2 - 4\Delta$ is positive, negative, or zero

Example Find the eigenvalues of the linear system

$$x' = -x + y$$
$$y' = cx - y$$

in terms of c, and use a numerical solver to discover the shapes of solutions corresponding to the cases $c = \frac{1}{4}$, 4, 0, and -9. $\mathbf{A}$ has trace $\tau = -2$ and determinant $\Delta = 1 - c$, and so the eigenvalues are

$$\lambda = -1 \pm \sqrt{c}$$

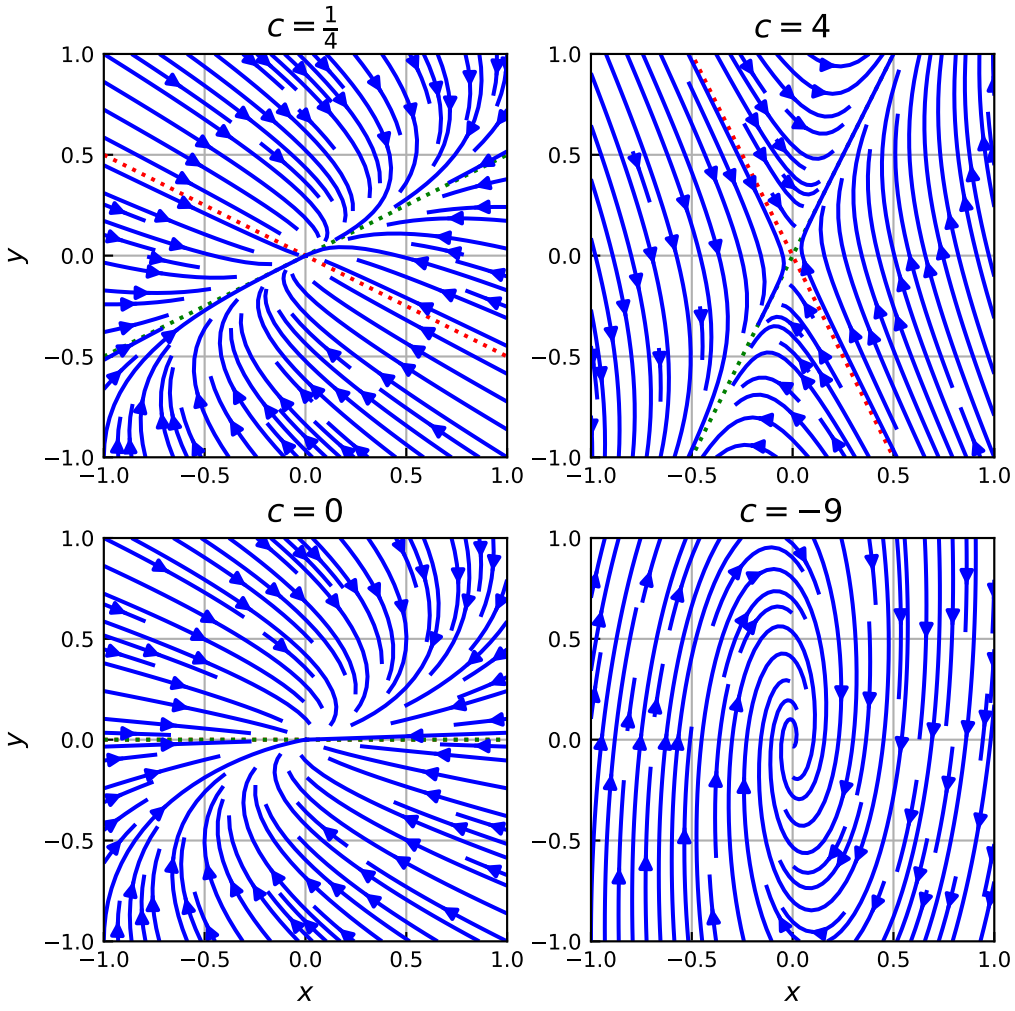
The nature of the eigenvalues is therefore determined by the value of c

```
import numpy as np
import sympy as sp
sp.init_printing(use_unicode=True)

c = sp.Symbol('c')
A = sp.Matrix([[-1, 1], [c, -1]])
A.eigenvects()
```

$$\left[\left(-\sqrt{c} - 1,\ 1,\ \left[\begin{bmatrix}\frac{-\sqrt{c}-1}{c} + \frac{1}{c}\\ 1\end{bmatrix}\right]\right),\ \left(\sqrt{c} - 1,\ 1,\ \left[\begin{bmatrix}\frac{\sqrt{c}-1}{c} + \frac{1}{c}\\ 1\end{bmatrix}\right]\right)\right]$$

```
import matplotlib.pyplot as plt

w = 1
xp = np.linspace(-w, w, 50)
yp = np.linspace(-w, w, 50)
x, y = np.meshgrid(xp, yp)

c_ = np.array([1/4, 4, 0, -9])
c_title = [r'$c=\frac{1}{4}$', r'$c=4$', r'$c=0$', r'$c=-9$']

fig = plt.figure(figsize=(6, 6))

for i in range(4):

    ax = fig.add_subplot(2, 2, i +1)

    xdot = -x +y
    ydot = c_[i]*x -y
```

```
    if c_[i] >= 0.0:
        y_1 = -np.sqrt(c_[i]) *xp
        y_2 = np.sqrt(c_[i]) *xp
        ax.plot(xp, y_1, 'r:', xp, y_2, 'g:')
    ax.streamplot(x, y, xdot, ydot, color='blue')

    ax.set_title(f'{c_title[i]}')

    ax.axis((-w, w, -w, w))
    ax.set_aspect(aspect='equal')
    ax.grid()

    ax.tick_params(axis='both', direction='in',
        labelsize=8)
    ax.xaxis.set_ticks(np.linspace(-w, w, 5))
    ax.yaxis.set_ticks(np.linspace(-w, w, 5))

    if i >= 2:
        ax.set_xlabel(r'$x$')
    if i == 0 or i == 2:
        ax.set_ylabel(r'$y$')
```



- Trajectory behaviors in phase portraits can be explained with eigenvalue-eigenvector of $\mathbf{A}$
- **Real Distinct Eigenvalues**, $\tau^2 - 4\Delta > 0$

$$\begin{aligned}\mathbf{x}(t) &= c_1\mathbf{k}_1 e^{\lambda_1 t} + c_2\mathbf{k}_2 e^{\lambda_2 t} \\ &\Big\Downarrow \lambda_1 > \lambda_2 \\ &= e^{\lambda_1 t}\left[c_1\mathbf{k}_1 + c_2\mathbf{k}_2 e^{(\lambda_2-\lambda_1)t}\right] \\ &\Big\Downarrow t\to\infty \\ &\simeq c_1\mathbf{k}_1 e^{\lambda_1 t}\end{aligned}$$

  - **Both eigenvalues negative**, $\tau^2 - 4\Delta > 0$, $\tau < 0$, $\Delta > 0$

    Stable Node: Since both eigenvalues are negative, it follows that $\lim_{t\to\infty}\mathbf{x}(t) = \mathbf{0}$ in the direction of $\mathbf{k}_1$ when $c_1 \neq 0$ or in the direction of $\mathbf{k}_2$ when $c_1 = 0$
  - **Both eigenvalues positive**, $\tau^2 - 4\Delta > 0$, $\tau > 0$, $\Delta > 0$

    Unstable Node: $\mathbf{x}(t)$ becomes unbounded in the direction of $\mathbf{k}_1$ when $c_1 \neq 0$ or in the direction of $\mathbf{k}_2$ when $c_1 = 0$
  - **Eigenvalues have opposite signs**, $\tau^2 - 4\Delta > 0$, $\Delta < 0$

    Saddle Point: When $c_1 = 0$, $\mathbf{x}(t)$ will approach $\mathbf{0}$ along the line determined by $\mathbf{k}_2$. If $\mathbf{x}(0)$ does not lie on the line determined by $\mathbf{k}_2$, the direction determined by $\mathbf{k}_1$ serves as an asymptote for $\mathbf{x}(t)$

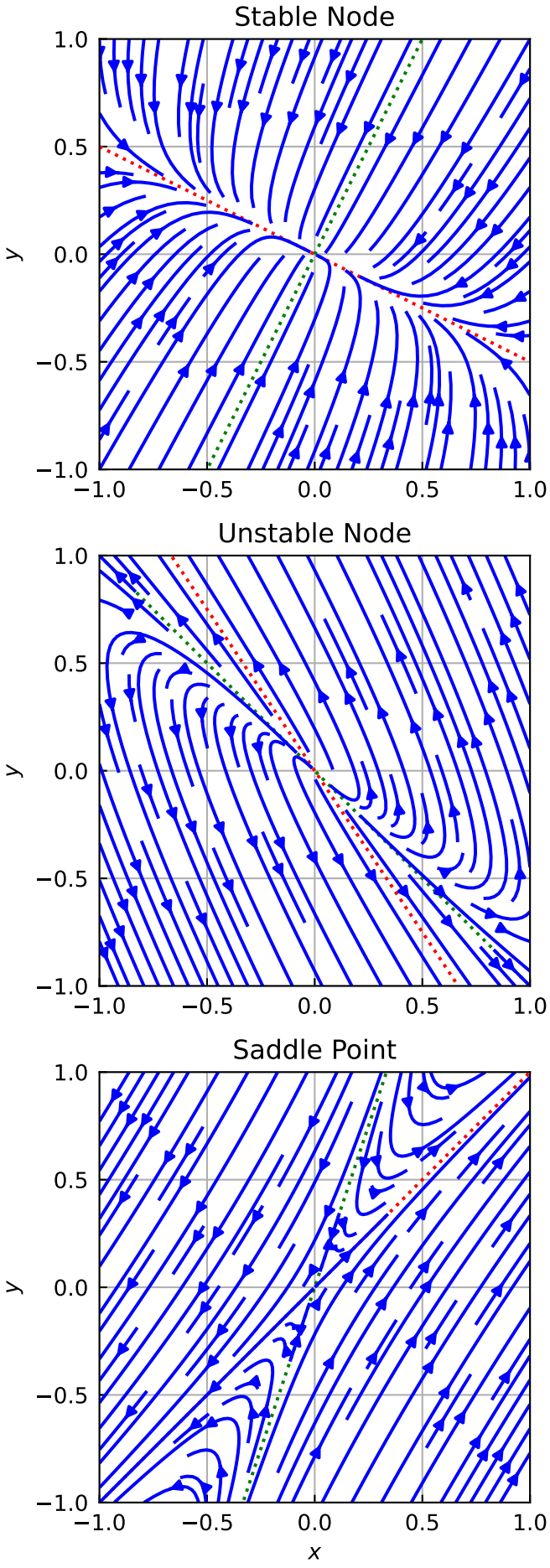
Example Classify the critical point $(0, 0)$ of each of the following linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ as either a stable node, an unstable node, or a saddle point

(a) $\begin{pmatrix}-2 & -2\\ -2 & -5\end{pmatrix}$, (b) $\begin{pmatrix}-1 & -2\\ 3 & 4\end{pmatrix}$, (c) $\begin{pmatrix}2 & -1\\ 3 & -2\end{pmatrix}$

Answer 2 (end of chapter) · Solution

```
import numpy as np

w = 1
xp = np.linspace(-w, w, 6)
yp = np.linspace(-w, w, 6)
x, y = np.meshgrid(xp, yp)

A = np.array([[[-2, -2], [-2, -5]],
              [[-1, -2], [3, 4]],
              [[2, -1], [3, -2]]])
A_title = ['Stable Node', 'Unstable Node', 'Saddle Point']

fig = plt.figure(figsize=(4, 12))

for i in range(3):
    xdot = A[i,0,0]*x +A[i,0,1]*y
    ydot = A[i,1,0]*x +A[i,1,1]*y

    lamda, v = np.linalg.eig(A[i])

    if lamda[0] >= lamda[1]:
```

```
        y_1 = v[1,0]/v[0,0]*xp
        y_2 = v[1,1]/v[0,1]*xp
    else:
        y_1 = v[1,1]/v[0,1]*xp
        y_2 = v[1,0]/v[0,0]*xp

    ax = fig.add_subplot(3, 1, i +1)

    ax.plot(xp, y_1, 'r:', xp, y_2, 'g:')
    ax.streamplot(x, y, xdot, ydot, color='blue')

    ax.set_title(A_title[i])

    ax.axis((-w, w, -w, w))
    ax.set_aspect(aspect='equal')
    ax.grid()

    ax.tick_params(axis='both', direction='in', pad=5)
    ax.xaxis.set_ticks(np.linspace(-w, w, 5))
    ax.set_ylabel(r'$y$')

    ax.yaxis.set_ticks(np.linspace(-w, w, 5))
    if i == 2:
        ax.set_xlabel(r'$x$')
```


Stable Node
Unstable Node
Saddle Point
1.0
0.5
0.0
−0.5
−1.0
y
x

- **A Repeated Real Eigenvalue**, $\tau^2 - 4\Delta = 0$

  The general solution takes on one of two different forms depending on whether **one** or **two** linearly independent eigenvectors can be found for the repeated eigenvalues λ_1

  - **Two linearly independent eigenvectors**

    If $\mathbf{k}_1$ and $\mathbf{k}_2$ are two linearly independent eigenvectors corresponding to λ_1, then the general solution is given by

    $$\mathbf{x}(t) = c_1\mathbf{k}_1 e^{\lambda_1 t} + c_2\mathbf{k}_2 e^{\lambda_1 t} = (c_1\mathbf{k}_1 + c_2\mathbf{k}_2)\, e^{\lambda_1 t}$$

    If $\lambda_1 < 0$, the $\mathbf{x}(t)$ approaches $\mathbf{0}$ along the line determined by the vector $c_1\mathbf{k}_1 + c_2\mathbf{k}_2$ and the critical point is a **degenerate stable node**. The arrows are reversed when $\lambda_1 > 0$, and the critical point is a **degenerate unstable node**

  - **A single linearly independent eigenvector**

    When only a single linearly independent eigenvector $\mathbf{k}_{11}$ exists, the general solution is given by

    $$\begin{aligned}\mathbf{x}(t) &= c_1\mathbf{k}_{11}e^{\lambda_1 t} + c_2\left(\mathbf{k}_{11}te^{\lambda_1 t} + \mathbf{k}_{12}e^{\lambda_1 t}\right)\\ &= te^{\lambda_1 t}\left[c_2\mathbf{k}_{11} + \frac{1}{t}\left(c_1\mathbf{k}_{11} + c_2\mathbf{k}_{12}\right)\right]\end{aligned}$$

    where $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{k}_{12} = \mathbf{k}_{11}$. If $\lambda_1 < 0$, then $\lim_{t\to\infty} te^{\lambda_1 t} = 0$ and it follows that $\mathbf{x}(t)$ approaches $\mathbf{0}$ in the line determined by $\mathbf{k}_{11}$. The critical point is again a **degenerate stable node**. When $\lambda_1 > 0$, $\mathbf{x}(t)$ becomes unbounded as t increases, and the critical point is a **degenerate unstable node**

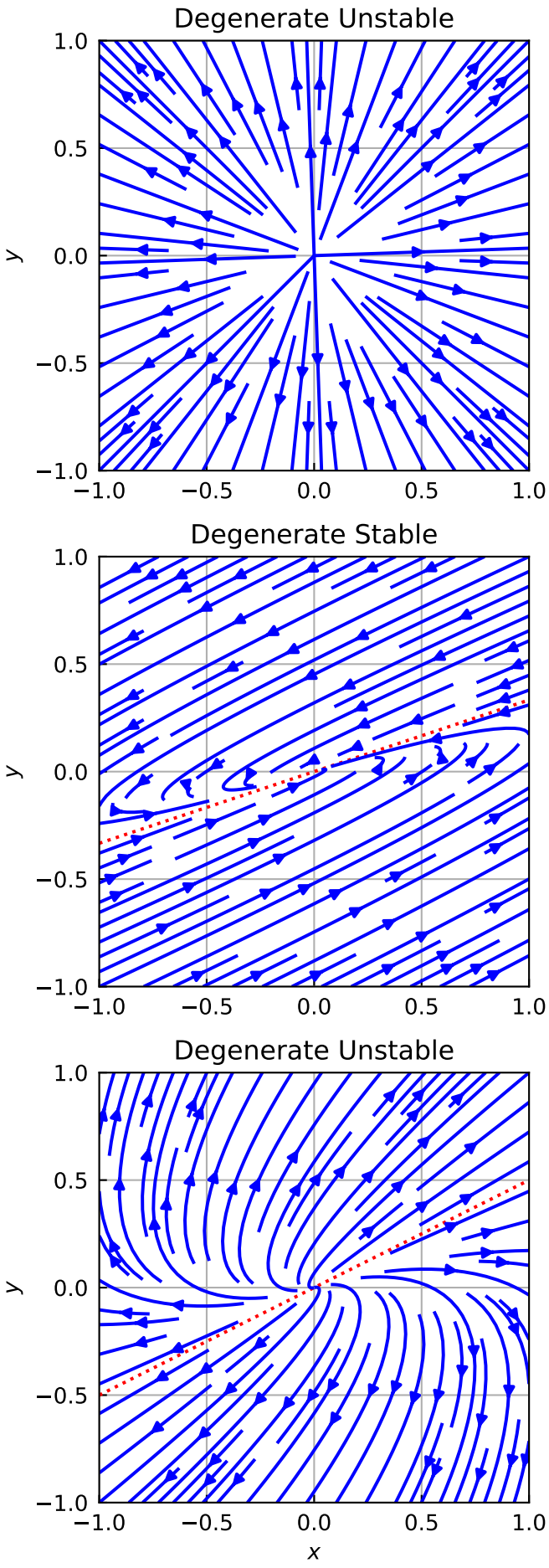
Example Classify the critical point $(0, 0)$ of each of the following linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$

(a) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, (b) $\begin{pmatrix}3 & -18\\ 2 & -9\end{pmatrix}$, (c) $\begin{pmatrix}2 & 4\\ -1 & 6\end{pmatrix}$

Answer 3 (end of chapter) · Solution

```
w = 1
xp = np.linspace(-w, w, 6)
yp = np.linspace(-w, w, 6)
x, y = np.meshgrid(xp, yp)

A = np.array([[[1, 0], [0, 1]],
              [[3, -18], [2, -9]],
              [[2, 4], [-1, 6]]])
A_title = ['Degenerate Unstable',
           'Degenerate Stable',
           'Degenerate Unstable']

fig = plt.figure(figsize=(4, 12))

for i in range(3):

    xdot = A[i,0,0]*x +A[i,0,1]*y
    ydot = A[i,1,0]*x +A[i,1,1]*y
```

```
    lamda, v = np.linalg.eig(A[i])

    ax = fig.add_subplot(3, 1, i +1)

    if i != 0:
        y_1 = v[1,0] /v[0,0] *xp
        ax.plot(xp, y_1, 'r:')

    ax.streamplot(x, y, xdot, ydot, color='blue')

    ax.set_title(A_title[i])

    ax.axis((-w, w, -w, w))
    ax.set_aspect(aspect='equal')
    ax.grid()

    ax.tick_params(axis='both', direction='in', pad=5)
    ax.xaxis.set_ticks(np.linspace(-w, w, 5))
    ax.set_ylabel(r'$y$')

    ax.yaxis.set_ticks(np.linspace(-w, w, 5))
    if i == 2: ax.set_xlabel(r'$x$')
```


Degenerate Unstable
Degenerate Stable
Degenerate Unstable
1.0
0.5
0.0
−0.5
−1.0
y
x

- **Complex Eigenvalues**, $\tau^2 - 4\Delta < 0$

  If $\lambda_1 = \alpha + i\beta$ and $\bar{\lambda}_1$ are the complex eigenvalues and $\mathbf{k}_1 = \mathbf{b}_1 + i\mathbf{b}_2$ is a complex eigenvector corresponding to λ_1, the general solution can be written as $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$

$$\mathbf{x}_1(t) = e^{\alpha t}\left(\mathbf{b}_1 \cos \beta t - \mathbf{b}_2 \sin \beta t\right)$$
$$\mathbf{x}_2(t) = e^{\alpha t}\left(\mathbf{b}_2 \cos \beta t + \mathbf{b}_1 \sin \beta t\right)$$

  - **Pure imaginary roots**, $\tau^2 - 4\Delta < 0, \ \ \tau = 0$

    Center: When $\alpha = 0$, all solutions are ellipses with center at the origin and are periodic with period $p = 2\pi/\beta$. The critical point is called a **center**

  - **Nonzero real part**, $\tau^2 - 4\Delta < 0, \ \ \tau \neq 0$

    Spiral Point: When $\alpha < 0$, $e^{\alpha t} \to 0$, and the elliptical-like solution spirals closer and closer to the origin. The critical point is called a **stable spiral point**. When $\alpha > 0$, the effect is the opposite. An elliptical-like solution is driven farther and farther from the origin, and the critical point is now called an **unstable spiral point**

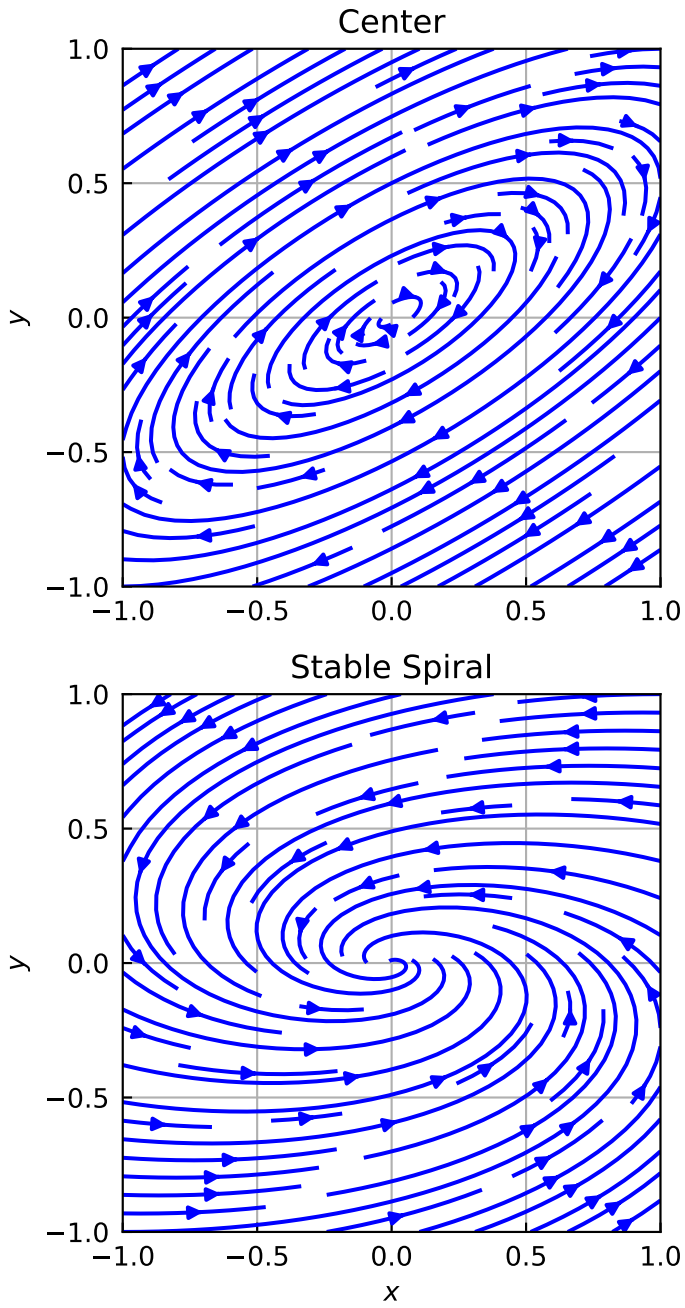
Example Classify the critical point $(0, 0)$ of each of the following linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$

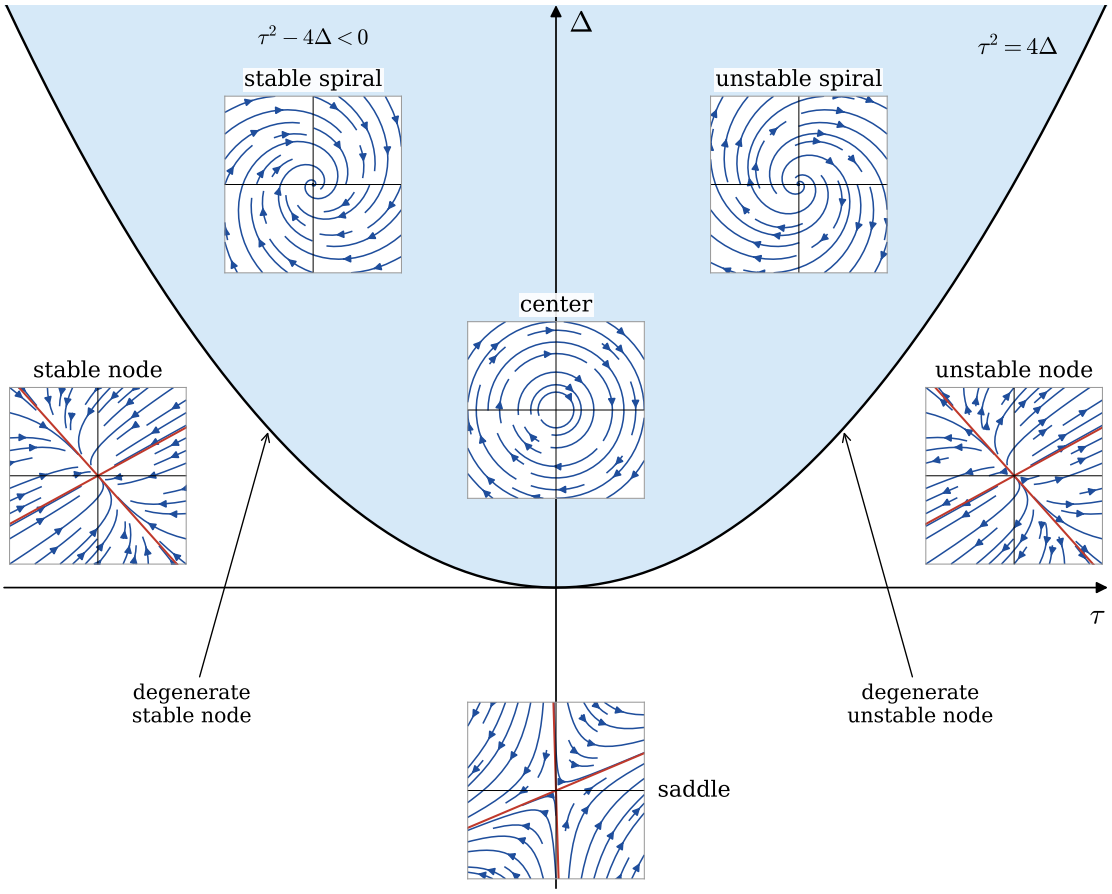
(a) $\begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$

Answer 4 (end of chapter) · Solution

```
w = 1
xp = np.linspace(-w, w, 6)
yp = np.linspace(-w, w, 6)
x, y = np.meshgrid(xp, yp)

A = np.array([[[-1, 2], [-1, 1]],
              [[-1, -4], [1, -1]]])
A_title = ['Center', 'Stable Spiral']

fig = plt.figure(figsize=(4, 8))

for i in range(2):

    ax = fig.add_subplot(2, 1, i +1)

    xdot = A[i,0,0]*x +A[i,0,1]*y
    ydot = A[i,1,0]*x +A[i,1,1]*y

    lamda, v = np.linalg.eig(A[i])

    ax.streamplot(x, y, xdot, ydot, color='blue')

    ax.set_title(A_title[i])

    ax.axis((-w, w, -w, w))
    ax.set_aspect(aspect='equal')
    ax.grid()
```

```
    ax.tick_params(axis='both', direction='in', pad=5)
    ax.xaxis.set_ticks(np.linspace(-w, w, 5))
    ax.set_ylabel(r'$y$')

    ax.yaxis.set_ticks(np.linspace(-w, w, 5))
    if i == 1: ax.set_xlabel(r'$x$')
```





For a linear plane autonomous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\det \mathbf{A} \neq 0$, let $\mathbf{x}$ denote the solution that satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$, where $\mathbf{x}_0 \neq \mathbf{0}$

1. $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ if and only if the eigenvalues of $\mathbf{A}$ have negative real parts. This will occur when $\Delta > 0$ and $\tau < 0$
2. $\mathbf{x}(t)$ is periodic if and only if the eigenvalues of $\mathbf{A}$ are pure imaginary. This will occur when $\Delta > 0$ and $\tau = 0$
3. In all other cases, given any neighborhood of the origin, there is at least one $\mathbf{x}_0$ in the neighborhood for which $\mathbf{x}(t)$ becomes unbounded as t increases

8.3 Linearization and Local Stability

Here we will use linearization as a means of analyzing nonlinear DEs and nonlinear systems; the idea is to replace them by linear DEs and linear systems. Let $\mathbf{x}_1$ be a critical point of an autonomous system, and let $\mathbf{x} = \mathbf{x}(t)$ denote the solution that satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$, where $\mathbf{x}_0 \neq \mathbf{x}_1$.

- $\mathbf{x}_1$ is a **stable critical point**

  when, given any $\rho > 0$, there is a $r > 0$ such that if $\mathbf{x}_0$ satisfies $|\mathbf{x}_0 - \mathbf{x}_1| < r$, then $\mathbf{x}(t)$ satisfies $|\mathbf{x}(t) - \mathbf{x}_1| < \rho$ for all $t > 0$. If, in addition, $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_1$ whenever $|\mathbf{x}_0 - \mathbf{x}_1| < r$, we call $\mathbf{x}_1$ an **asymptotically stable critical point**

- $\mathbf{x}_1$ is an **unstable critical point**

if there is $\rho > 0$ with the property that, for any $r > 0$, there is at least one $\mathbf{x}_0$ that satisfies $|\mathbf{x}_0 - \mathbf{x}_1| < r$, yet the corresponding solution $\mathbf{x}(t)$ satisfies $|\mathbf{x}(t) - \mathbf{x}_1| \geq \rho$ for at least one $t > 0$

Example Show that $(0, 0)$ is a stable critical point of the nonlinear plane autonomous system

$$x' = -y - x\sqrt{x^2 + y^2}, \quad y' = x - y\sqrt{x^2 + y^2}$$

From the formulas $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$, we obtain

$$\frac{dr}{dt} = \frac{1}{r}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right), \quad \frac{d\theta}{dt} = \frac{1}{r^2}\left(-y\frac{dx}{dt} + x\frac{dy}{dt}\right)$$

Substituting for dx/dt and dy/dt in the expressions for dr/dt and $d\theta/dt$, we obtain

$$\frac{dr}{dt} = -r^2, \quad \frac{d\theta}{dt} = 1$$

Using separation of variables, we see that the solution of the system is

$$r = \frac{1}{t + c_1}, \quad \theta = t + c_2$$

for $r \neq 0$. If $\mathbf{x}(0) = (r_0, \theta_0)$ is the initial condition in polar coordinates, then

$$r = \frac{r_0}{r_0 t + 1}, \quad \theta = t + \theta_0$$

Note that $r \leq r_0$ for $t \geq 0$, and r approaches 0 as t increases. Therefore, given $\rho > 0$, a solution that starts less than ρ from the origin remains within ρ of the origin for all $t \geq 0$. Hence the critical point $(0, 0)$ is asymptotically stable

Example When expressed in polar coordinates, a plane autonomous system takes the form

$$\frac{dr}{dt} = 0.05\, r(3 - r)$$
$$\frac{d\theta}{dt} = -1$$

Show that $(x, y) = (0, 0)$ is an unstable critical point

We see that $dr/dt = 0$ when $r = 0$ and can conclude that $(x, y) = (0, 0)$ is a critical point. The differential equation can be solved using separation of variables. If $r(0) = r_0$ and $r_0 \neq 0$, then

$$r = \frac{3}{1 + c_0 e^{-0.15t}}$$

where $c_0 = (3 - r_0)/r_0$. Since

$$\lim_{t \to \infty} \frac{3}{1 + c_0 e^{-0.15t}} = 3$$

it follows that no matter how close to $(0, 0)$ a solution starts, the solution will leave a disk of radius ρ $(0 < \rho < 3)$ about the origin. Therefore $(0, 0)$ is an unstable critical point

- **Linearization**

We replace the term $\mathbf{g}(\mathbf{x})$ in the original autonomous system $\mathbf{x}' = \mathbf{g}(\mathbf{x})$ by a linear term $\mathbf{A}(\mathbf{x} - \mathbf{x}_1)$ that most closely approximates $\mathbf{g}(\mathbf{x})$ in a neighborhood of $\mathbf{x}_1$. This replacement process is called **linearization**

When $\mathbf{x}_1$ is a critical point of a plane autonomous system

$$\begin{aligned} x' &= P(x, y) \\ y' &= Q(x, y) \end{aligned}$$

$P(x_1, y_1) = Q(x_1, y_1) = 0$ and we have

$$\begin{aligned} x' = P(x, y) &\simeq \left.\frac{\partial P}{\partial x}\right|_{(x_1, y_1)} (x - x_1) + \left.\frac{\partial P}{\partial y}\right|_{(x_1, y_1)} (y - y_1) \\ y' = Q(x, y) &\simeq \left.\frac{\partial Q}{\partial x}\right|_{(x_1, y_1)} (x - x_1) + \left.\frac{\partial Q}{\partial y}\right|_{(x_1, y_1)} (y - y_1) \end{aligned}$$

The original system $\mathbf{x}' = \mathbf{g}(\mathbf{x})$ may be approximated in a neighborhood of $\mathbf{x}_1$ by the linear system $\mathbf{x}' = \mathbf{A}(\mathbf{x} - \mathbf{x}_1)$, where

$$\mathbf{A} = \begin{pmatrix} \left.\frac{\partial P}{\partial x}\right|_{(x_1, y_1)} & \left.\frac{\partial P}{\partial y}\right|_{(x_1, y_1)} \\ \left.\frac{\partial Q}{\partial x}\right|_{(x_1, y_1)} & \left.\frac{\partial Q}{\partial y}\right|_{(x_1, y_1)} \end{pmatrix} = \mathbf{g}'(\mathbf{x}_1)$$

- **Stability Criteria for Plane Autonomous System**

1. If the eigenvalues of $\mathbf{A} = \mathbf{g}'(\mathbf{x}_1)$ all have negative real parts, then $\mathbf{x}_1$ is an asymptotically stable critical point
2. If $\mathbf{A} = \mathbf{g}'(\mathbf{x}_1)$ has an eigenvalue with positive real part, then $\mathbf{x}_1$ is an unstable critical point

Example Classify the critical points of the following plane autonomous system

$$\begin{aligned} x' &= x^2 + y^2 - 6 \\ y' &= x^2 - y \end{aligned}$$

The critical points are $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$ and the Jacobian matrix is

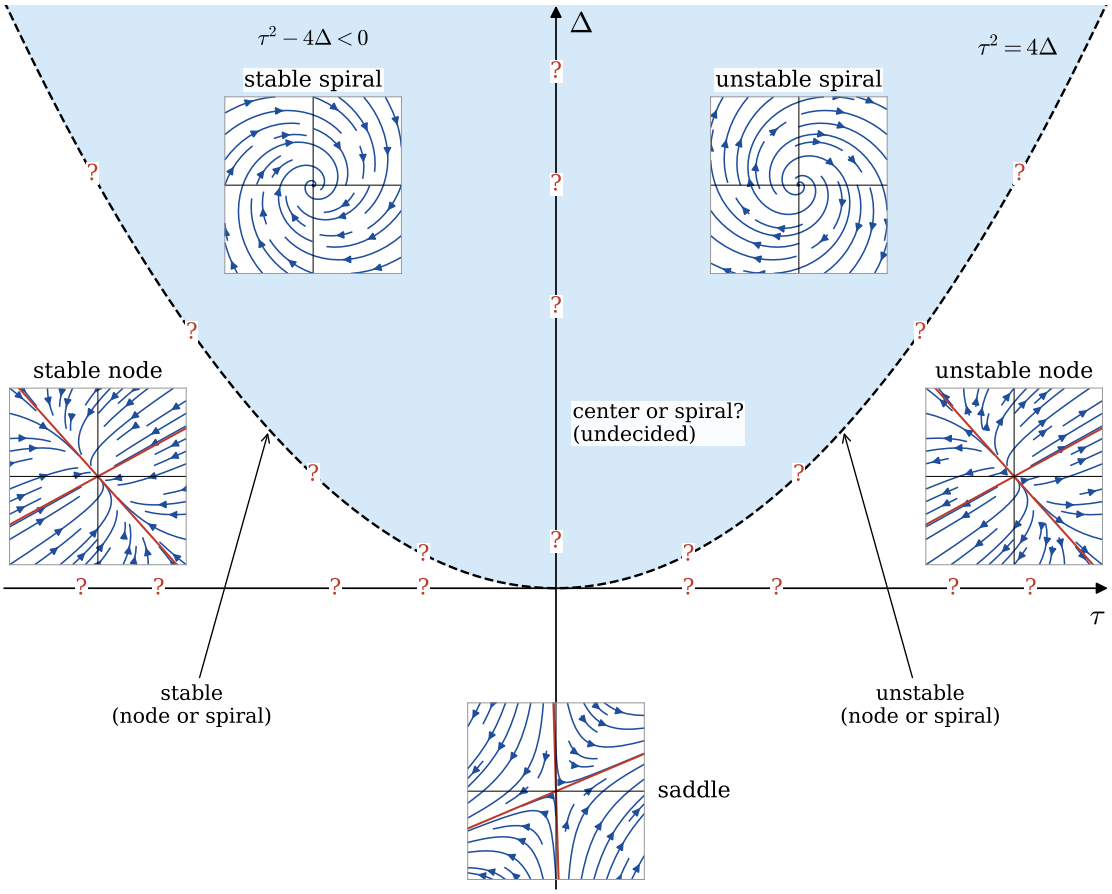
$$\mathbf{g}'(\mathbf{x}) = \begin{pmatrix} 2x & 2y \\ 2x & -1 \end{pmatrix}$$

and so

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{g}'((\sqrt{2}, 2)) = \begin{pmatrix} 2\sqrt{2} & 4 \\ 2\sqrt{2} & -1 \end{pmatrix} \\ \mathbf{A}_2 &= \mathbf{g}'((-\sqrt{2}, 2)) = \begin{pmatrix} -2\sqrt{2} & 4 \\ -2\sqrt{2} & -1 \end{pmatrix} \end{aligned}$$

Since the determinant of $\mathbf{A}_1$ is negative, $\mathbf{A}_1$ has a positive real eigenvalue. Therefore $(\sqrt{2}, 2)$ is an unstable critical point. $\mathbf{A}_2$ has a positive determinant and a negative trace, and so both eigenvalues have negative real parts. It follows that $(-\sqrt{2}, 2)$ is a stable critical point

- **Classifying Critical Points**



Example The second-order differential equation $mx'' + kx + k_1x^3 = 0$, for $k > 0$, represents a general model for the free, undamped oscillations of a mass m attached to a nonlinear spring. If $k = 1$ and $k_1 = -1$, the spring is called soft and the plane autonomous system corresponding to the nonlinear second-order equation $x'' + x - x^3 = 0$ is

$$\begin{aligned} x' &= y \\ y' &= x^3 - x \end{aligned}$$

Find and classify the critical points

Since $x^3 - x = x(x^2 - 1)$, the critical points are $(0, 0)$, $(1, 0)$, and $(-1, 0)$. The corresponding Jacobian matrices are

$$\mathbf{A}_1 = \mathbf{g}'((0,0)) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\mathbf{A}_2 = \mathbf{g}'((1,0)) = \mathbf{g}'((-1,0)) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

Since $\det \mathbf{A}_2 < 0$, critical points $(1, 0)$ and $(-1, 0)$ are both saddle points. The eigenvalues of $\mathbf{A}_1$ are $\pm i$, and the status of the critical points at $(0, 0)$ remains in doubt. It may be either a stable spiral, an unstable spiral, or a center

- **Phase-Plane Method**

The linearization method can provide useful information on the local behavior of solutions near critical points. It is of little help if we are interested in solutions whose initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is not close to a critical point or if we wish to obtain a global view of the family of solution curves. The **phase-plane method** is based on the fact that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{Q(x,y)}{P(x,y)}$$

and it attempts to find y as a function of x using one of the methods available for solving first-order DEs

Example Use the phase-plane method to determine the nature of the solutions to $x'' + x - x^3 = 0$ in a neighborhood of $(0,0)$

If we let $dx/dt = y$, then $dy/dt = x^3 - x$. From this we obtain the first-order DE

$$\frac{dy}{dx} = \frac{x^3 - x}{y}$$

which can be solved by separation of variables. Integrating gives

$$y^2 = \frac{x^4}{2} - x^2 + c$$

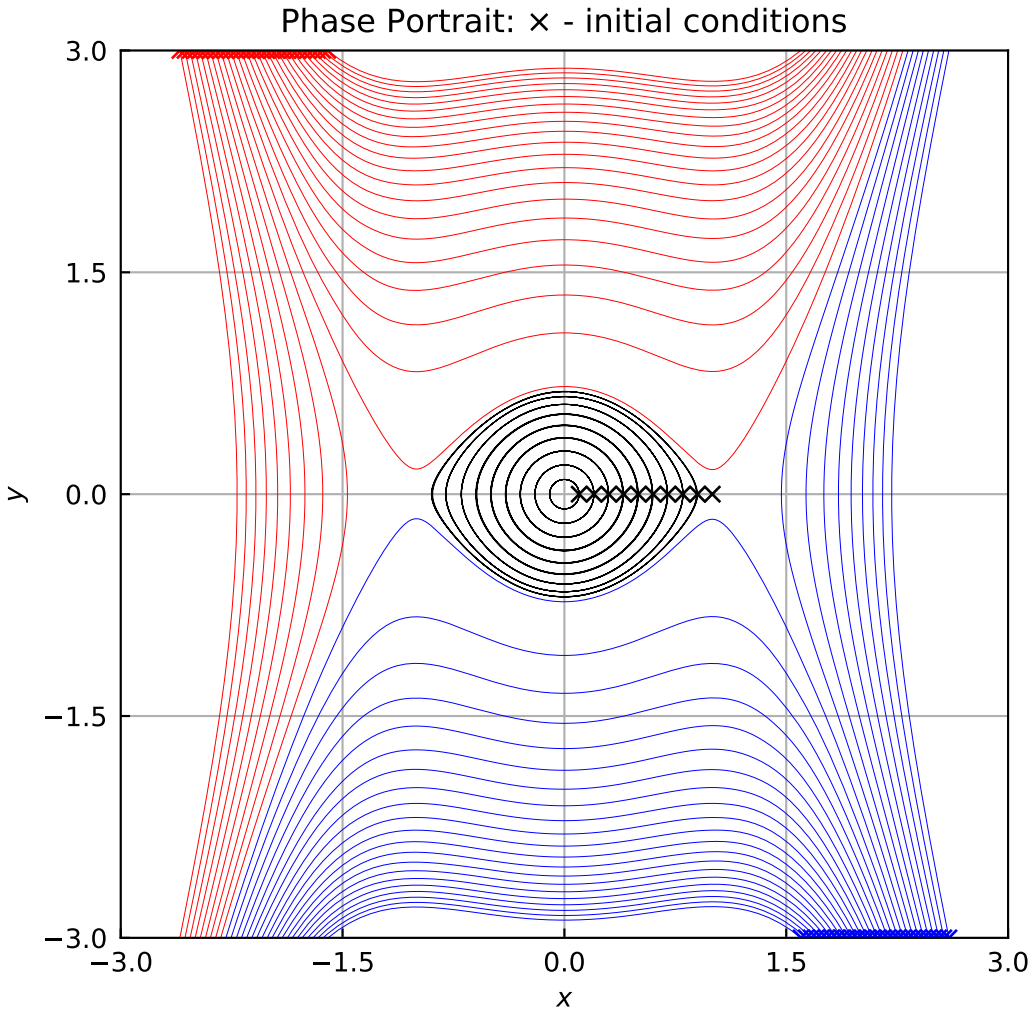
If $\mathbf{x}(0) = (x_0, 0)$, where $0 < x_0 < 1$, then $c = -\frac{x_0^4}{2} + x_0^2$, and so

$$y^2 = \frac{(2 - x^2 - x_0^2)(x_0^2 - x^2)}{2}$$

Note that $y = 0$ when $x = -x_0$. In addition, the right-hand side is positive when $-x_0 < x < x_0$, and so each x has two corresponding values of y. The solution $\mathbf{x}(t)$ that satisfies $\mathbf{x}(0) = (x_0, 0)$ is therefore periodic, and so $(0,0)$ is a center

```
from scipy.integrate import solve_ivp

def myODE(t, y):
    return [y[1], y[0]**3 -y[0]]

def plotPhasePortrait(ax, g, td, x0, y0, color):

    for i in range(x0.shape[0]):
        sol = solve_ivp(g, [0, td[-1]],
               [x0[i], y0], t_eval = td)
        ax.plot(x0[i], y0, color +'x')
        ax.plot(sol.y[0], sol.y[1], color +'-', lw=0.2)

td = np.linspace(0, 50, 2000)

x0_0, y0_0 = np.linspace( 0.1, 1.0, 10),  0.0
x0_1, y0_1 = np.linspace( 1.6, 2.6, 30), -3.0
x0_2, y0_2 = np.linspace(-2.6,-1.6, 30),  3.0
```

```
fig, ax = plt.subplots(figsize=(6, 6))

plotPhasePortrait(ax, myODE, td, x0_0,  y0_0, 'k')
plotPhasePortrait(ax, myODE, td, x0_1,  y0_1, 'b')
plotPhasePortrait(ax, myODE, td, x0_2,  y0_2, 'r')

ax.set_title(r'Phase Portrait: $\times$ - initial conditions')

w = 3
ax.axis((-w, w, -w, w))
ax.set_aspect(aspect='equal')
ax.grid()

ax.tick_params(axis='both', direction='in', pad=5)
ax.xaxis.set_ticks(np.linspace(-w, w, 5))
ax.yaxis.set_ticks(np.linspace(-w, w, 5))
ax.set_xlabel(r'$x$')
ax.set_ylabel(r'$y$')

plt.show()
```



8.4 Autonomous Systems as Mathematical Models

Many applications from physics and biology give rise to nonlinear autonomous second-order DEs $x'' = g(x, x')$

- Nonlinear Pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

When we let $x = \theta$ and $y = \theta'$, this second-order differential equation may be rewritten as the dynamical system

$$\begin{aligned} x' &= y \\ y' &= -\frac{g}{l}\sin x \end{aligned}$$

The critical points are $(\pm k\pi, 0)$, and the Jacobian matrix is easily shown to be

$$\mathbf{g}'((\pm k\pi, 0)) = \begin{pmatrix} 0 & 1 \\ (-1)^{k+1}\frac{g}{l} & 0 \end{pmatrix}$$

If $k = 2n + 1$, $\Delta < 0$, and so all critical points $(\pm(2n + 1)\pi, 0)$ are saddle points. When $k = 2n$, the eigenvalues are pure imaginary, and so the nature of these critical points remains in doubt. From

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{g}{l}\frac{\sin x}{y}$$

it follows that

$$y^2 = \frac{2g}{l}\cos x + c$$

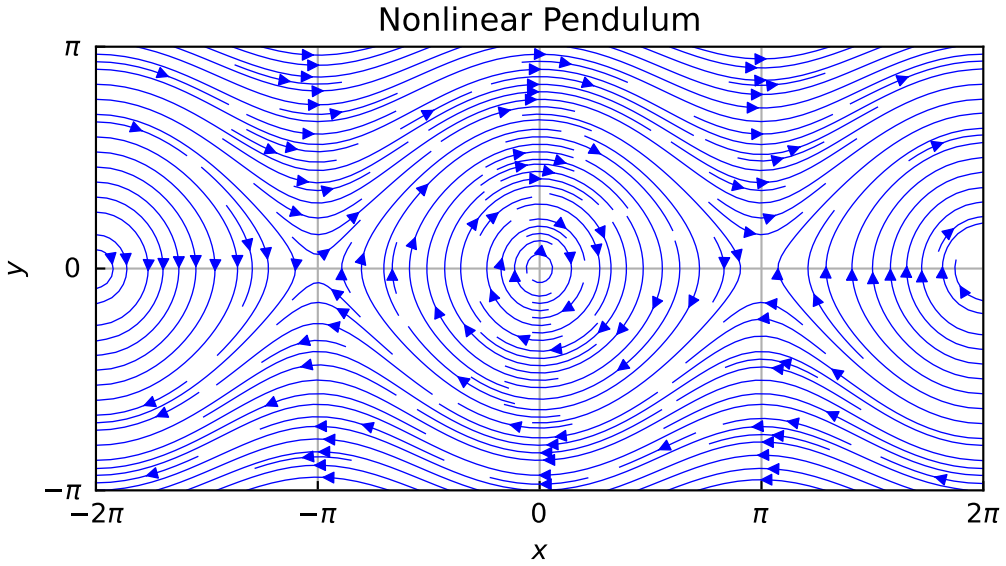
If $\mathbf{x}(0) = (x_0, 0)$, then

$$y^2 = \frac{2g}{l}(\cos x - \cos x_0)$$

Note that $y = 0$ when $x = -x_0$, and that $(2g/l)(\cos x - \cos x_0) > 0$ for $|x| < |x_0| < \pi$. Thus each such x has two corresponding values of y, and so the solution $\mathbf{x}(t)$ that satisfies $\mathbf{x}(0) = (x_0, 0)$ is periodic. We may conclude that $(0, 0)$ is a center. In the case of large initial velocities, the pendulum spins in complete circles about the pivot

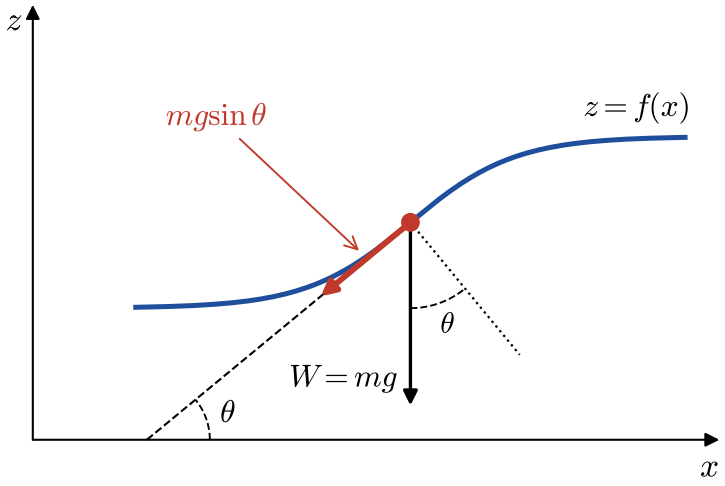
```
w1 = 2*np.pi
w2 = np.pi
xp = np.linspace(-w1, w1, 100)
yp = np.linspace(-w2, w2, 100)
x, y = np.meshgrid(xp, yp)

g = 9.8
l = 8
```

```
xdot = y
ydot = -g/l*np.sin(x)

fig, ax = plt.subplots(figsize=(6, 3))

ax.streamplot(x, y, xdot, ydot,
              linewidth=0.5, density=2, color='blue')

ax.set_title('Nonlinear Pendulum')

ax.axis((-w1, w1, -w2, w2))
ax.set_aspect(aspect='equal')
ax.grid()

ax.tick_params(axis='both', direction='in', pad=5)
ax.xaxis.set_ticks(np.linspace(-w1, w1, 5))
ax.xaxis.set_ticklabels([r'$-2\pi$', r'$-\pi$', r'$0$',
                         r'$\pi$', r'$2\pi$'])
ax.set_xlabel(r'$x$')
ax.yaxis.set_ticks(np.linspace(-w2, w2, 3))
ax.yaxis.set_ticklabels([r'$-\pi$', r'$0$', r'$\pi$'])
ax.set_ylabel(r'$y$')

plt.show()
```



- Nonlinear Oscillations: The Sliding Bead

  Suppose a bead with mass m slides along a thin wire whose shape is described by the function $z = f(x)$. A wide variety of nonlinear oscillations can be obtained by changing the shape of the wire and by making different assumptions about the forces acting on the bead



The tangential force $\mathbf{F}$ due to the weight $W = mg$ has a magnitude $mg\sin\theta$, and therefore the x-component of $\mathbf{F}$ is $F_x = -mg\sin\theta\cos\theta$. Since $f'(x) = \tan\theta$, we can use the identity $1 + \tan^2\theta = \sec^2\theta$ to conclude that

$$\begin{aligned} F_x &= -mg\sin\theta\cos\theta = -mg\tan\theta\cos^2\theta \\ &= -mg\frac{f'(x)}{1+[f'(x)]^2} \end{aligned}$$

A damping force $\mathbf{D}$, in the direction opposite to the motion, is a constant multiple of the velocity of the bead. The x-component of $\mathbf{D}$ is therefore

$$D_x = -\beta\frac{dx}{dt}$$

It follows from Newton's second law that

$$mx'' = -mg\frac{f'(x)}{1+[f'(x)]^2} - \beta x'$$

and the corresponding plane autonomous system is

$$\begin{aligned} x' &= y \\ y' &= -g\frac{f'(x)}{1+[f'(x)]^2} - \frac{\beta}{m}y \end{aligned}$$

If $\mathbf{x}_1 = (x_1, y_1)$ is a critical point of the system, $y_1 = 0$ and therefore $f'(x_1) = 0$. The bead must therefore be at rest at a point on the wire where the tangent line is horizontal. When f is twice differentiable, the Jacobian matrix at $\mathbf{x}_1$ is

$$\mathbf{g}'(\mathbf{x}_1) = \begin{pmatrix} 0 & 1 \\ -gf''(x_1) & -\frac{\beta}{m} \end{pmatrix}$$

and so

$$\tau = -\beta/m$$
$$\Delta = gf''(x_1)$$
$$\tau^2 - 4\Delta = \beta^2/m^2 - 4gf''(x_1)$$

We can make the following conclusions:

- $f''(x_1) < 0$: A relative maximum therefore occurs at $x = x_1$, and since $\Delta < 0$, an unstable saddle point occurs at $\mathbf{x}_1 = (x_1, 0)$
- $f''(x_1) > 0$ and $\beta > 0$: A relative minimum therefore occurs at $x = x_1$, and since $\tau < 0$ and $\Delta > 0$, $\mathbf{x}_1 = (x_1, 0)$ is a stable critical point.

  If $\beta^2 > 4gm^2 f''(x_1)$, the system is **overdamped** and the critical point is a stable node. If $\beta^2 < 4gm^2 f''(x_1)$, the system is **underdamped** and the critical point is a stable spiral point. The exact nature of the stable critical point is still in doubt if $\beta^2 = 4gm^2 f''(x_1)$
- $f''(x_1) > 0$ and the system is undamped ($\beta = 0$):

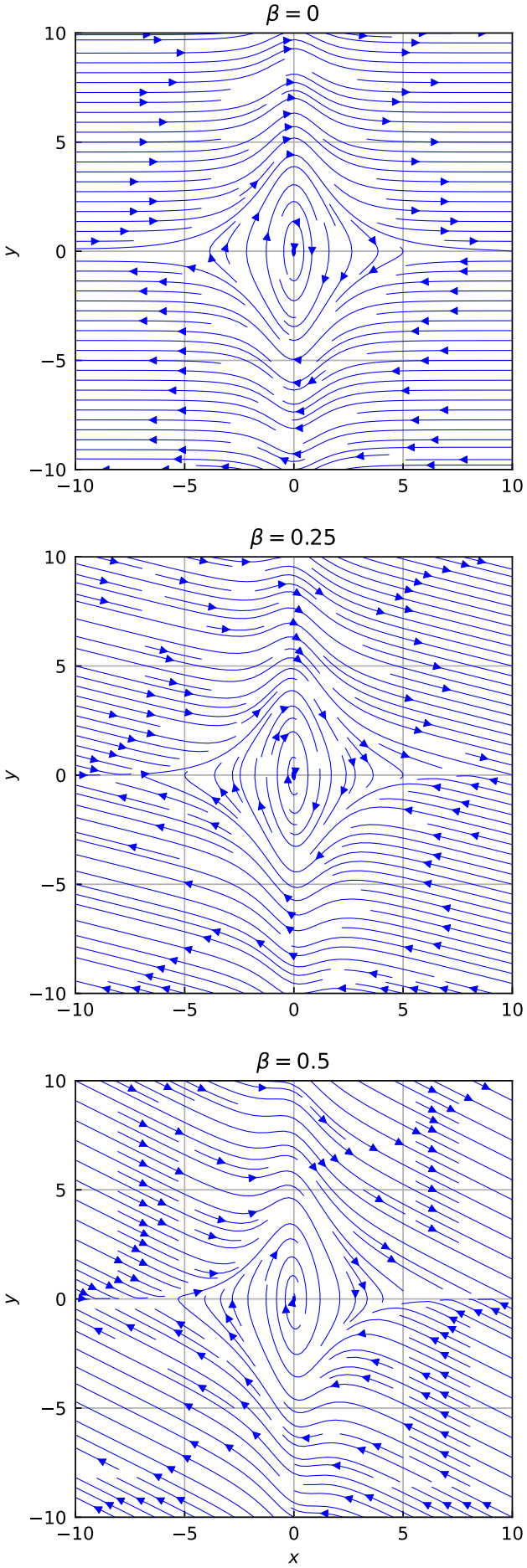
  In this case the eigenvalues are pure imaginary, but the phase-plane method can be used to show that the critical point is a center. Therefore solutions with $\mathbf{x}(0) = (x(0), x'(0))$ near $\mathbf{x}_1 = (x_1, 0)$ are periodic

Example A bead is released from the position $x(0) = x_0$ on the curve $z = \cosh(x)$ with initial velocity $x'(0) = \nu_0$. Use the phase-plane method to show that the resulting solution is periodic when the system is undamped

Answer 5 (end of chapter) · Solution

```
w = 10
xp = np.linspace(-w, w, 200)
yp = np.linspace(-w, w, 200)
x, y = np.meshgrid(xp, yp)

g = 9.8
m = 1
beta = [0, 0.25, 0.5]

sb_title = [r'$\beta=0$', r'$\beta=0.25$', r'$\beta=0.5$']

def df(x):
    return np.sinh(x)

fig = plt.figure(figsize=(5, 15))

for i in range(3):

    xdot = y
    ydot = -g*df(x) /(1 +df(x)**2) -beta[i] /m *y

    ax = fig.add_subplot(3, 1, i +1)
    ax.streamplot(x, y, xdot, ydot, density=1.5,
                  linewidth=0.5, color='blue')

    ax.set_title(sb_title[i])
```

```
    ax.axis((-w, w, -w, w))
    ax.set_aspect(aspect='equal')
    ax.grid()

    ax.tick_params(axis='both', direction='in', pad=5)
    ax.xaxis.set_ticks(np.linspace(-w, w, 5))
    ax.set_ylabel(r'$y$')
    ax.yaxis.set_ticks(np.linspace(-w, w, 5))
    if i == 2:
        ax.set_xlabel(r'$x$')

plt.show()
```


$\beta = 0$
$\beta = 0.25$
$\beta = 0.5$
10
5
0
−5
−10
y
x
−10
−5
0
5
10

- Lotka-Volterra Predator-Prey Model

  There are many predator-prey models that lead to plane autonomous systems with at least one periodic solution. The first such model was constructed independently by pioneer biomathematicians A. Lotka (1925) and V. Volterra (1926). If x denotes the number of predators and y denotes the number of prey, then the Lotka-Volterra model takes the form

$$\begin{aligned} x' &= -ax + bxy = x(-a + by) \\ y' &= -cxy + dy = y(-cx + d) \end{aligned}$$

  where a, b, c, and d are positive constants. The critical points of this plane autonomous system are $(0, 0)$ and $(d/c, a/b)$, and the corresponding Jacobian matrices are

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{g}'((0,0)) = \begin{pmatrix} -a & 0 \\ 0 & d \end{pmatrix} \\ \mathbf{A}_2 &= \mathbf{g}'((d/c, a/b)) = \begin{pmatrix} 0 & bd/c \\ -ac/b & 0 \end{pmatrix} \end{aligned}$$

  The critical point $(0, 0)$ is a saddle point. Since $\mathbf{A}_2$ has pure imaginary eigenvalues $\lambda = \pm i\sqrt{ad}$, the critical point $(d/c, a/b)$ may be a center. This possibility can be investigated using the phase-plane method. Since

$$\frac{dy}{dx} = \frac{y(-cx + d)}{x(-a + by)}$$

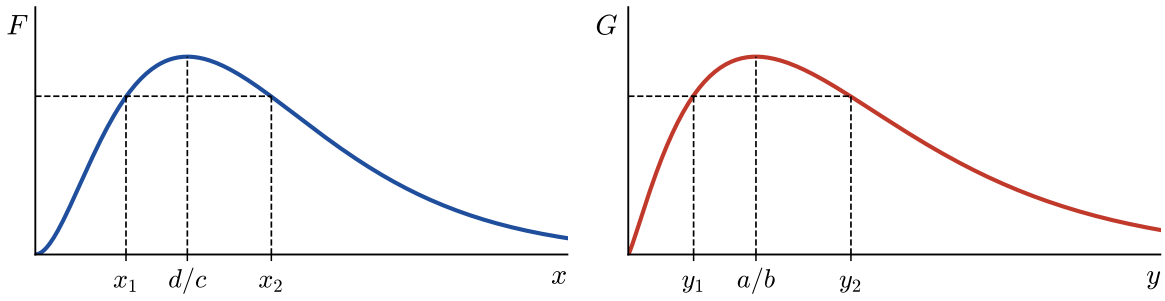
  we may separate variables and obtain

$$\int \frac{-a + by}{y}\, dy = \int \frac{-cx + d}{x}\, dx$$

  so that

$$(x^d e^{-cx})\,(y^a e^{-by}) = F(x)G(y) = c_0$$

  Typical graphs of the nonnegative functions $F(x) = x^d e^{-cx}$ and $G(y) = y^a e^{-by}$ are shown in



(a) graph of $F(x)$ (b) graph of $G(y)$

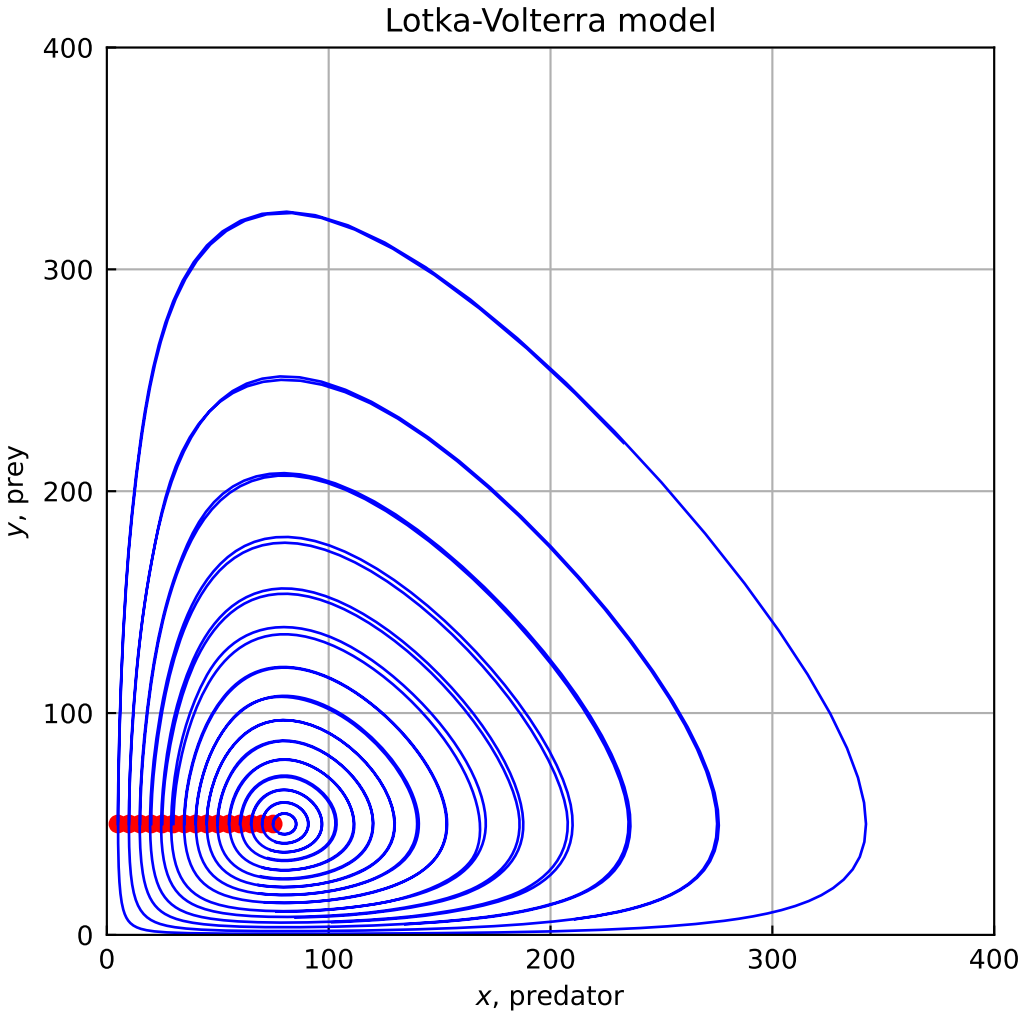
  It is not hard to show that $F(x)$ has an absolute maximum at $x = d/c$, whereas $G(y)$ has an absolute maximum at $y = a/b$. These graphs can be used to establish the following properties of a solution curve that originates at a noncritical point (x_0, y_0) in the first quadrant

  – If $y = a/b$, $F(x)G(a/b) = c_0$ has two solutions x_m and x_M that satisfy $x_m < d/c < x_M$

- If $x_m < x_1 < x_M$ and $x = x_1$, then $F(x_1)G(y) = c_0$ has exactly two solutions y_1 and y_2 that satisfy $y_1 < a/b < y_2$
- If x is outside the interval $[x_m, x_M]$, then $F(x)G(y) = c_0$ has no solutions

Example If we let $a = 0.1$, $b = 0.002$, $c = 0.0025$, and $d = 0.2$ in the Lotka-Volterra predator-prey model, the critical point in the first quadrant is $(d/c, a/b) = (80, 50)$, and we know that this critical point is a center. The eigenvalues of $\mathbf{g}'((80, 50))$ are $\lambda = \pm(\sqrt{2}/10)i$, and so the solutions near the critical point have period $p \simeq 10\sqrt{2}\pi$

```
a = 0.1
b = 0.002
c = 0.0025
d = 0.2

def myODE(t, y):
    return [-a*y[0] +b*y[0]*y[1], -c*y[0]*y[1] +d*y[1]]

fig, ax = plt.subplots(figsize=(6, 6))

td = 80
t_eval = np.linspace(0, td, 300)
x0 = 0
y0 = 50
for i in range(15):
    x0 = x0 +5
    sol = solve_ivp(myODE, [0, td], [x0, y0],
         t_eval=t_eval)
    ax.plot(x0, y0, 'ro')
    ax.plot(sol.y[0], sol.y[1], 'b-', linewidth=1)

ax.set_title('Lotka-Volterra model')

w = 400
ax.axis((0, w, 0, w))
ax.set_aspect(aspect='equal')
ax.grid()

ax.tick_params(axis='both', direction='in', pad=5)
ax.xaxis.set_ticks(np.linspace(0, w, 5))
ax.set_xlabel(r'$x$, predator')
ax.yaxis.set_ticks(np.linspace(0, w, 5))
ax.set_ylabel(r'$y$, prey')
plt.show()
```



- Lotka-Volterra Competition Model

  A **competitive interaction** occurs when two or more species compete for the food, water, light, and space resources of an ecosystem. A number of mathematical models have been constructed that offer insights into conditions that permit *coexistence*. If x denotes the number in species I and y denotes the number in species II, then the Lotka-Volterra model takes the form

$$x' = \frac{r_1}{K_1} x(K_1 - x - \alpha_{12} y)$$
$$y' = \frac{r_2}{K_2} y(K_2 - y - \alpha_{21} x)$$

  Note that in the absence of species II ($y = 0$), $x' = (r_1/K_1)x(K_1 - x)$, and so the first population grows logistically and approaches the steady-state population K_1. A similar statement holds for species II growing in the absence of species I. The term $-\alpha_{21}xy$ in the second equation stems from the competitive effect of species I on species II. The model therefore assumes that this rate of inhibition is directly proportional to the number of possible competitive pairs xy at a particular t

  This plane autonomous system has critical points at $(0, 0)$, $(K_1, 0)$, and $(0, K_2)$. When $\alpha_{12}\alpha_{21} \neq 1$, the lines $K_1 - x - \alpha_{12}y = 0$ and $K_2 - y - \alpha_{21}x = 0$ intersect to produce a fourth critical point $\hat{\mathbf{x}} = (\hat{x}, \hat{y})$

$$\hat{x} = \frac{K_1 - \alpha_{12}K_2}{1 - \alpha_{12}\alpha_{21}},\ \hat{y} = \frac{K_2 - \alpha_{21}K_1}{1 - \alpha_{12}\alpha_{21}}$$

Example A competitive interaction is described by the Lotka-Volterra competition model

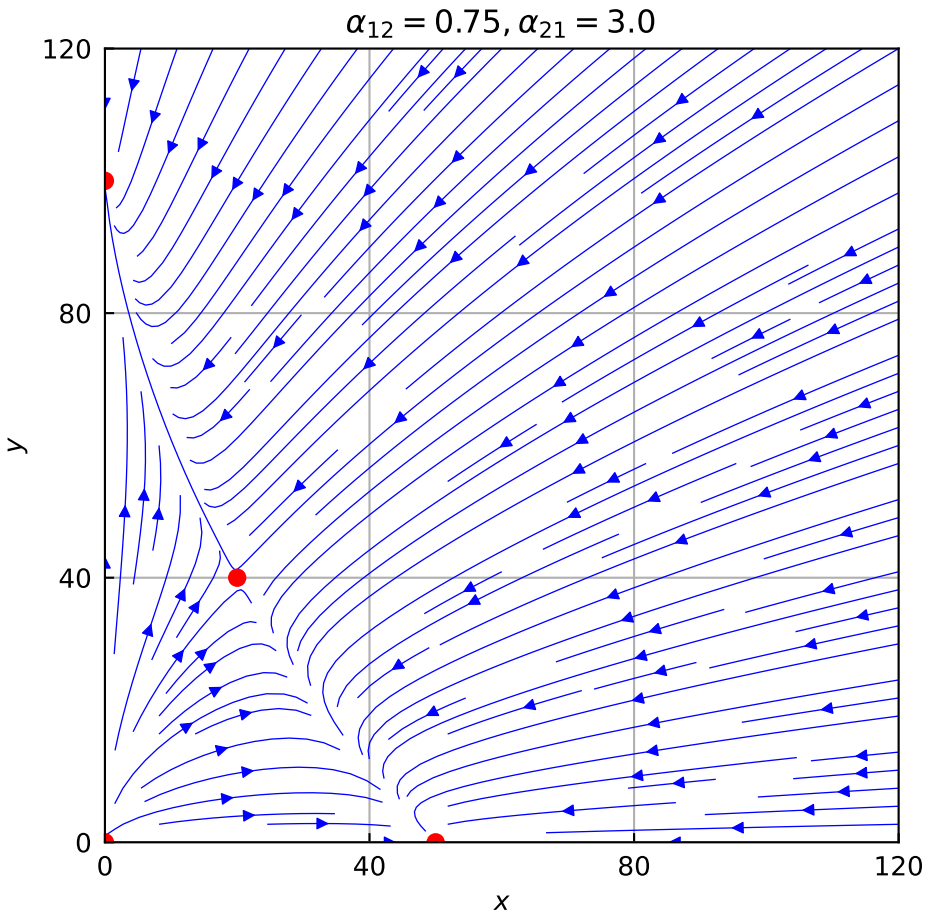
$$x' = 0.004x\,(50 - x - 0.75y)$$
$$y' = 0.001y\,(100 - y - 3.0x)$$

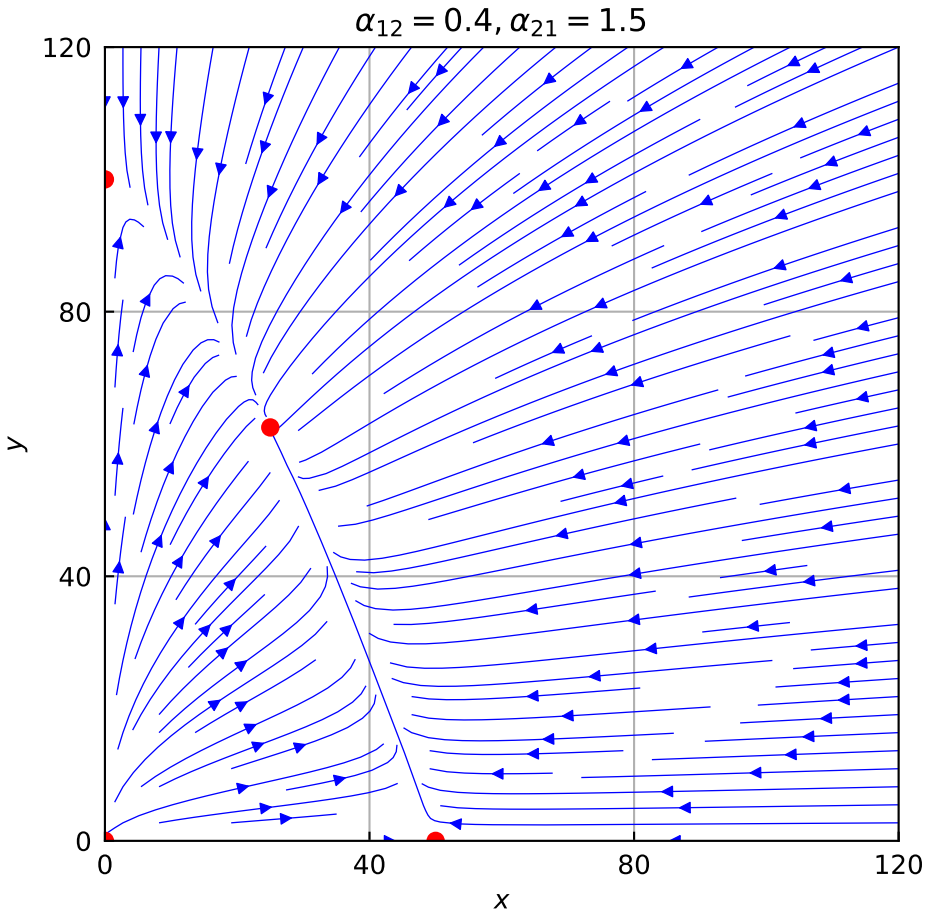
Find and classify all critical points of the system

Answer 6 (end of chapter) · Solution

```
w = 120
xp = np.linspace(0, w, 200)
yp = np.linspace(0, w, 200)
x, y = np.meshgrid(xp, yp)

r1_K1 = 0.004
r2_K2 = 0.001
K1 = 50
K2 = 100
a_12 = np.array([0.75, 0.4])
a_21 = np.array([3.00, 1.5])

fig = plt.figure(figsize=(6, 12))

for i in range(2):

    ax = fig.add_subplot(2, 1, i +1)

    xdot = r1_K1*x*(K1 -x -a_12[i]*y)
    ydot = r2_K2*y*(K2 -y -a_21[i]*x)

    cpt_x = np.array([0, K1, 0, (K1 -a_12[i]*K2)
                      /(1 -a_12[i]*a_21[i])])
    cpt_y = np.array([0, 0, K2, (K2 -a_21[i]*K1)
                      /(1 -a_12[i]*a_21[i])])

    ax.streamplot(x, y, xdot, ydot, density=1.5,
                  linewidth=0.5, color='blue')
    ax.plot(cpt_x, cpt_y, 'ro')
    ax.set_title(r'$\alpha_{12}=$' +str(a_12[i])
                 +r'$, \alpha_{21}=$' +str(a_21[i]))
    ax.axis((0, w, 0, w))
    ax.set_aspect(aspect='equal')
    ax.grid()

    ax.tick_params(axis='both', direction='in', pad=5)
    ax.xaxis.set_ticks(np.linspace(0, w, 4))
    ax.set_xlabel(r'$x$')
    ax.yaxis.set_ticks(np.linspace(0, w, 4))
    ax.set_ylabel(r'$y$')
```

```
plt.show()
```


$\alpha_{12} = 0.75, \alpha_{21} = 3.0$
120
80
40
0
y
0
40
80
120
x


$\alpha_{12} = 0.4, \alpha_{21} = 1.5$
120
80
40
0
y
0
40
80
120
x

Try it in Python: Predator-prey and Lorenz systems with `solve_ivp`: Numerical integration of ODEs using 'scipy'; the chaotic double pendulum: Double Pendulum

8.5 Periodic Solutions, Limit Cycles, and Global Stability

In this section, we will investigate the existence of periodic solutions of nonlinear plane autonomous systems and introduce special periodic solutions called **limit cycles**

An analysis of critical points using linearization can provide valuable information on the behavior of solutions near critical points and insight into a variety of biological and physical phenomena. However there are some inherent limitations to this approach. When the eigenvalues of the Jacobian matrix are pure imaginary, we cannot conclude that there are periodic solutions near the critical point

The first goal of this section is *to determine conditions under which we can either exclude the possibility of periodic solutions or assert their existence*

A second goal is *to determine conditions under which an asymptotically stable critical point is globally stable*: $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_1$ for all initial conditions in a simply connected region $\mathbb{R}$

- **Negative Criteria**

Theorem If a plane autonomous system has a periodic solution $\mathbf{x} = \mathbf{x}(t)$ in a simply connected region $\mathbb{R}$, then the system has at least one critical point inside the corresponding simple closed curve C. If there is a single critical point inside C, then that critical point cannot be a saddle point

Corollary If a simply connected region $\mathbb{R}$ either contains no critical points of a plane autonomous system or contains a single saddle point, then there are no periodic solutions in $\mathbb{R}$

Example Show that the Lotka-Volterra competition model

$$\begin{aligned} x' &= 0.004x\,(50 - x - 0.75y) \\ y' &= 0.001y\,(100 - y - 3.0x) \end{aligned}$$

has no periodic solutions in the first quadrant

Answer 7 (end of chapter) · Solution

Another sometimes useful result can be formulated in terms of the divergence of the vector field

$$\mathbf{v}(x, y) = (P(x, y), Q(x, y))$$

- **Bendixson Negative Criterion**

If $\operatorname{div} \mathbf{v} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ does not change sign in a simply connected region $\mathbb{R}$, then the plane autonomous system has no periodic solutions in $\mathbb{R}$

Proof Suppose, to the contrary, that there is a periodic solution $\mathbf{x} = \mathbf{x}(t)$ in $\mathbb{R}$, and let C be the resulting simple closed curve and R_1 the region bounded by C. By using Green's theorem, we obtain

$$\int_C -Q(x,y)\,dx + P(x,y)\,dy = \iint_{R_1} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dxdy$$

Since $\mathbf{x} = \mathbf{x}(t)$ is a solution with period p, we have $x'(t) = P(x(t), y(t))$ and $y'(t) = Q(x(t), y(t))$, and so

$$\begin{aligned}\int_C &-Q(x,y)\,dx + P(x,y)\,dy \\ &= \int_0^p \left[-Q(x(t),y(t))\,x'(t) + P(x(t),y(t))\,y'(t)\right]\,dt \\ &= \int_0^p -QP + PQ\,dt = 0\end{aligned}$$

Since $\operatorname{div}\mathbf{v} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ is continuous and does not change sign in $\mathbb{R}$, it follows that either $\operatorname{div}\mathbf{v} \geq 0$ or $\operatorname{div}\mathbf{v} \leq 0$ in $\mathbb{R}$, and so

$$\iint_{R_1} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dxdy \neq 0$$

This contradiction establishes that there are no periodic solutions in $\mathbb{R}$

Example The sliding bead satisfies the differential equation

$$mx'' = -mg\frac{f'(x)}{1 + [f'(x)]^2} - \beta x'$$

Show that there are no periodic solutions

Solution The corresponding plane autonomous system is

$$\begin{aligned}x' &= y \\ y' &= -g\frac{f'(x)}{1 + [f'(x)]^2} - \frac{\beta}{m}y\end{aligned}$$

and so $\operatorname{div}\mathbf{v} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} = -\dfrac{\beta}{m} < 0 \;\; (\beta > 0)$

- **Dulac Negative Criterion**

If $\delta(x,y)$ has continuous first partial derivatives in a simply connected region $\mathbb{R}$ and $\dfrac{\partial(\delta P)}{\partial x} + \dfrac{\partial(\delta Q)}{\partial y}$ does not change sign in $\mathbb{R}$, then the plane autonomous system has no periodic solutions in $\mathbb{R}$

Example Use $\delta(x,y) = 1/(xy)$ to show that the Lotka-Volterra competition equations

$$\begin{aligned}x' &= \frac{r_1}{K_1}x(K_1 - x - \alpha_{12}y) \\ y' &= \frac{r_2}{K_2}y(K_2 - y - \alpha_{21}x)\end{aligned}$$

have no periodic solutions in the first quadrant

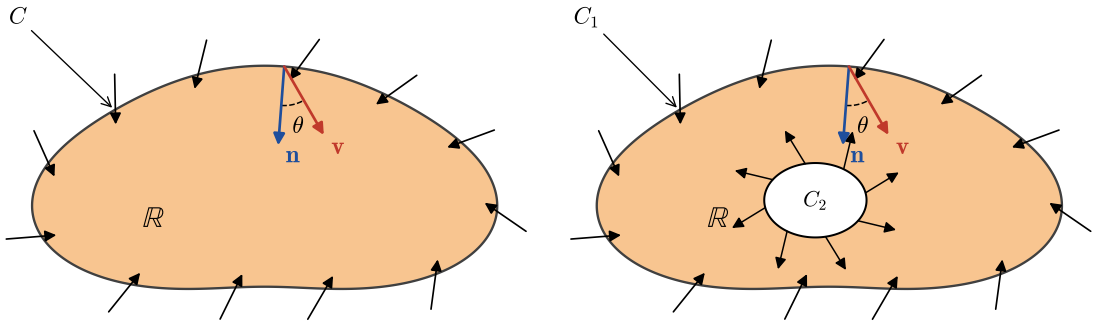
Solution If $\delta(x, y) = 1/(xy)$, then

$$\frac{\partial(\delta P)}{\partial x} + \frac{\partial(\delta Q)}{\partial y} = \frac{r_1}{K_1}\left(-\frac{1}{y}\right) + \frac{r_2}{K_2}\left(-\frac{1}{x}\right) < 0$$

in the first quadrant

- **Positive Criteria**

  A region $\mathbb{R}$ is called an **invariant region** for a plane autonomous system if, whenever $\mathbf{x}_0$ is in $\mathbb{R}$, the solution $\mathbf{x} = \mathbf{x}(t)$ remains in $\mathbb{R}$. If $\mathbf{n}(x, y)$ denotes a normal vector on the boundary that points inside the region, then $\mathbb{R}$ will be an invariant region for the plane autonomous system provided $\mathbf{v}(x, y) \cdot \mathbf{n}(x, y) \geq 0$ for all points (x, y) in the boundary



(a) type I invariant region

(b) type II invariant region

- **Poincaré–Bendixson I**

  Let $\mathbb{R}$ be an invariant region for a plane autonomous system and suppose that $\mathbb{R}$ has no critical points on its boundary

  (a) If $\mathbb{R}$ is a type I region that has a single unstable node or an unstable spiral point in its interior, then there is at least one periodic solution in $\mathbb{R}$

  (b) If $\mathbb{R}$ is a type II region that contains no critical points of the system, then there is at least one periodic solution in $\mathbb{R}$

  In either of the two cases, if $\mathbf{x} = \mathbf{x}(t)$ is a nonperiodic solution in $\mathbb{R}$, then $\mathbf{x}(t)$ spirals toward a cycle that is a solution to the system. This periodic solution is called a **limit cycle**

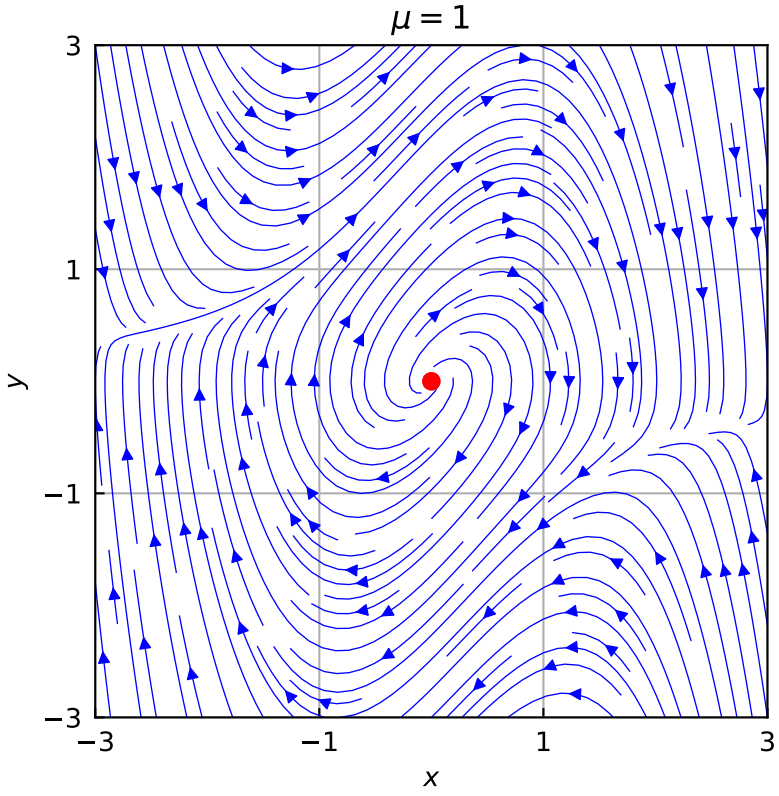
Example The Van der Pol equation is a nonlinear second-order differential equation that arises in electronics, and as a plane autonomous system it takes the form

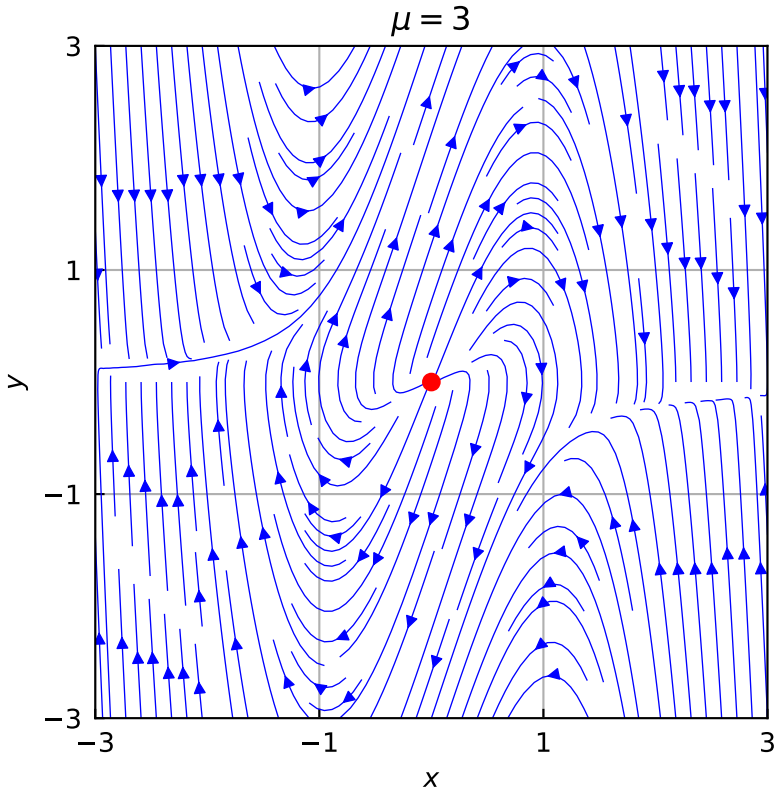
$$\begin{aligned} x' &= y \\ y' &= -\mu(x^2 - 1)y - x \end{aligned}$$

Show that the Van der Pol DE has a periodic solution when $\mu > 0$

```
w = 3
xp = np.linspace(-w, w, 100)
yp = np.linspace(-w, w, 100)
x, y = np.meshgrid(xp, yp)

fig = plt.figure(figsize=(5, 10))
```

```
mu = np.array([1, 3])
vdp_title = [r'$\mu=1$', r'$\mu=3$']

for i in range(2):

    ax = fig.add_subplot(2, 1, i +1)

    xdot = y
    ydot = -mu[i]*(x*x -1)*y -x

    ax.streamplot(x, y, xdot, ydot, density=1.5,
       linewidth=0.5, color='blue')
    ax.plot(0, 0, 'ro')
    ax.set_title(vdp_title[i])

    ax.axis((-w, w, -w, w))
    ax.set_aspect(aspect='equal')
    ax.grid()

    ax.tick_params(axis='both', direction='in', pad=5)
    ax.xaxis.set_ticks(np.linspace(-w, w, 4))
    ax.set_xlabel(r'$x$')
    ax.yaxis.set_ticks(np.linspace(-w, w, 4))
    ax.set_ylabel(r'$y$')

plt.show()
```



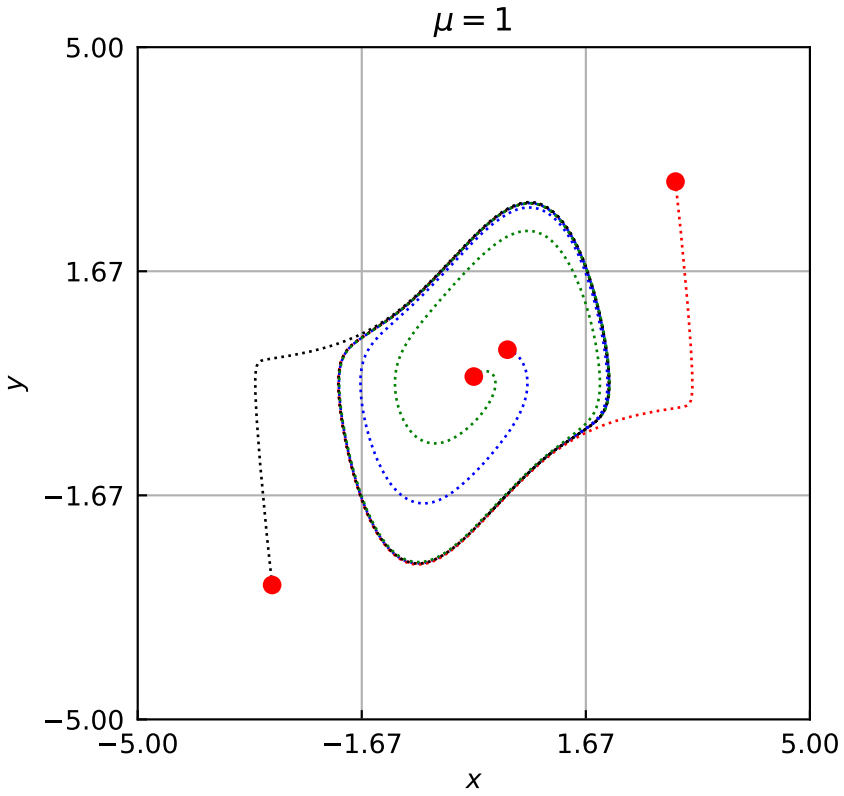


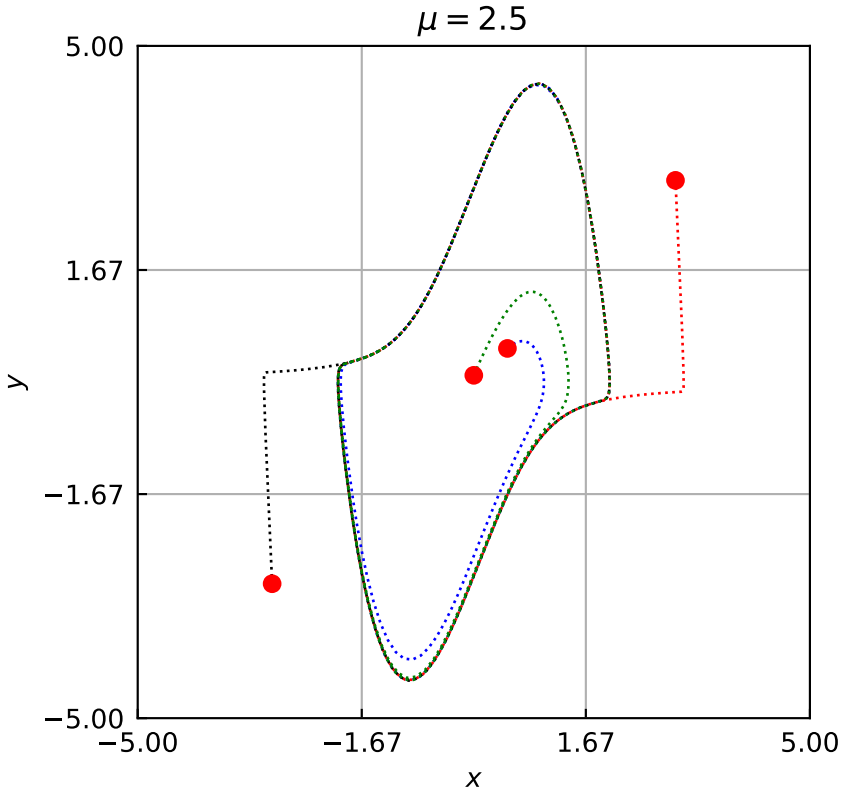
We will assume that there is a Type I invariant region $\mathbb{R}$ for the corresponding plane autonomous system and that this region contains $(0, 0)$ in its interior. The only critical point is $(0, 0)$, and the Jacobian matrix is given by

$$\mathbf{g}'((0,0)) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$$

Therefore, $\tau = \mu$, $\Delta = 1$, and $\tau^2 - 4\Delta = \mu^2 - 4$. Since $\mu > 0$, the critical point is either an unstable spiral point or an unstable node. By part (a) of **Poincaré–Bendixson I**, the system has at least one periodic solution in $\mathbb{R}$

```
td = 15
w = 5
mu_ = np.array([1, 2.5])
x0 = np.array([[0.5, 0.5], [3, 3], [-3,-3], [0, 0.1]])

def myODE(t, y):
    return [y[1], -mu*(y[0]*y[0] -1)*y[1] -y[0]]

vdp_title = [r'$\mu=1$', r'$\mu=2.5$']
vdp_line = ['b:', 'r:', 'k:', 'g:']

fig = plt.figure(figsize=(5, 10))

for i in range(2):

    ax = fig.add_subplot(2, 1, i +1)

    mu = mu_[i]
    for j in range(4):
        y0 = x0[j]
        t_eval = np.linspace(0, td, 500)
        sol = solve_ivp(myODE, [0, td], y0, t_eval=t_eval)
        ax.plot(sol.y[0], sol.y[1], vdp_line[j],
            linewidth=1)
        ax.plot(y0[0], y0[1], 'ro')

    ax.set_title(vdp_title[i])

    ax.axis((-w, w, -w, w))
    ax.set_aspect(aspect='equal')
    ax.grid()

    ax.tick_params(axis='both', direction='in', pad=5)
    ax.xaxis.set_ticks(np.linspace(-w, w, 4))
    ax.set_xlabel(r'$x$')
    ax.yaxis.set_ticks(np.linspace(-w, w, 4))
    ax.set_ylabel(r'$y$')

plt.show()
```





- **Poincaré–Bendixson II**

  Let $\mathbb{R}$ be a type I invariant region for a plane autonomous system that has no periodic solutions in $\mathbb{R}$

  (a) If $\mathbb{R}$ has a finite number of nodes or spiral points, then given any initial position $\mathbf{x}_0$ in $\mathbb{R}$, $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_1$ for some critical point $\mathbf{x}_1$

(b) If $\mathbb{R}$ has a single stable node or stable spiral point $\mathbf{x}_1$ in its interior and no critical points on its boundary, then $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_1$ for all initial positions $\mathbf{x}_0$ in $\mathbb{R}$: **globally stable**

Example Empirical evidence suggests that the plane autonomous system

$$x' = x^2 y - x + 1$$
$$y' = -x^2 y + \frac{1}{2}$$

has a type I invariant region $\mathbb{R}$ that lies inside the rectangle defined by $0 \le x \le 2$, $0 \le y \le 1$

(a) Use the Bendixson negative criterion to show that there are no periodic solutions in R

(b) If $\mathbf{x}_0$ is in $\mathbb{R}$ and $\mathbf{x} = \mathbf{x}(t)$ is the solution satisfying $\mathbf{x}(0) = \mathbf{x}_0$, use **Poincaré–Bendixson II** to find $\lim_{t\to\infty} \mathbf{x}(t)$

Answer 8 (end of chapter) · Solution

Summary

Case	Critical point $(0,0)$ of $\mathbf{x}' = \mathbf{A}\mathbf{x}$
Eigenvalues	$\lambda = \dfrac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$
$\tau^2 - 4\Delta > 0$	$\Delta < 0$: saddle point; $\Delta > 0$, $\tau < 0$: stable node; $\Delta > 0$, $\tau > 0$: unstable node
$\tau^2 - 4\Delta = 0$	degenerate node, stable if $\tau < 0$ and unstable if $\tau > 0$
$\tau^2 - 4\Delta < 0$	$\tau = 0$: center; $\tau < 0$: stable spiral point; $\tau > 0$: unstable spiral point
Linearization	$\mathbf{A} = \mathbf{g}'(\mathbf{x}_1)$: all eigenvalues with negative real part ⇒ asymptotically stable; one with positive real part ⇒ unstable
Bendixson	$\operatorname{div} \mathbf{v} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ does not change sign in a simply connected $\mathbb{R}$ ⇒ no periodic solution
Dulac	the same with $\dfrac{\partial(\delta P)}{\partial x} + \dfrac{\partial(\delta Q)}{\partial y}$ for some $\delta(x, y)$
Poincaré-Bendixson I	invariant region of type I with a single unstable node or spiral point inside, or of type II with no critical points ⇒ a periodic solution (limit cycle)

Worked Exercises

Answers

Answer 1 (Section 8.1)

$(\sqrt{2},\, 2)$ and $(-\sqrt{2},\, 2)$

Answer 2 (Section 8.2)

(a) stable node $(\lambda = -1, -6)$ **(b)** unstable node $(\lambda = 1, 2)$ **(c)** saddle point $(\lambda = \pm 1)$

Answer 3 (Section 8.2)

(a) degenerate unstable node $(\lambda = 1,\ \mathbf{A} = \mathbf{I})$ **(b)** degenerate stable node $(\lambda = -3)$ **(c)** degenerate unstable node $(\lambda = 4)$

Answer 4 (Section 8.2)

(a) center $(\lambda = \pm i$, clockwise) **(b)** stable spiral point $(\lambda = -1 \pm 2i$, counterclockwise)

Answer 5 (Section 8.4)

Trajectories $y^2 = \nu_0^2 + 2g\,(\operatorname{sech} x - \operatorname{sech} x_0)$ are closed curves around $(0, 0)$, so the motion is periodic, provided $|\nu_0| < \sqrt{2g \operatorname{sech} x_0}$ (always true for $\nu_0 = 0$)

Answer 6 (Section 8.4)

$(0, 0)$: unstable node $(50, 0)$: stable node $(0, 100)$: degenerate stable node $(20, 40)$: saddle point

The coexistence point is unstable, so almost every solution tends to $(50, 0)$ or $(0, 100)$ (competitive exclusion)

Answer 7 (Section 8.5)

With the Dulac function $\delta = 1/(xy)$: $\dfrac{\partial(\delta P)}{\partial x} + \dfrac{\partial(\delta Q)}{\partial y} = -\dfrac{0.004}{y} - \dfrac{0.001}{x} < 0$ in the first quadrant

Answer 8 (Section 8.5)

(a) $\operatorname{div} \mathbf{v} = -(x - y)^2 - (1 - y^2) \le 0$ in the rectangle (zero only at $(1, 1)$)

(b) $\lim\limits_{t \to \infty} \mathbf{x}(t) = \left(\frac{3}{2},\ \frac{2}{9}\right)$ for every $\mathbf{x}_0 \in \mathbb{R}$

Part II

Second Semester

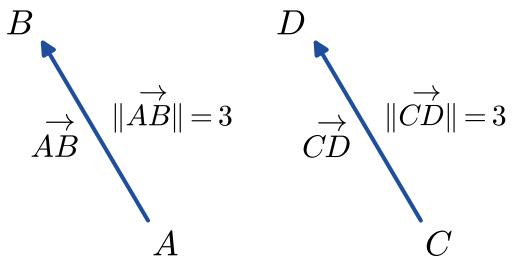
9 Vectors

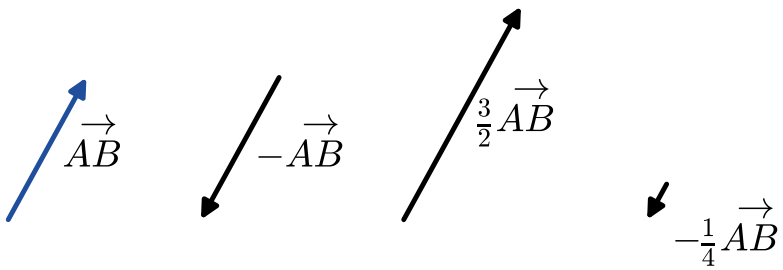
Learning objectives After this chapter you should be able to

- carry out vector algebra in 2- and 3-space with components, norms and unit vectors
- use the dot product for angles, orthogonality and projections, and the cross product for normals, areas and volumes
- write equations of lines and planes in 3-space
- decide whether a set is a vector space or a subspace, test linear independence, and find a basis and the dimension
- construct an orthogonal (orthonormal) basis by the Gram-Schmidt process

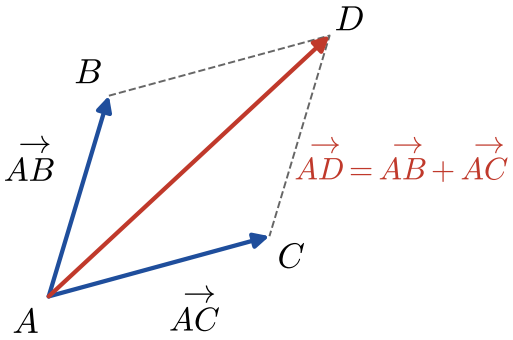
9.1 Vectors in 2-Space

- We distinguish two important quantities: **scalars** and **vectors**
  - A scalar is simply a real number or a quantity that has *magnitude*
  - A vector is usually described as a quantity that has both *magnitude* and *direction*
- Equal, Scalar Multiplication

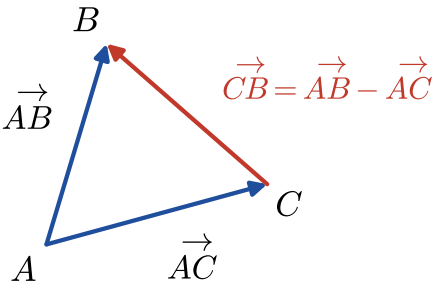




- Addition and Subtraction



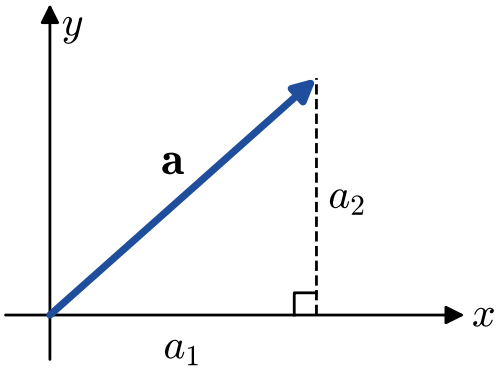
(a) sum



(b) difference

- Vectors in a Coordinate Plane

$$\mathbf{a} = \langle a_1, a_2 \rangle$$



- Let $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ be vectors in $\mathbb{R}^2$
  - Equality: $\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1,\ a_2 = b_2$
  - Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$
  - Scalar multiplication: $k\mathbf{a} = \langle ka_1, ka_2 \rangle$
- **Properties of Vectors**
  - $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
  - $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
  - $\mathbf{a} + \mathbf{0} = \mathbf{a}$
  - $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
  - $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$
  - $(k_1 + k_2)\mathbf{a} = k_1\mathbf{a} + k_2\mathbf{a}$
  - $k_1(k_2\mathbf{a}) = (k_1 k_2)\mathbf{a}$
  - $1\mathbf{a} = \mathbf{a}$
  - $0\mathbf{a} = \mathbf{0}$

- Magnitude, Unit Vector

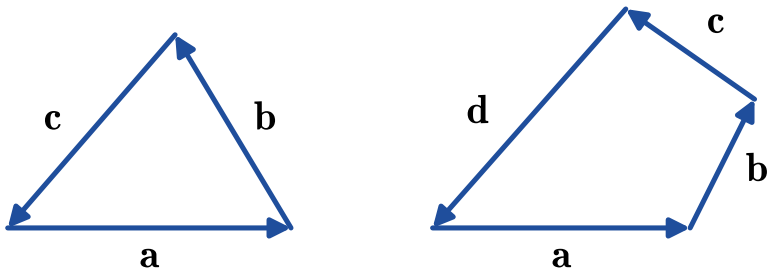
  $$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2} \quad \text{for} \quad \mathbf{a} = \langle a_1, a_2 \rangle$$
  $$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

- $\mathbf{i}$, $\mathbf{j}$ vectors

  $$\begin{aligned}\mathbf{a} &= \langle a_1, a_2 \rangle \\ &= \langle a_1, 0 \rangle + \langle 0, a_2 \rangle \\ &= a_1 \langle 1, 0 \rangle + a_2 \langle 0, 1 \rangle \\ &= a_1\mathbf{i} + a_2\mathbf{j}\end{aligned}$$

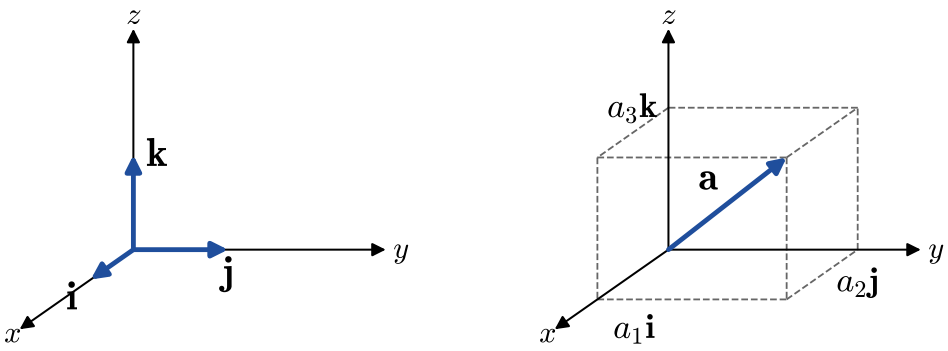
Example Use the given figure to prove the given result

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0} \quad \text{and} \quad \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = \mathbf{0}$$



Answer 1 (end of chapter) · Solution

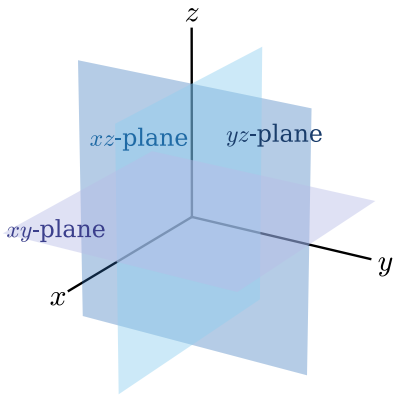
9.2 Vectors in 3-Space



- $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ vectors

  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$
  $= \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$
  $= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle$
  $= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

- Octants



Axes	Coordinates	Plane	Coordinates
x	$(a, 0, 0)$	xy	$(a, b, 0)$
y	$(0, b, 0)$	yz	$(0, b, c)$
z	$(0, 0, c)$	xz	$(a, 0, c)$

- Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ be vectors in $\mathbb{R}^3$
  - Equality: $\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$
  - Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
  - Scalar multiplication: $k\mathbf{a} = \langle ka_1, ka_2, ka_3 \rangle$
  - Negative: $-\mathbf{b} = (-1)\mathbf{b} = \langle -b_1, -b_2, -b_3 \rangle$
  - Subtraction: $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$
  - Zero vector: $\mathbf{0} = \langle 0, 0, 0 \rangle$
  - Magnitude: $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Example Describe the locus of points $P(x, y, z)$ that satisfy the given equations

- $xyz = 0$
- $(x + 1)^2 + (y - 2)^2 + (z + 3)^2 = 0$
- $(x - 2)(z - 8) = 0$

- $z^2 - 25 = 0$
- $x = y = z$

Answer 2 (end of chapter) · Solution

9.3 Dot Product (Inner or Scalar Product)

- The dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is

  $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

- Properties of the Dot Product
  - $\mathbf{a} \cdot \mathbf{b} = 0$ if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$
  - $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
  - $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
  - $\mathbf{a} \cdot (k\mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b})$
  - $\mathbf{a} \cdot \mathbf{a} \geq 0$
  - $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$

- Alternative Form

  The dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is

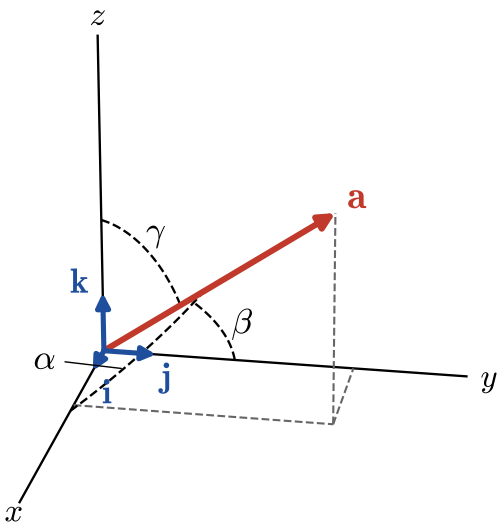
  $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$

  where θ is the angle between the vectors, $0 \leq \theta \leq \pi$

- Angle between two vectors

  $$\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\|\mathbf{a}\| \, \|\mathbf{b}\|}$$

  - Direction cosines



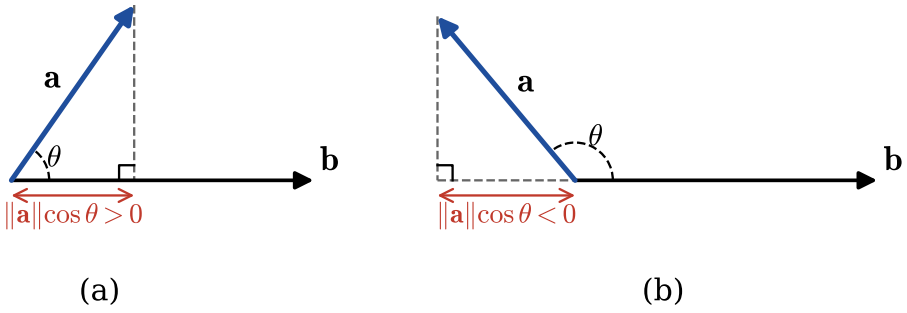
$$\cos\alpha = \frac{a_1}{\|\mathbf{a}\|}, \quad \cos\beta = \frac{a_2}{\|\mathbf{a}\|}, \quad \cos\gamma = \frac{a_3}{\|\mathbf{a}\|}$$

- Orthogonal vectors

  Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$

- Component of $\mathbf{a}$ on $\mathbf{b}$

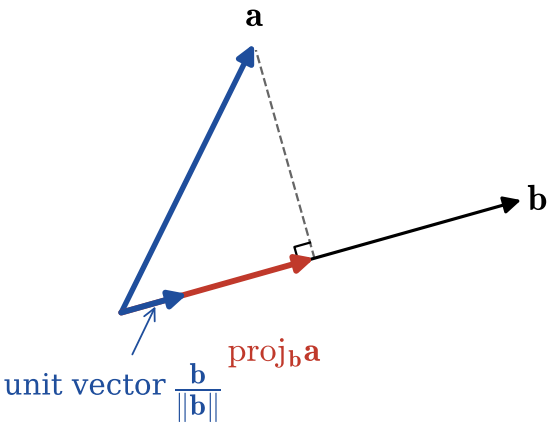
  To find the component of $\mathbf{a}$ on a vector $\mathbf{b}$, we dot $\mathbf{a}$ with a unit vector in the direction of $\mathbf{b}$



(a) (b)

$$\text{comp}_{\mathbf{b}}\mathbf{a} = \|\mathbf{a}\| \cos\theta = \|\mathbf{a}\| \cos\theta \frac{\|\mathbf{b}\|}{\|\mathbf{b}\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} = \mathbf{a} \cdot \frac{\mathbf{b}}{\|\mathbf{b}\|}$$

- Projection of $\mathbf{a}$ onto $\mathbf{b}$



$$\text{proj}_{\mathbf{b}}\mathbf{a} = (\text{comp}_{\mathbf{b}}\mathbf{a}) \left(\frac{\mathbf{b}}{\|\mathbf{b}\|} \right) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \right) \mathbf{b}$$

Example Verify that the vector $\mathbf{c} = \mathbf{b} - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \right) \mathbf{a}$ is orthogonal to the vector $\mathbf{a}$

Answer 3 (end of chapter) · Solution

Example In the methane molecule CH_4, the hydrogen atoms are located at the four vertices of a regular tetrahedron. The distance between the center of a hydrogen atom and the center of a carbon atom is 1.10 angstroms, and the hydrogen-carbon-hydrogen bond angle is $\theta = 109.5^\circ$. Using vector methods only, find the distance between two hydrogen atoms

Answer 4 (end of chapter) · Solution

Try it in Python: Dot (inner), outer and Kronecker products with `numpy`: Matrix and vector operations

9.4 Cross Product

- Cross product of Two vectors

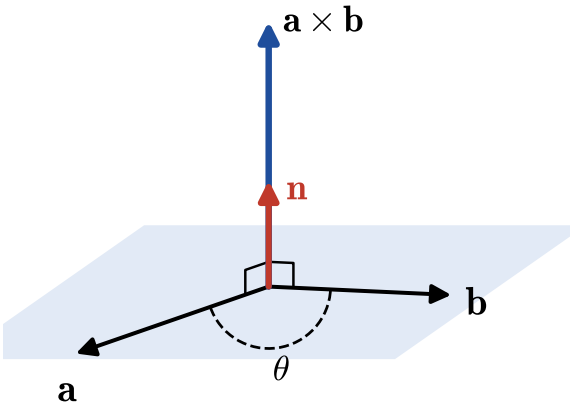
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

- Alternative Form and Magnitude of the Cross Product

  $\mathbf{a} \times \mathbf{b} = (\|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta) \, \mathbf{n} = \|\mathbf{a} \times \mathbf{b}\| \, \mathbf{n}$

- Right-Hand Rule

  The vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} \times \mathbf{b}$ form a right-hand system:



- Parallel Vectors

  Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

- Properties of the Cross Product

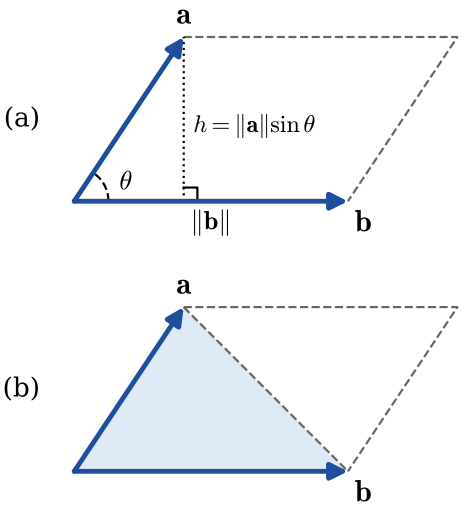
  - $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$
  - $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
  - $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$
  - $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$

  - $\mathbf{a} \times (k\mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b})$
  - $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
  - $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$
  - $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$

- Areas of a Parallelogram and a Triangle

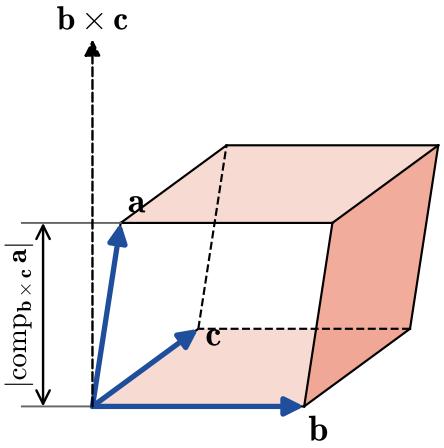
  (a) $A = \|\mathbf{a} \times \mathbf{b}\|$

  (b) $A = \frac{1}{2}\|\mathbf{a} \times \mathbf{b}\|$



- Volume of a Parallelepiped

  $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$



- Special Products

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

Example Prove or disprove

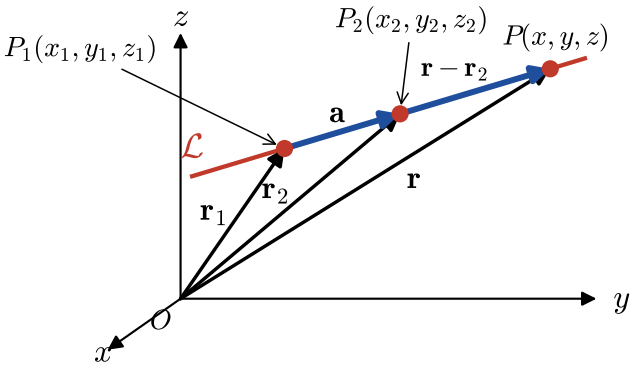
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$
- Lagrange's identity $\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2$

Answer 5 (end of chapter) · Solution

9.5 Lines and Planes in 3-Space

- Lines

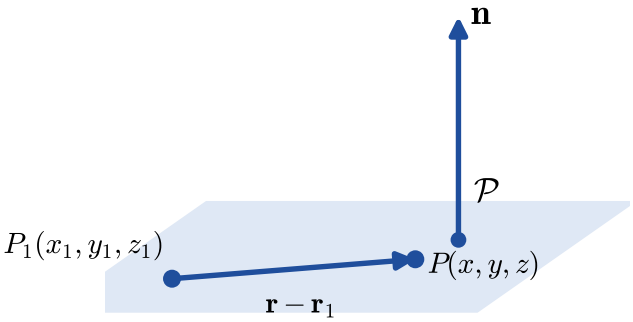


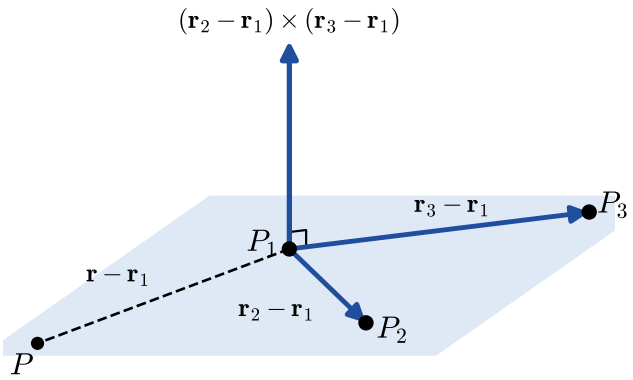
$$\mathbf{r} - \mathbf{r}_2 = t(\mathbf{r}_2 - \mathbf{r}_1) = t\mathbf{a}$$
$$\Rightarrow\ t = \frac{x - x_2}{a_1} = \frac{y - y_2}{a_2} = \frac{z - z_2}{a_3}$$

- Planes

  If the normal vector is $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$, then

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$
$$\Rightarrow\ n_1(x - x_1) + n_2(y - y_1) + n_3(z - z_1) = 0$$





Example Find parametric equation for the line that contains $(-4, 1, 7)$ and is perpendicular to the plane

$$-7x + 2y + 3z = 1$$

Answer 6 (end of chapter) · Solution

Example Find parametric equations for the line of intersection of the given planes

$$\begin{aligned} 5x - 4y - 9z &= 8 \\ x + 4y + 3z &= 4 \end{aligned}$$

Answer 7 (end of chapter) · Solution

Example Find an equation of the plane that contains the given line and is orthogonal to the indicated plane

$$\frac{2-x}{3} = \frac{y+2}{5} = \frac{z-8}{2}; \quad 2x - 4y - z + 16 = 0$$

Answer 8 (end of chapter) · Solution

9.6 Vector Spaces

- n-Space
  - A **vector in n-space** is any ordered n-tuple $\mathbf{a} = \langle a_1, a_2, \cdots, a_n \rangle$ of real numbers called the components of $\mathbf{a}$
  - The set of all vectors in n-space is denoted by $\mathbb{R}^n$
  - The component definitions in 3-space carry over to $\mathbb{R}^n$ in a natural way

- The standard **dot** or **inner product** of two n-vectors is the real number defined by

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \langle a_1, a_2, \cdots, a_n\rangle \cdot \langle b_1, b_2, \cdots, b_n\rangle \\ &= a_1b_1 + a_2b_2 + \cdots + a_nb_n\end{aligned}$$

- Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ are said to be **orthogonal** if and only if $\mathbf{a}\cdot\mathbf{b} = 0$
- **Vector Space**

  Let V be a set of elements on which two operations called **vector addition** and **scalar multiplication** are defined. Then V is said to be a **vector space** if the following 10 properties are satisfied
- **Axioms for Vector Addition**
  - If $\mathbf{x}$ and $\mathbf{y}$ are in V, then $\mathbf{x}+\mathbf{y}$ is in V
  - For all $\mathbf{x}$, $\mathbf{y}$ in V, $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$
  - For all $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in V, $\mathbf{x}+(\mathbf{y}+\mathbf{z}) = (\mathbf{x}+\mathbf{y})+\mathbf{z}$
  - There is a unique vector $\mathbf{0}$ in V such that $\mathbf{0}+\mathbf{x} = \mathbf{x}+\mathbf{0} = \mathbf{x}$
  - For each $\mathbf{x}$ in V, there exists a vector $-\mathbf{x}$ such that $\mathbf{x}+(-\mathbf{x}) = (-\mathbf{x})+\mathbf{x} = \mathbf{0}$
- **Axioms for Scalar Multiplication**
  - If k is any scalar and $\mathbf{x}$ is in V, then $k\mathbf{x}$ is in V
  - $k(\mathbf{x}+\mathbf{y}) = k\mathbf{x}+k\mathbf{y}$
  - $(k_1+k_2)\mathbf{x} = k_1\mathbf{x}+k_2\mathbf{x}$
  - $k_1(k_2\mathbf{x}) = (k_1k_2)\mathbf{x}$
  - $1\mathbf{x} = \mathbf{x}$
- Here are some important vector spaces:
  - The set $\mathbb{R}$ of real numbers
  - The set $\mathbb{R}^2$ of ordered pairs
  - The set $\mathbb{R}^3$ of ordered triples
  - The set $\mathbb{R}^n$ of ordered n-tuples
  - The set P_n of polynomials of degree less than or equal to n
  - The set P of all polynomials
  - The set of real-valued functions f defined on the set $\mathbb{R}$
  - The set $C[a,b]$ of real-valued functions f continuous on the closed interval $[a,b]$
  - The set $C(-\infty,\infty)$ of real-valued functions f continuous on the set $\mathbb{R}$
  - The set $C^n[a,b]$ of all real-valued functions f, for which f, f', f'', $\cdots$, $f^{(n)}$ exist and are continuous on the closed interval $[a,b]$

Example Consider the set V of positive real numbers. If x and y denote positive real numbers, then we write vectors in V as $\mathbf{x} = x$ and $\mathbf{y} = y$

- Now, addition of vectors is defined by

  $\mathbf{x} + \mathbf{y} = xy$

- and scalar multiplication is defined by

  $k\mathbf{x} = x^k$

  Determine whether V is a vector space

Answer 9 (end of chapter) · Solution

- **Subspace**
  - A nonempty subset W of a vector space V is a **subspace** of V if and only if W is closed under vector addition and scalar multiplication defined on V:
    * If $\mathbf{x}$ and $\mathbf{y}$ are in W, then $\mathbf{x} + \mathbf{y}$ is in W
    * If $\mathbf{x}$ is in W and k is any scalar, then $k\mathbf{x}$ is in W
- **Linear Independence**

  A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ is said to be **linearly independent** if the only constants satisfying the equation

  $$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_n\mathbf{x}_n = \mathbf{0}$$

  are $k_1 = k_2 = \cdots = k_n = 0$. If the set of vectors is not linearly independent, then it is said to be **linearly dependent**
- **Basis**

  Consider a set of vectors $B = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ in a vector space V. If the set B is linearly independent and if every vector in V can be expressed as a linear combination of these vectors, then B is said to be a **basis** in V
- **Dimension**

  The number of vectors in a basis B for a vector space V is said to be the **dimension** of the space
- **Span**

  If S denotes any set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ in a vector space V, then the set of all linear combinations of the vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ in S,

  $$\{k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_n\mathbf{x}_n\}$$

  where the k_i are scalars, is called the **span** of the vectors and written $\text{span}(S)$ or $\text{span}(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$

  If $V = \text{span}(S)$, then we say that S is a spanning set for the vector space V, or that S spans V

Example Determine whether the given set is a vector space:

- The set of complex numbers $a+bi$, where $i^2=-1$, addition and scalar multiplication defined by

$$(a_1+b_1i)+(a_2+b_2i)=(a_1+a_2)+(b_1+b_2)i$$
$$k(a+bi)=ka+kbi$$

  where k is a real number

- The set of arrays of real numbers $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, addition and scalar multiplication defined by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} a_{12}+b_{12} & a_{11}+b_{11} \\ a_{22}+b_{22} & a_{21}+b_{21} \end{pmatrix}$$
$$k\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}=\begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix}$$

Answer 10 (end of chapter) · Solution

9.7 Gram-Schmidt Orthogonalization Process

- Every vector $\mathbf{u}$ in $\mathbb{R}^n$ can be written as a linear combination of the vectors in the standard basis $B=\{\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n\}$:

$$\mathbf{e}_1=\langle 1,0,0,\cdots,0\rangle$$
$$\mathbf{e}_2=\langle 0,1,0,\cdots,0\rangle$$
$$\vdots$$
$$\mathbf{e}_n=\langle 0,0,0,\cdots,1\rangle$$

  This standard basis $B=\{\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n\}$ is also an example of an **orthonormal basis**

$$\mathbf{e}_i\cdot\mathbf{e}_j=0,\ i\neq j\ \text{ and }\ \|\mathbf{e}_i\|=1,\ i=1,2,\cdots,n$$

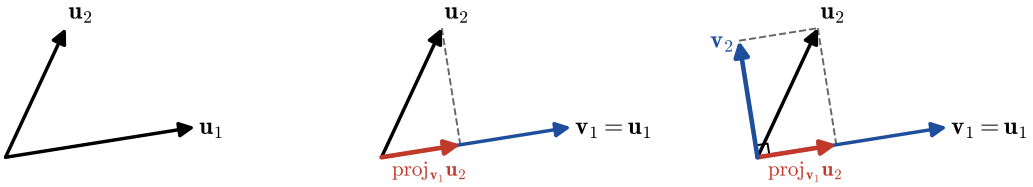
- Coordinates Relative to an Orthonormal Basis

  Suppose $B=\{\mathbf{w}_1,\mathbf{w}_2,\cdots,\mathbf{w}_n\}$ is an orthonormal basis for $\mathbb{R}^n$. If $\mathbf{u}$ is any vector in $\mathbb{R}^n$, then

$$\mathbf{u}=(\mathbf{u}\cdot\mathbf{w}_1)\mathbf{w}_1+(\mathbf{u}\cdot\mathbf{w}_2)\mathbf{w}_2+\cdots+(\mathbf{u}\cdot\mathbf{w}_n)\mathbf{w}_n$$

- Constructing an Orthogonal Basis for $\mathbb{R}^2$

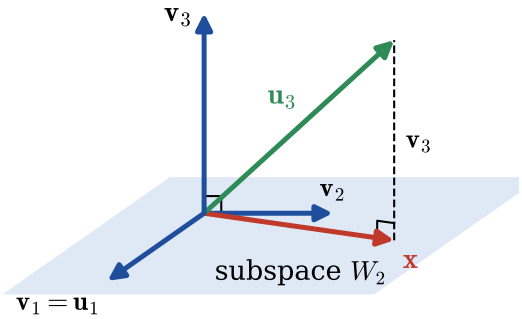
$$\mathbf{v}_1=\mathbf{u}_1$$
$$\mathbf{v}_2=\mathbf{u}_2-\left(\frac{\mathbf{u}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1$$



(a) linearly independent $\mathbf{u}_1, \mathbf{u}_2$ (b) projection of $\mathbf{u}_2$ onto $\mathbf{v}_1$ (c) $\mathbf{v}_1 \perp \mathbf{v}_2$

- Constructing an Orthogonal Basis for $\mathbb{R}^3$

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 \\
\mathbf{v}_2 &= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2
\end{aligned}$$



- **Gram-Schmidt Orthogonalization Process**

  Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m\}$, $m \leq n$, be a basis for a subspace W_m of $\mathbb{R}^n$. Then $B' = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$, where

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 \\
\mathbf{v}_2 &= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_m &= \mathbf{u}_m - \left(\frac{\mathbf{u}_m \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_m \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \cdots - \left(\frac{\mathbf{u}_m \cdot \mathbf{v}_{m-1}}{\mathbf{v}_{m-1} \cdot \mathbf{v}_{m-1}}\right)\mathbf{v}_{m-1}
\end{aligned}$$

  is an orthogonal basis for W_m

- An orthonormal basis for W_m is

$$B'' = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\} = \left\{\frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1, \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2, \cdots, \frac{1}{\|\mathbf{v}_m\|}\mathbf{v}_m\right\}$$

Example An inner product defined on the vector space P_2 of all polynomials of degree less than or equal to 2, is given by

$$\langle p, q\rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$$

Use the Gram-Schmidt orthogonalization process to transform the given basis for P_2 into an orthogonal basis B'

$$B = \{1, x, x^2\}$$

Answer 11 (end of chapter) · Solution

Summary

Topic	Formula
Norm	$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, unit vector $\mathbf{a}/\|\mathbf{a}\|$
Dot product	$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$; orthogonal iff $\mathbf{a}\cdot\mathbf{b} = 0$
Projection	$\text{comp}_{\mathbf{b}}\mathbf{a} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|}$, $\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b}$
Cross product	$\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} = (\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta)\,\mathbf{n}$; area $\|\mathbf{a}\times\mathbf{b}\|$, volume $\|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\|$
Line	$\mathbf{r} = \mathbf{r}_2 + t\mathbf{a}$, $\dfrac{x-x_2}{a_1} = \dfrac{y-y_2}{a_2} = \dfrac{z-z_2}{a_3}$
Plane	$\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_1) = 0$, $n_1(x-x_1) + n_2(y-y_1) + n_3(z-z_1) = 0$
Basis and dimension	a linearly independent set that spans V; the dimension is the number of basis vectors
Gram-Schmidt	$\mathbf{v}_1 = \mathbf{u}_1$, $\mathbf{v}_k = \mathbf{u}_k - \sum_{j<k}\dfrac{\mathbf{u}_k\cdot\mathbf{v}_j}{\mathbf{v}_j\cdot\mathbf{v}_j}\mathbf{v}_j$; orthonormal: $\mathbf{v}_k/\|\mathbf{v}_k\|$

Worked Exercises

1. Determine whether the given vectors are linearly independent or linearly dependent:

$$1, (x+1), (x+1)^2, x^2 \text{ in } P_2$$

Solution

Step 1: Understand the space P_2

The vector space P_2 consists of all real polynomials of degree at most 2. So any element in P_2 has the form:

$$a_0 + a_1 x + a_2 x^2$$

That means the dimension of P_2 is 3. Therefore, any set of more than 3 vectors in P_2 must be linearly dependent

Step 2: Count the number of vectors

We are given 4 vectors:

$$1, \quad (x+1), \quad (x+1)^2, \quad x^2$$

Since $\dim P_2 = 3$, and we are given 4 vectors, they must be linearly dependent

(Optional) Step 3: Verify dependence explicitly

To see this more concretely, write each vector in terms of the standard basis $\{1, x, x^2\}$:

- $1 = 1 + 0x + 0x^2 \Rightarrow (1, 0, 0)$
- $x + 1 = 1x + 1 \Rightarrow (1, 1, 0)$
- $(x+1)^2 = x^2 + 2x + 1 \Rightarrow (1, 2, 1)$
- $x^2 = x^2 \Rightarrow (0, 0, 1)$

Write these as column vectors:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

You can now see, for instance:

$$(x+1)^2 = x^2 + 2(x) + 1 = x^2 + 2(x+1) - 1 \Rightarrow (x+1)^2 - 2(x+1) + x^2 - 1 = 0$$

So there is a nontrivial linear combination:

$$1 \cdot (x+1)^2 - 2 \cdot (x+1) + 1 \cdot x^2 - 1 = 0$$

2. Explain why $f(x) = \dfrac{x}{x^2 + 4x + 3}$ is a vector in $C[0, 3]$ but not a vector in $C[-3, 0]$

Solution

Step 1: Understand the vector space $C[a, b]$

- $C[a, b]$ is the vector space of all continuous real-valued functions defined on the interval $[a, b]$
- So, a function $f(x)$ is a vector in $C[a, b]$ if and only if it is continuous on the entire interval $[a, b]$

Step 2: Analyze $f(x) = \dfrac{x}{x^2 + 4x + 3}$

Factor the denominator:

$$x^2 + 4x + 3 = (x + 1)(x + 3)$$

So the function becomes:

$$f(x) = \frac{x}{(x + 1)(x + 3)}$$

This function is undefined at $x = -1$ and $x = -3$, since the denominator becomes 0 at those points.

Step 3: Behavior on $[0, 3]$

- The interval $[0, 3]$ lies entirely to the right of both singularities $x = -1$ and $x = -3$
- On this interval, $f(x)$ is:
  - Defined (denominator never zero),
  - Continuous (since it's a rational function with no discontinuities on this domain)

Step 4: Behavior on $[-3, 0]$

- This interval includes the point $x = -1$, where the denominator becomes 0.
- At $x = -1$, the function is undefined, so it is not continuous on the entire interval

Conclusion:

- $f(x) \in C[0, 3]$ — it is a continuous function on that interval, so it is a vector in $C[0, 3]$
- $f(x) \notin C[-3, 0]$ — because it is not defined (and hence not continuous) at $x = -1$, so it is not a vector in $C[-3, 0]$

3. The given vectors span a subspace W of $\mathbb{R}^4$. Use the Gram-Schmidt orthogonalization process to construct an orthonormal basis for the subspace

$$\mathbf{u}_1 = \langle 4, 0, 2, -1 \rangle\,,\ \mathbf{u}_2 = \langle 2, 1, -1, 1 \rangle\,,\ \text{and } \mathbf{u}_3 = \langle 1, 1, -1, 0 \rangle$$

Solution

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_3 = \langle 1, 1, -1, 0\rangle \\
\mathbf{v}_2 &= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
&= \langle 2, 1, -1, 1\rangle - \frac{4}{3}\langle 1, 1, -1, 0\rangle = \frac{1}{3}\langle 2, -1, 1, 3\rangle \\
&\Rightarrow \ \langle 2, -1, 1, 3\rangle \\
\mathbf{v}_3 &= \mathbf{u}_1 - \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&= \langle 4, 0, 2, -1\rangle - \frac{2}{3}\langle 1, 1, -1, 0\rangle - \frac{7}{15}\langle 2, -1, 1, 3\rangle \\
&= \frac{1}{5}\langle 12, -1, 11, -12\rangle \ \Rightarrow \ \langle 12, -1, 11, -12\rangle \\
&\Downarrow \\
\mathbf{w}_1 &= \frac{1}{\sqrt{3}}\langle 1, 1, -1, 0\rangle \\
\mathbf{w}_2 &= \frac{1}{\sqrt{15}}\langle 2, -1, 1, 3\rangle \\
\mathbf{w}_3 &= \frac{1}{\sqrt{410}}\langle 12, -1, 11, -12\rangle
\end{aligned}
$$

4. The set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \langle 1, 1, 3\rangle\,,\ \mathbf{u}_2 = \langle 1, 4, 1\rangle\,,\ \text{and}\ \mathbf{u}_3 = \langle 1, 10, -3\rangle$$

is linearly dependent in $\mathbb{R}^3$. Carry out the Gram-Schmidt orthogonalization process and then discuss that result

Solution

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 = \langle 1, 1, 3\rangle \\
\mathbf{v}_2 &= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = \langle 1, 4, 1\rangle - \frac{1+4+3}{1+1+9}\langle 1, 1, 3\rangle \\
&= \frac{1}{11}\langle 3, 36, -13\rangle \ \ \Rightarrow \ \ \langle 3, 36, -13\rangle \\
\mathbf{v}_3 &= \mathbf{u}_1 - \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&= \langle 1, 10, -3\rangle - \frac{1+10-9}{1+1+9}\langle 1, 1, 3\rangle - \frac{93+360+39}{9+36^2+13^2}\langle 3, 36, -13\rangle \\
&= \langle 0, 0, 0\rangle \\
&\Downarrow \\
\mathbf{w}_1 &= \frac{1}{\sqrt{11}}\langle 1, 1, 3\rangle \\
\mathbf{w}_2 &= \frac{1}{\sqrt{1474}}\langle 3, 36, -13\rangle
\end{aligned}
$$

Answers

Answer 1 (Section 9.1)

Vectors joined head to tail around a closed path add up to the vector from the starting point back to itself, i.e. the zero vector, whatever the number of vertices.

Answer 2 (Section 9.2)

(a) the union of the three coordinate planes **(b)** the single point $(-1, 2, -3)$ **(c)** the union of the planes $x = 2$ and $z = 8$

(d) the two planes $z = 5$ and $z = -5$ **(e)** the line through the origin with direction $\langle 1, 1, 1 \rangle$

Answer 3 (Section 9.3)

$$\mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a} \cdot \mathbf{a} = 0, \quad \text{so } \mathbf{c} \perp \mathbf{a}$$

Answer 4 (Section 9.3)

$$d = \|\mathbf{a} - \mathbf{b}\| = 1.10\sqrt{2(1 - \cos 109.5^\circ)} \approx 1.80 \text{ Å}$$

Answer 5 (Section 9.4)

(a) true **(b)** false (counterexample $\mathbf{a} = \mathbf{b} = \mathbf{i}$, $\mathbf{c} = \mathbf{j}$) **(c)** true (Jacobi identity) **(d)** true

Answer 6 (Section 9.5)

$$x = -4 - 7t, \quad y = 1 + 2t, \quad z = 7 + 3t, \quad -\infty < t < \infty$$

Answer 7 (Section 9.5)

$$x = 2 + t, \quad y = \tfrac{1}{2} - t, \quad z = t, \quad -\infty < t < \infty$$

Answer 8 (Section 9.5)

$$3x + y + 2z = 20$$

Answer 9 (Section 9.6)

Yes, all ten axioms hold. The zero vector is the number 1, and the negative of $\mathbf{x}$ is $1/x$.

Answer 10 (Section 9.6)

(a) a vector space (the same as $\mathbb{R}^2$) **(b)** not a vector space (there is no zero vector, and addition is not associative)

Answer 11 (Section 9.7)

$$B' = \{\, 1, \quad x, \quad x^2 - \tfrac{1}{3} \,\}$$

10 Vector Calculus

Learning objectives After this chapter you should be able to

- differentiate vector functions and describe motion on a curve: velocity, acceleration, curvature and its components
- compute partial and directional derivatives, gradients, tangent planes, and the divergence and curl of a vector field
- evaluate line integrals, and recognize conservative fields and path independence
- evaluate double, triple and surface integrals, using polar, cylindrical and spherical coordinates and a change of variables
- apply Green's theorem, Stokes' theorem and the divergence theorem

10.1 Vector Functions

- A parametric curve in space is a set of ordered triples (x, y, z), where

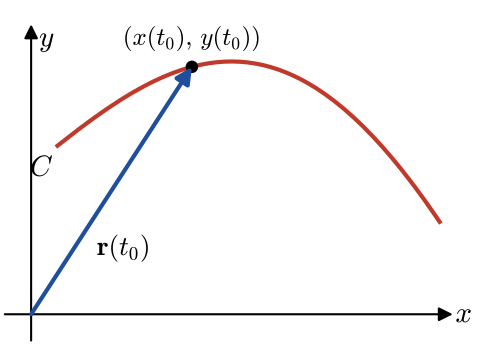
$$x = f(t),\ y = g(t),\ z = h(t)$$

  are continuous on an interval: $a \le t \le b$

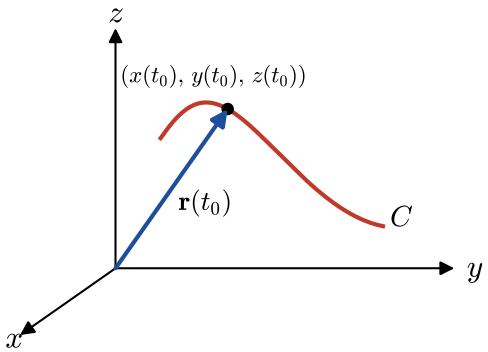
- The following

$$\mathbf{r}(t) = \langle f(t),\, g(t),\, h(t)\rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$$

  are **vector-valued functions**



(a) 2-space



(b) 3-space

- **Limits**
  - $\lim\limits_{t\to a} \mathbf{r}(t) = \left\langle \lim\limits_{t\to a} f(t),\ \lim\limits_{t\to a} g(t),\ \lim\limits_{t\to a} h(t) \right\rangle$

- If $\lim_{t\to a}\mathbf{r}_1(t)=\mathbf{L}_1$ and $\lim_{t\to a}\mathbf{r}_2(t)=\mathbf{L}_2$, then

  $\lim_{t\to a} c\mathbf{r}_1(t)=c\mathbf{L}_1$

  $\lim_{t\to a}[\mathbf{r}_1(t)+\mathbf{r}_2(t)]=\mathbf{L}_1+\mathbf{L}_2$

  $\lim_{t\to a}[\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)]=\mathbf{L}_1\cdot\mathbf{L}_2$

- **Continuity**

  - A vector function $\mathbf{r}$ is said to be **continuous** at $t=a$ if

    $\mathbf{r}(a)$ is defined,

    $\lim_{t\to a}\mathbf{r}(t)$ exists, and

    $\lim_{t\to a}\mathbf{r}(t)=\mathbf{r}(a)$

- **Derivatives**

  - The **derivative** of a vector function $\mathbf{r}$ is

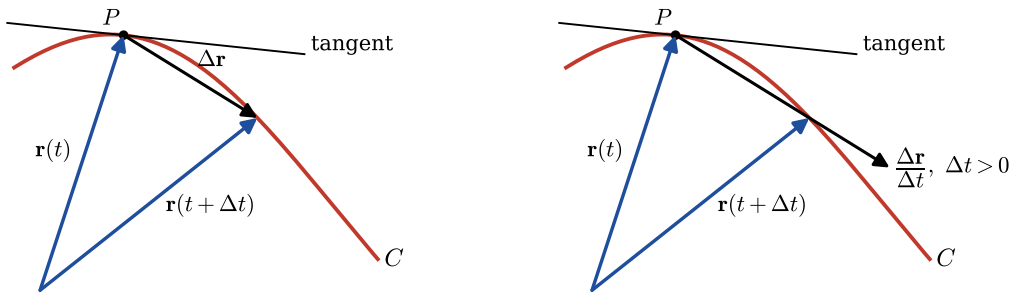
    $$\mathbf{r}'(t)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}[\mathbf{r}(t+\Delta t)-\mathbf{r}(t)]$$

    for all t for which the limit exists

  - If $\mathbf{r}(t)=\langle f(t),g(t),h(t)\rangle$, where f, g, and h are differentiable, then

    $\mathbf{r}'(t)=\langle f'(t),g'(t),h'(t)\rangle$

  - When $\mathbf{r}$ has a continuous first derivative and $\mathbf{r}'(t)\neq\mathbf{0}$ for all t in the open interval (a,b), then $\mathbf{r}$ is said to be a **smooth function**



  - In the case of the **second derivative**, we have

    $$\mathbf{r}''(t)=\langle f''(t),g''(t),h''(t)\rangle=f''(t)\mathbf{i}+g''(t)\mathbf{j}+h''(t)\mathbf{k}$$

  - Chain rule

    If $\mathbf{r}$ is a differentiable vector function and $s=u(t)$ is a differentiable scalar function, then the derivative of $\mathbf{r}(s)$ with respect to t is

    $$\frac{d\mathbf{r}}{dt}=\frac{d\mathbf{r}}{ds}\frac{ds}{dt}=\mathbf{r}'(s)u'(t)$$

  - Rules of Differentiation

    Let $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ be differentiable vector functions and $u(t)$ a differentiable scalar function

* $\dfrac{d}{dt}\left[\mathbf{r}_1(t)+\mathbf{r}_2(t)\right]=\mathbf{r}_1'(t)+\mathbf{r}_2'(t)$
* $\dfrac{d}{dt}\left[u(t)\mathbf{r}_1(t)\right]=u(t)\mathbf{r}_1'(t)+u'(t)\mathbf{r}_1(t)$
* $\dfrac{d}{dt}\left[\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)\right]=\mathbf{r}_1(t)\cdot\mathbf{r}_2'(t)+\mathbf{r}_1'(t)\cdot\mathbf{r}_2(t)$
* $\dfrac{d}{dt}\left[\mathbf{r}_1(t)\times\mathbf{r}_2(t)\right]=\mathbf{r}_1(t)\times\mathbf{r}_2'(t)+\mathbf{r}_1'(t)\times\mathbf{r}_2(t)$

- Integrals

$$\int \mathbf{r}(t)\,dt=\left[\int f(t)\,dt\right]\mathbf{i}+\left[\int g(t)\,dt\right]\mathbf{j}+\left[\int h(t)\,dt\right]\mathbf{k}$$

- Length of a Space Curve

If $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$ is a smooth function, then it can be shown that the **length** of the smooth curve traced by $\mathbf{r}$ is given by

$$s=\int_a^b \sqrt{\left[f'(t)\right]^2+\left[g'(t)\right]^2+\left[h'(t)\right]^2}\,dt=\int_a^b \|\mathbf{r}'(t)\|\,dt$$

10.2 Motion on a Curve

- Position

Suppose a body moves along a curve C so that its position at time t is given by the vector function

$$\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$$

- Velocity and Acceleration

If f, g, and h have second derivatives, then the vectors

$$\mathbf{v}(t)=\mathbf{r}'(t)=f'(t)\mathbf{i}+g'(t)\mathbf{j}+h'(t)\mathbf{k}$$
$$\mathbf{a}(t)=\mathbf{r}''(t)=f''(t)\mathbf{i}+g''(t)\mathbf{j}+h''(t)\mathbf{k}$$

are called the **velocity** and **acceleration** of the particle, respectively

  - The scalar function $\|\mathbf{v}(t)\|$ is the **speed** of the particle
  - The speed is related to arc length s by $s'(t)=\|\mathbf{v}(t)\|$
  - If a particle moves with a constant speed c, then its acceleration vector is perpendicular to the velocity vector $\mathbf{v}$
  - To see this, note that $\mathbf{v}\cdot\mathbf{v}=c^2$. We differentiate both sides with respect to t and obtain

$$\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v})=\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}+\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}=2\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}=0$$

Thus, $\dfrac{d\mathbf{v}}{dt}\cdot\mathbf{v}=0$ or $\mathbf{a}(t)\cdot\mathbf{v}(t)=0$ for all t

- Centripetal Acceleration

  For circular motion in the plane, described by $\mathbf{r}(t) = r_0 \cos \omega t\, \mathbf{i} + r_0 \sin \omega t\, \mathbf{j}$, it is evident that

$$\mathbf{r}'' = -\omega^2 \mathbf{r}$$

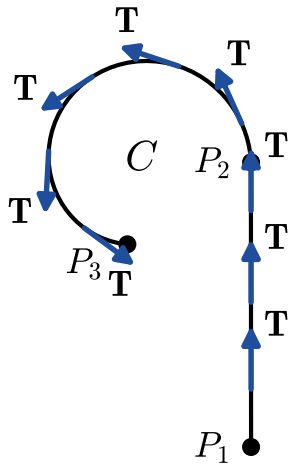
  – This means that the acceleration vector $\mathbf{a}(t) = \mathbf{r}''(t)$ points in the direction opposite to that of the position vector $\mathbf{r}(t)$. We then say $\mathbf{a}(t)$ is **centripetal acceleration**

10.3 Curvature and Components of Acceleration

- We know that $\mathbf{r}'(t)$ is a tangent vector to the curve C, and consequently

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

  is a **unit tangent**



$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} \quad \text{and so}$$

$$\frac{d\mathbf{r}}{ds} = \frac{\dfrac{d\mathbf{r}}{dt}}{\dfrac{ds}{dt}} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \mathbf{T}(t)$$

- Let $\mathbf{r}(t)$ be a vector function defining a smooth curve C. If s is the arc length parameter and $\mathbf{T}$ is the unit tangent vector, then the **curvature** of C at a point is

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$$

- Using the chain rule, we can write

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \quad \Rightarrow \quad \frac{d\mathbf{T}}{ds} = \frac{\dfrac{d\mathbf{T}}{dt}}{\dfrac{ds}{dt}}$$

  In other words, curvature is given by $\kappa(t) = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$

- Tangential and Normal Components of Acceleration

  – The velocity of the particle on C is $\mathbf{v}(t) = \mathbf{r}'(t)$, whereas its speed is $\dfrac{ds}{dt} = v = \|\mathbf{v}(t)\|$. Thus,

$$\mathbf{v}(t) = v\mathbf{T}$$

  – Differentiating this last expression with respect to t gives acceleration

$$\mathbf{a}(t) = v\frac{d\mathbf{T}}{dt} + \frac{dv}{dt}\mathbf{T}$$

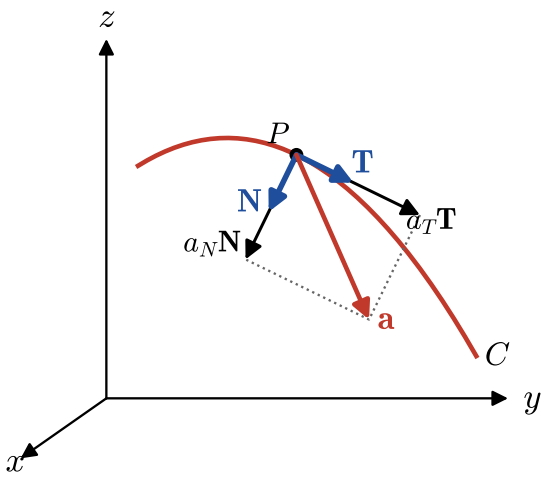
  – It follows from the differentiation of $\mathbf{T} \cdot \mathbf{T} = 1$ that $\mathbf{T} \cdot \dfrac{d\mathbf{T}}{dt} = 0$. If $\left\|\dfrac{d\mathbf{T}}{dt}\right\| \neq 0$, the vector

$$\mathbf{N}(t) = \frac{\dfrac{d\mathbf{T}}{dt}}{\left\|\dfrac{d\mathbf{T}}{dt}\right\|}$$

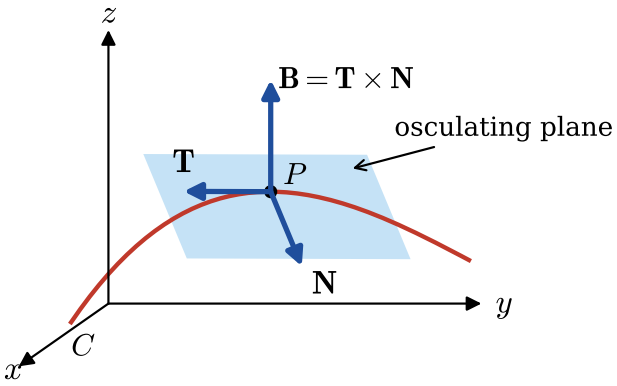
  is a **unit normal** to the curve C. The vector $\mathbf{N}$ is also called the **principal normal**

  – Since curvature is $\kappa = \dfrac{\left\|\frac{d\mathbf{T}}{dt}\right\|}{v}$, it follows that $\dfrac{d\mathbf{T}}{dt} = \kappa v\mathbf{N}$. Thus

$$\mathbf{a}(t) = \kappa v^2\mathbf{N} + \frac{dv}{dt}\mathbf{T} = a_N\mathbf{N} + a_T\mathbf{T}$$



- Binormal: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$



- Formulas for a_T, a_N, and Curvature

$$\mathbf{v} \cdot \mathbf{a} = a_N(v\mathbf{T} \cdot \mathbf{N}) + a_T(v\mathbf{T} \cdot \mathbf{T}) = a_T v$$

$$\Rightarrow \; a_T = \frac{dv}{dt} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}$$

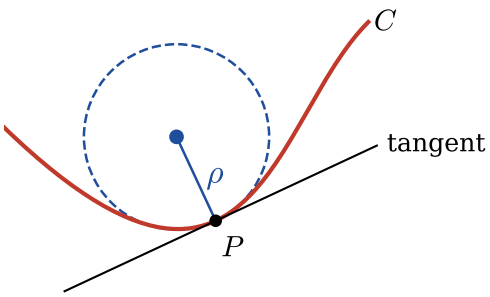
$$\mathbf{v} \times \mathbf{a} = a_N(v\mathbf{T} \times \mathbf{N}) + a_T(v\mathbf{T} \times \mathbf{T}) = a_N v\mathbf{B}$$

$$\Rightarrow \; a_N = \kappa v^2 = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}$$

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

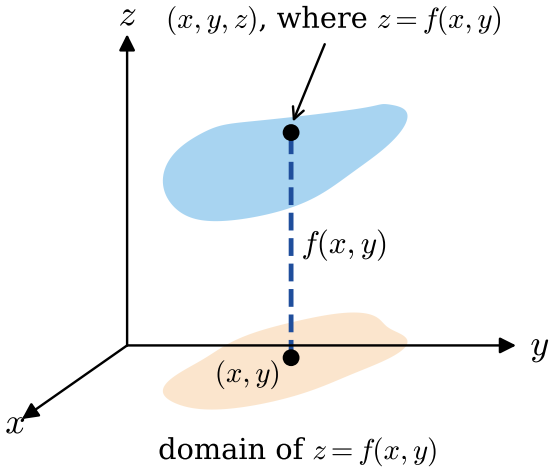
- Radius of Curvature

$$\rho = \frac{1}{\kappa}$$

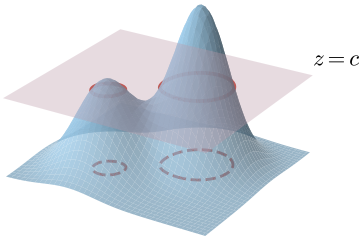


10.4 Partial Derivatives

- Functions of Two Variables

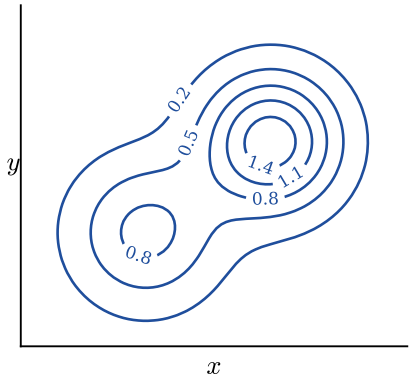


- Level Curves



(a) surface $z = f(x, y)$ and the plane $z = c$

(b) level curves $f(x, y) = c$

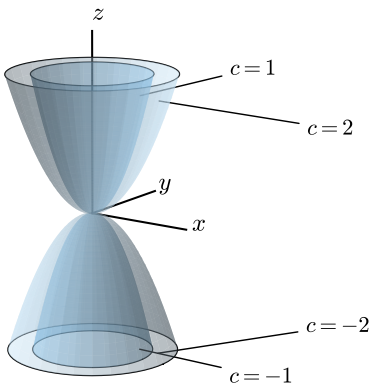


- Functions of Three Variables, Level Surfaces

$$w = F(x, y, z) \;\Rightarrow\; c = F(x, y, z)$$

ex) $w = \dfrac{x^2 + y^2}{z}$

$$w = \frac{x^2 + y^2}{z} = c$$



- Partial Derivatives

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = f_x$$
$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = f_y$$
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx}$$
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{yx}$$

- Chain Rule

  If $z = f(u, v)$ is differentiable and $u = g(x, y)$ and $v = h(x, y)$ have continuous first partial derivatives, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

  If $z = f(u, v)$ is differentiable and $u = g(t)$ and $v = h(t)$ are differentiable functions of a single variable t, the ordinary derivative $\dfrac{dz}{dt}$ is

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}$$

Try it in Python: Partial and higher derivatives with `sympy`: Derivatives

10.5 Directional Derivatives

- **Vector Differential Operator**

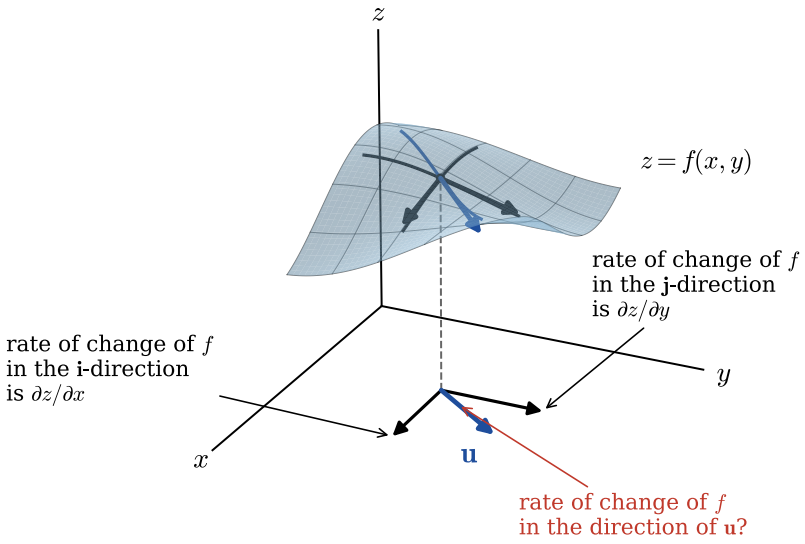
$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

- **Gradient of a Function**

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y}$$

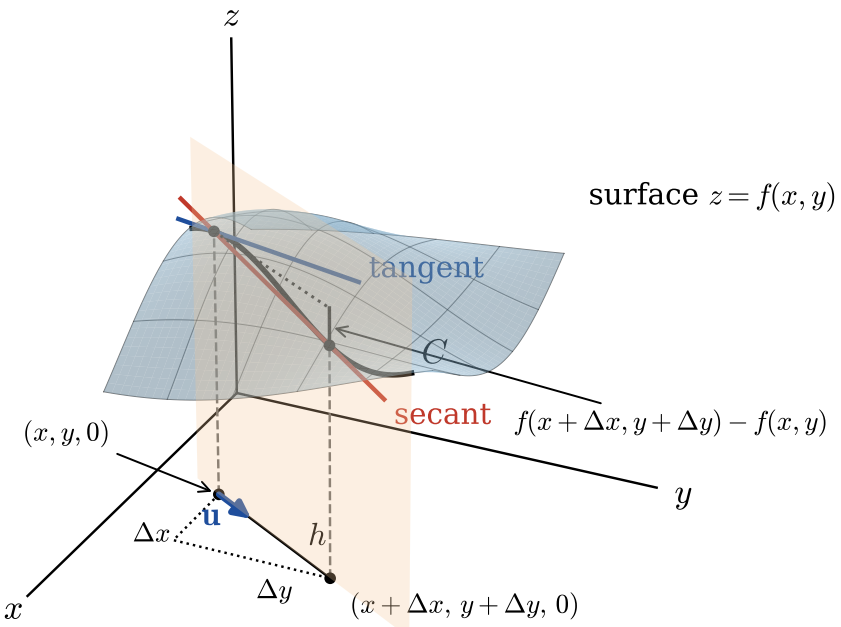
$$\nabla F = \mathbf{i}\frac{\partial F}{\partial x} + \mathbf{j}\frac{\partial F}{\partial y} + \mathbf{k}\frac{\partial F}{\partial z}$$



- Generalization of Partial Differentiation

  The **directional derivative** of $z = f(x, y)$ in the direction of a unit vector $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= \lim_{h \to 0} \frac{f(x + \Delta x,\, y + \Delta y) - f(x, y)}{h} \\ &= \lim_{h \to 0} \frac{f(x + h\cos\theta,\, y + h\sin\theta) - f(x, y)}{h} = \nabla f(x, y) \cdot \mathbf{u} \end{aligned}$$



- **Partial Proof** Let x, y, and θ be fixed so that

$$g(t) = f(x + t\cos\theta,\ y + t\sin\theta)$$

is a function of one variable. We wish to compare the value of $g'(0)$, which is found by two different methods

  - First, by the definition of a derivative,

$$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x + h\cos\theta,\ y + h\sin\theta) - f(x, y)}{h}$$

  - Second, by the chain rule,

$$\begin{aligned} g'(t) &= f_1(x + t\cos\theta,\ y + t\sin\theta)\frac{d}{dt}(x + t\cos\theta) \\ &\quad + f_2(x + t\cos\theta,\ y + t\sin\theta)\frac{d}{dt}(y + t\sin\theta) \\ &= f_1(x + t\cos\theta,\ y + t\sin\theta)\cos\theta + f_2(x + t\cos\theta,\ y + t\sin\theta)\sin\theta \\ &\Downarrow\ t \to 0 \\ g'(0) &= f_x(x, y)\cos\theta + f_y(x, y)\sin\theta \\ &= \left[\, f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}\,\right] \cdot (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \\ &= \nabla f(x, y) \cdot \mathbf{u} \end{aligned}$$

- For a function $w = F(x, y, z)$, the directional derivative is defined by

$$\begin{aligned} D_{\mathbf{u}}F(x, y, z) &= \lim_{h\to 0} \frac{F(x + h\cos\alpha,\ y + h\cos\beta,\ z + h\cos\gamma) - F(x, y, z)}{h} \\ &= \nabla F(x, y, z) \cdot \mathbf{u} \end{aligned}$$

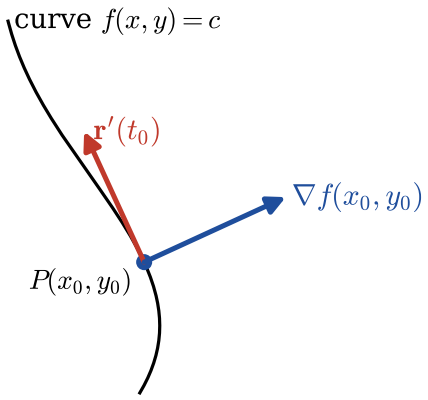
- Maximum Value of the Directional Derivative

$$D_{\mathbf{u}} f = \|\nabla f\|\|\mathbf{u}\| \cos\phi = \|\nabla f\| \cos\phi$$
$$\Rightarrow \; -\|\nabla f\| \le D_{\mathbf{u}} f \le \|\nabla f\|$$

- The gradient vector ∇f points in the direction in which f increases most rapidly, whereas $-\nabla f$ points in the direction of the most decrease of f

10.6 Tangent Planes and Normal Lines

- ∇f is orthogonal to the level curve at P



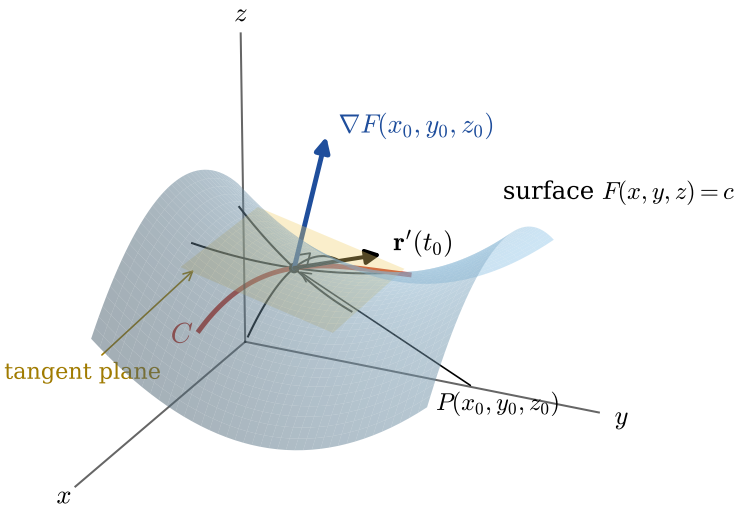
- The derivative of $f\,(x(t),\, y(t)) = c$ with respect to t is

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0 = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \cdot \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right) = \nabla f \cdot \mathbf{r}'$$

- The derivative of $F\,(x(t), y(t), z(t)) = c$ implies that

$$\begin{aligned}\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} &= 0 \\ = \left(\frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}\right) &\cdot \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right) = \nabla F \cdot \mathbf{r}'\end{aligned}$$

- ∇F is normal to the level surface at P



- Let $P(x_0, y_0, z_0)$ be a point on the graph of $F(x, y, z) = c$. The **tangent plane** at P is

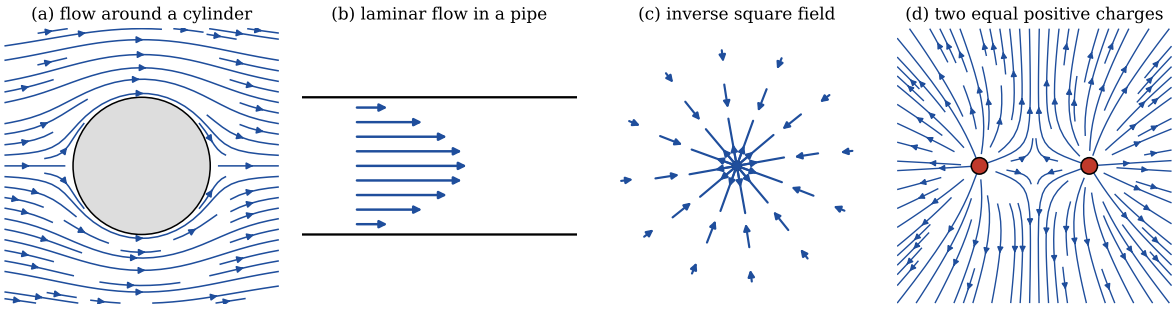
$$\begin{aligned} &F_x(x_0, y_0, z_0)(x - x_0) \\ &\quad + F_y(x_0, y_0, z_0)(y - y_0) \\ &\qquad + F_z(x_0, y_0, z_0)(z - z_0) = 0 \end{aligned}$$

  The line containing $P(x_0, y_0, z_0)$ that is parallel to $\nabla F(x_0, y_0, z_0)$ is called the **normal line** to the surface at P

10.7 Gradient, Curl and Divergence

- Vector Fields - Vector functions of two or three variables

$$\begin{aligned} \mathbf{f}(x, y) &= P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} \\ \mathbf{f}(x, y, z) &= P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} \end{aligned}$$



- The del operator combined with a scalar function $\phi(x, y, z)$ produces a vector field

$$\mathbf{f}(x, y, z) = \nabla\phi = \operatorname{grad}\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \;\Rightarrow\; \delta_{ij}\mathbf{e}_i\frac{\partial\phi}{\partial x_j}$$

  called the **gradient of ϕ**, where δ_{ij} is *Kronecker Delta*

- The **curl** of a vector field $\mathbf{f} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is the vector field

$$\nabla \times \mathbf{f} = \operatorname{curl} \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \Rightarrow \varepsilon_{ijk} \mathbf{e}_i \frac{\partial}{\partial x_j} f_k$$

  where ε_{ijk} is *Levi-Civita Symbol*

- The **divergence** of a vector field $\mathbf{f} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is the scalar function

$$\nabla \cdot \mathbf{f} = \operatorname{div} \mathbf{f} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \Rightarrow \delta_{ij} \frac{\partial}{\partial x_i} f_j$$

- Two important properties and plus alpha

$$\nabla \times \nabla f = \operatorname{curl}(\operatorname{grad} f) = \mathbf{0}$$
$$\Rightarrow \varepsilon_{ijk} \mathbf{e}_i \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_k} = 0\,\mathbf{e}_i = \mathbf{0}$$

$$\nabla \cdot (\nabla \times \mathbf{f}) = \operatorname{div}(\operatorname{curl} \mathbf{f}) = 0$$
$$\Rightarrow \delta_{ij} \frac{\partial}{\partial x_i} \varepsilon_{jlm} \frac{\partial}{\partial x_l} f_m = \varepsilon_{ilm} \frac{\partial^2}{\partial x_i \partial x_l} f_m = 0$$

$$\nabla \cdot \nabla f = \nabla^2 f \Rightarrow \delta_{ij} \frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_j} = \frac{\partial^2 f}{\partial x_i^2}$$

$$\nabla \cdot \nabla \mathbf{f} = \nabla^2 \mathbf{f} \Rightarrow \delta_{ij} \frac{\partial}{\partial x_i} \mathbf{e}_m \frac{\partial f_m}{\partial x_j} = \mathbf{e}_m \frac{\partial^2 f_m}{\partial x_i^2}$$

$$\nabla \times (\nabla \times \mathbf{f}) = \nabla(\nabla \cdot \mathbf{f}) - \nabla^2 \mathbf{f}$$
$$\Rightarrow \varepsilon_{ijk} \mathbf{e}_i \frac{\partial}{\partial x_j} \varepsilon_{klm} \frac{\partial}{\partial x_l} f_m$$
$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \mathbf{e}_i \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} f_m$$
$$= \mathbf{e}_i \frac{\partial}{\partial x_i} \frac{\partial f_j}{\partial x_j} - \mathbf{e}_i \frac{\partial^2 f_i}{\partial x_j^2}$$

- Product Rules with ∇

$$\begin{aligned}
\nabla(fg) &= f\nabla g + g\nabla f \\
\nabla(\mathbf{u} \cdot \mathbf{v}) &= \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}) \\
&\quad + (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} = (\nabla \mathbf{u}) \cdot \mathbf{v} + (\nabla \mathbf{v}) \cdot \mathbf{u} \\
\nabla \cdot (f\mathbf{u}) &= f(\nabla \cdot \mathbf{u}) + \mathbf{u} \cdot \nabla f \\
\nabla \cdot (\mathbf{u} \times \mathbf{v}) &= \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v}) \\
\nabla \times (f\mathbf{u}) &= \nabla f \times \mathbf{u} + f(\nabla \times \mathbf{u}) \\
\nabla \times (\mathbf{u} \times \mathbf{v}) &= \mathbf{u}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{u}) + (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v}
\end{aligned}$$

- Generalized Curvilinear Coordinates: $\mathbf{q}_i,\ q_i,\ h_i$

$$\nabla f = \mathbf{q}_1 \frac{1}{h_1}\frac{\partial f}{\partial q_1} + \mathbf{q}_2 \frac{1}{h_2}\frac{\partial f}{\partial q_2} + \mathbf{q}_3 \frac{1}{h_3}\frac{\partial f}{\partial q_3}$$

$$\nabla \cdot \mathbf{u} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(u_1 h_2 h_3) + \frac{\partial}{\partial q_2}(u_2 h_1 h_3) + \frac{\partial}{\partial q_3}(u_3 h_1 h_2)\right]$$

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$$

$$\nabla \times \mathbf{u} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\mathbf{q}_1 & h_2\mathbf{q}_2 & h_3\mathbf{q}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1 u_1 & h_2 u_2 & h_3 u_3 \end{vmatrix}$$

- Cylindrical Coordinates

  $q_1 = r,\ q_2 = \theta,\ q_3 = z$

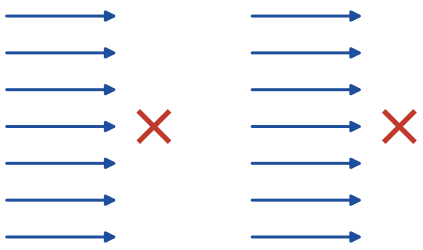
  $h_1 = h_r = 1,\ \ h_2 = h_\theta = r,\ \ h_3 = h_z = 1$

- Spherical Coordinates

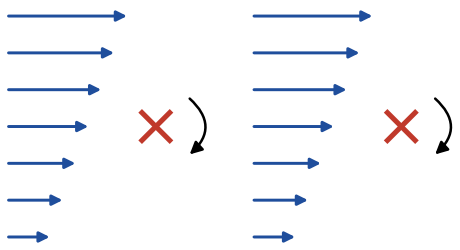
  $q_1 = \rho,\ q_2 = \phi,\ q_3 = \theta$

  $h_1 = h_\rho = 1,\ \ h_2 = h_\phi = \rho,\ \ h_3 = h_\theta = \rho \sin\phi$

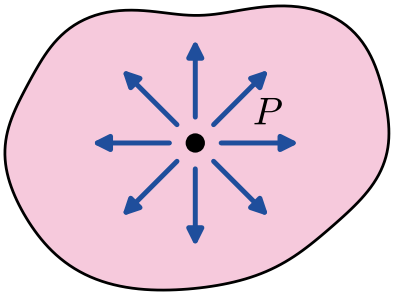
- Physical Interpretations



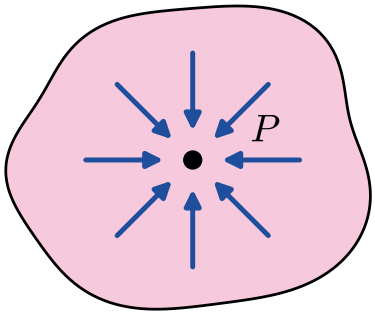
(a) irrotational flow: wheel does not rotate



(b) rotational flow: wheel rotates



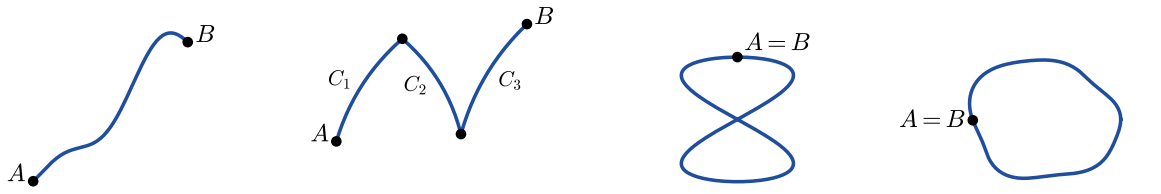
(a) $\operatorname{div}\mathbf{F}(P) > 0$: P a source



(b) $\operatorname{div}\mathbf{F}(P) < 0$: P a sink

10.8 Line Integrals

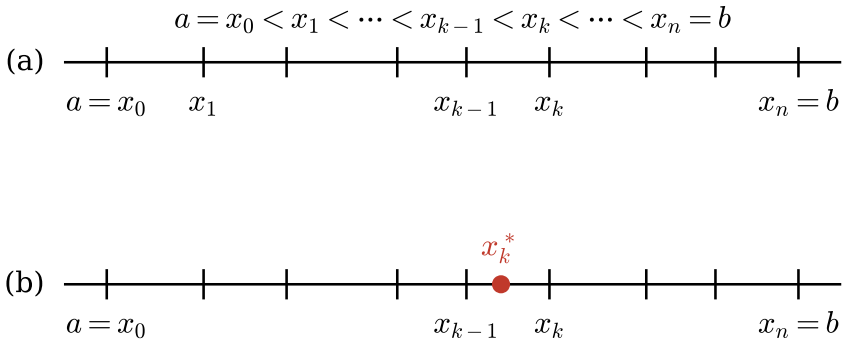
- Integration of a function defined along a curve



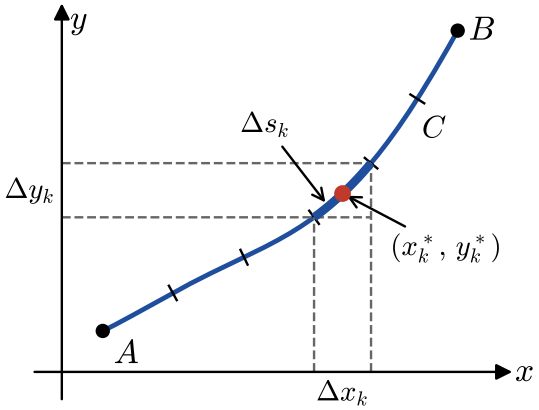
(a) smooth curve (b) piecewise-smooth curve (c) closed but not simple (d) simple closed curve

- Definite Integral

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx$$



- Line Integrals in the Plane



$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta x_k = \int_C G(x,y)\,dx$$
$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta y_k = \int_C G(x,y)\,dy$$
$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta s_k = \int_C G(x,y)\,ds$$

- Method of Evaluation
  - Curve Defined Parametrically, $x = f(t),\ y = g(t),\ a \le t \le b$

$$\int_C G(x,y)\,dx = \int_a^b G(\,f(t), g(t))\,f'(t)\,dt$$
$$\int_C G(x,y)\,dy = \int_a^b G(\,f(t), g(t))\,g'(t)\,dt$$
$$\int_C G(x,y)\,ds = \int_a^b G(\,f(t), g(t))\,\sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

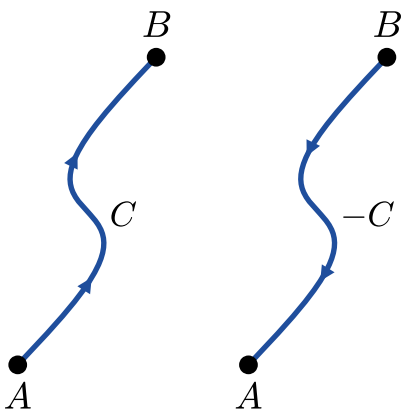
  - Curve Defined by an Explicit Function, $y = f(x),\ a \le x \le b$

$$\int_C G(x,y)\,dx = \int_a^b G(x, f(x))\,dx$$
$$\int_C G(x,y)\,dy = \int_a^b G(x, f(x))\,f'(x)\,dx$$
$$\int_C G(x,y)\,ds = \int_a^b G(x, f(x))\,\sqrt{1 + [f'(x)]^2}\,dx$$

- Notation

$$\int_C P\,dx + \int_C Q\,dy \;\Rightarrow\; \int_C P\,dx + Q\,dy$$

$$\int_{-C} P\,dx + Q\,dy = -\int_C P\,dx + Q\,dy$$



- A line integral along a closed curve C

$$\oint_C P\,dx + Q\,dy$$

- Line Integrals in Space

$$\int_C G(x,y,z)\,ds = \int_a^b G(\,f(t),g(t),h(t))\,\sqrt{[f'(t)]^2+[g'(t)]^2+[h'(t)]^2}\,dt$$

- Suppose the vector-valued function

$$\mathbf{f}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

is defined along a curve C and

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

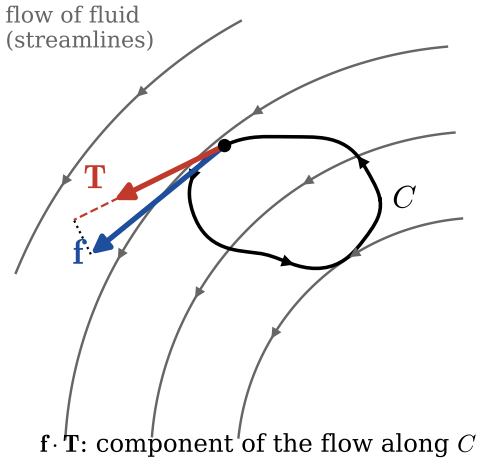
is the displacement vector of points on C, then

$$\int_C \mathbf{f}\cdot\,d\mathbf{r} = \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

- Work and Circulation

$$W = \int_C \mathbf{f}\cdot d\mathbf{r} = \int_C \mathbf{f}\cdot\mathbf{T}\,ds$$

$$\text{Circulation} = \oint_C \mathbf{f}\cdot d\mathbf{r} = \oint_C \mathbf{f}\cdot\mathbf{T}\,ds$$



$\mathbf{f}\cdot\mathbf{T}$: component of the flow along C

10.9 Independence of the Path

- The value of a line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

  depends on the path of integration

- Stated another way, if C_1 and C_2 are two different paths between the same points A and B, then we expect that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

- **Path Independence**

  A line integral $\int_C \mathbf{f} \cdot d\mathbf{r}$ is independent of the path if

$$\int_{C_1} \mathbf{f} \cdot d\mathbf{r} = \int_{C_2} \mathbf{f} \cdot d\mathbf{r}$$

  for any two paths C_1 and C_2 between A and B

- A vector function $\mathbf{f}$ in 2- or 3-space is said to be **conservative** if $\mathbf{f}$ can be written as the gradient of a scalar function ϕ. The function ϕ is called a **potential function** for $\mathbf{f}$

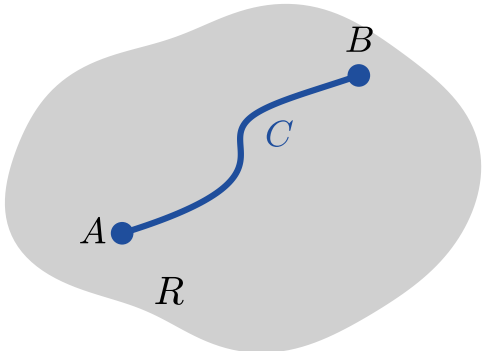
- In other words, $\mathbf{f}$ is conservative if there exists a function ϕ such that $\mathbf{f} = \nabla\phi$. A conservative vector field is also called a **gradient vector field**

- **Fundamental Theorem**

  If $\mathbf{f}(x, y, z)$ is a conservative vector field in R and ϕ is a potential function for $\mathbf{f}$, then

$$\int_C \mathbf{f} \cdot d\mathbf{r} = \int_C \nabla\phi \cdot d\mathbf{r} = \phi(B) - \phi(A)$$

  where $A = (x(a), y(a), z(a))$ and $B = (x(b), y(b), z(b))$



connected region R

- **Equivalent Concepts:** In an open connected region R, $\int_C \mathbf{f} \cdot d\mathbf{r}$ is independent of the path C

- if and only if the vector field $\mathbf{f}$ is conservative in R
- if and only if $\oint_{C'} \mathbf{f} \cdot d\mathbf{r} = 0$ for every closed path C' in R

- **Test for a Conservative Field:**
  - Suppose

$$\mathbf{f}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

    is a conservative vector field in an open region of 3-space, and that P, Q, and R are continuous and have continuous first partial derivatives in that region
  - Then $\mathbf{f} = \nabla\phi$ and $\nabla \times \mathbf{f} = \nabla \times \nabla\phi = \mathbf{0}$, that is

$$\nabla \times \mathbf{f} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} = \mathbf{0}$$

$$\Downarrow$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \ \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

    for all (x, y, z) in that region

10.10 Double Integrals

- The Double Integral

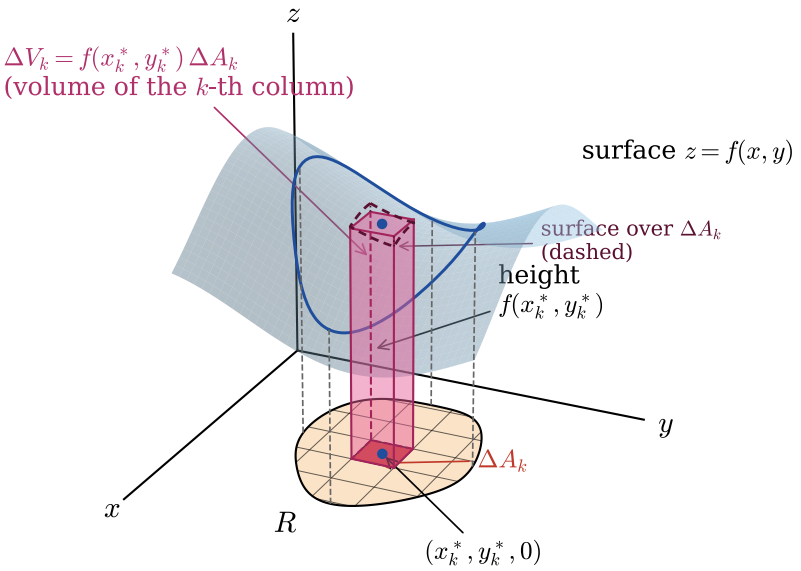
  Let f be a function of two variables defined on a closed region R. Then the **double integral of f over R** is given by

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f\left(x_k^*, y_k^*\right) \Delta A_k = \iint_R f(x, y)\, dA$$

- Area and Volume

$$A = \iint_R dA$$

$$V = \iint_R f(x, y)\, dA$$

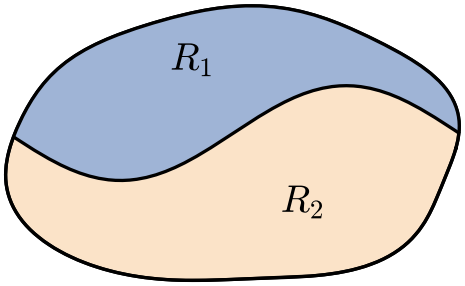


- Properties of Double Integrals

$$\iint_R kf(x,y)\,dA = k\iint_R f(x,y)\,dA$$

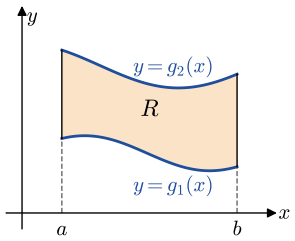
$$\iint_R f(x,y) \pm g(x,y)\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

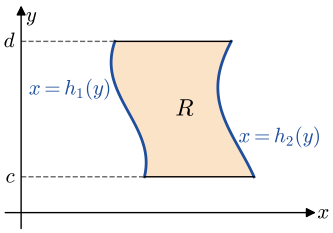


$$R = R_1 \cup R_2$$

- **Evaluation of Double Integrals**



(a) Type I region



(b) Type II region

  - If R is of Type I, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx$$

  - If R is of Type II, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$$

- Center of Mass and Moments of Inertia

$$\bar{x} = \frac{\iint_R x\rho(x,y)\,dA}{M} \quad \text{and} \quad \bar{y} = \frac{\iint_R y\rho(x,y)\,dA}{M}$$

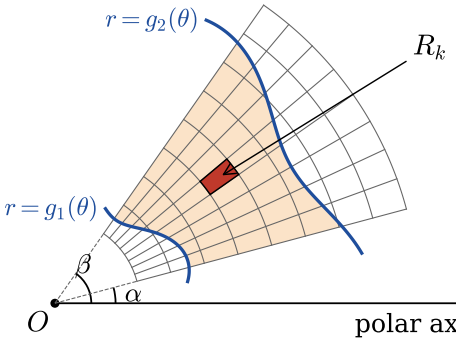
$$I_y = \iint_R x^2\rho(x,y)\,dA \quad \text{and} \quad I_x = \iint_R y^2\rho(x,y)\,dA$$

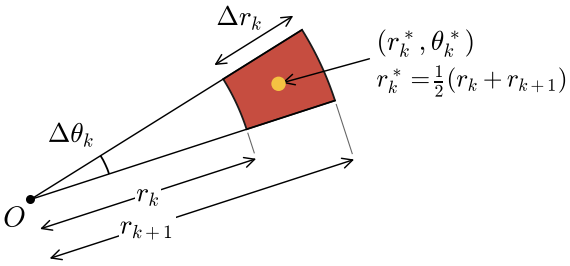
Try it in Python: Double and triple integrals numerically with `scipy` (`dblquad`, `tplquad`, `nquad`): Multiple integration; symbolically with `sympy`: Integrals

10.11 Double Integrals in Polar Coordinates

- Polar Rectangles

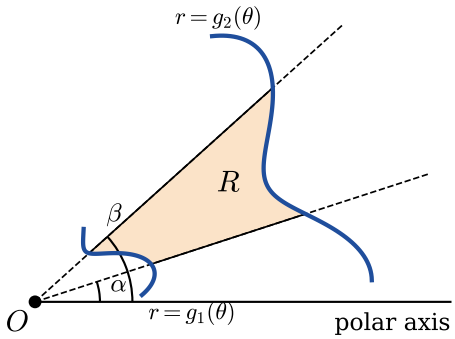
$$\begin{aligned}\Delta A_k &= \frac{1}{2}(r_{k+1}^2 - r_k^2)\Delta\theta_k \\ &= \frac{1}{2}(r_{k+1} + r_k)(r_{k+1} - r_k)\Delta\theta_k \\ &= r_k^* \Delta r_k \Delta\theta_k\end{aligned}$$

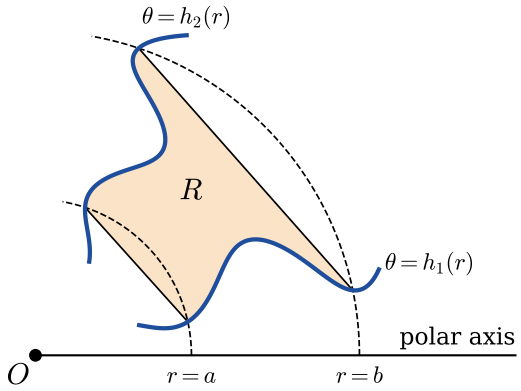




- Double Integrals in Polar Coordinates

$$\begin{aligned}\lim_{\|P\|\to 0} \sum_{k=1}^{n} f\,(r_k^*, \theta_k^*)\, r_k^* \Delta r_k \Delta\theta_k \\ &= \iint_R f(r,\theta)\, dA \\ &= \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta \\ &= \int_a^b \int_{h_1(r)}^{h_2(r)} f(r,\theta)\, r\, d\theta\, dr\end{aligned}$$



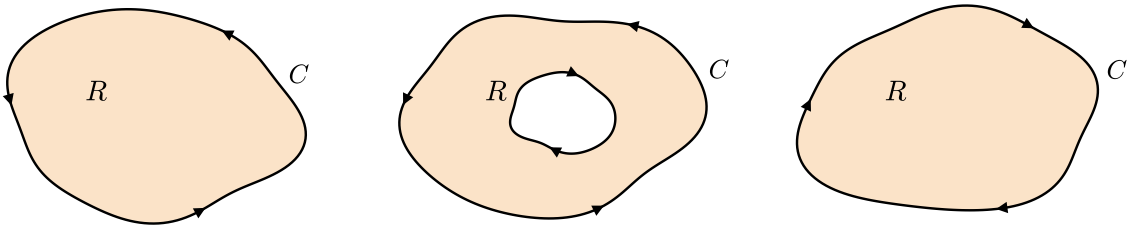


10.12 Green's Theorem

- Suppose that C is a piecewise-smooth simple closed curve bounding a simply connected region R

- If P, Q, $\dfrac{\partial P}{\partial y}$, and $\dfrac{\partial Q}{\partial x}$ are continuous on R, then

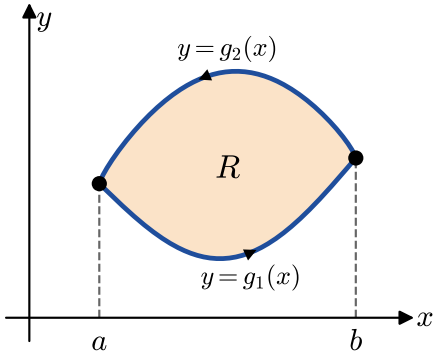
$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$



(a) positive direction (b) positive direction (c) negative direction

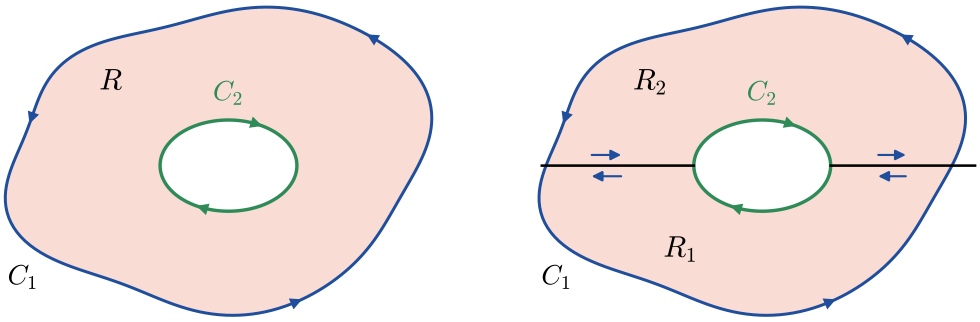
- **Partial Proof**

$$\begin{aligned}
-\iint_R \frac{\partial P}{\partial y}\,dA &= -\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy dx \\
&= -\int_a^b [P(x, g_2(x)) - P(x, g_1(x))]\,dx \\
&= \int_a^b P(x, g_1(x))\,dx + \int_b^a P(x, g_2(x))\,dx \\
&= \oint_C P(x, y)\,dx
\end{aligned}$$



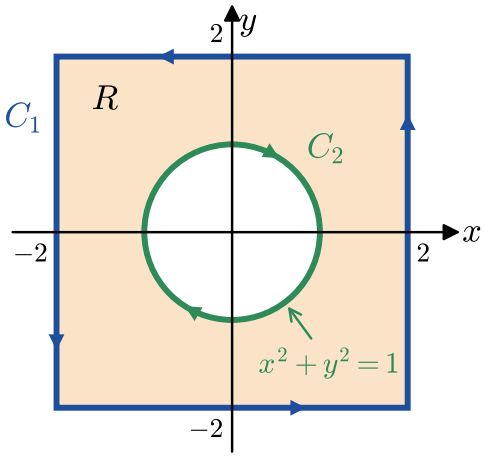
- **Region with Holes**

$$\begin{aligned}
&\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\
&\quad = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\
&\quad = \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy = \oint_C P\,dx + Q\,dy
\end{aligned}$$



(a) (b)

Example Evaluate $\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$, where $C = C_1 \cup C_2$ is the boundary of the shaded region R



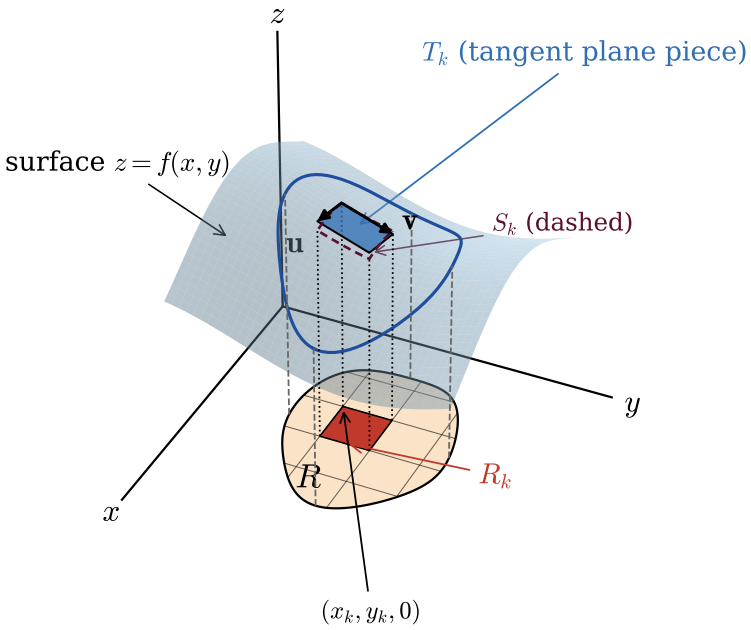
Answer 1 (end of chapter) · Solution

Example Evaluate $\oint_{C_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$

Answer 2 (end of chapter) · Solution

10.13 Surface Integrals

- Surface Area



$$\begin{aligned}
\mathbf{u} &= \Delta x_k \mathbf{i} + f_x(x_k, y_k)\Delta x_k \mathbf{k} \\
\mathbf{v} &= \Delta y_k \mathbf{j} + f_y(x_k, y_k)\Delta y_k \mathbf{k} \\
&\Downarrow \\
\mathbf{u} \times \mathbf{v} &= \left[-f_x(x_k, y_k)\mathbf{i} - f_y(x_k, y_k)\mathbf{j} + \mathbf{k}\right] \Delta x_k \Delta y_k \\
&\Downarrow \\
\Delta S_k &= \sqrt{1 + \left[f_x(x_k, y_k)\right]^2 + \left[f_y(x_k, y_k)\right]^2}\,\Delta A_k
\end{aligned}$$

Then the area of the surface over R is given by

$$S = \iint_R \sqrt{1 + \left[f_x(x, y)\right]^2 + \left[f_y(x, y)\right]^2}\, dA$$

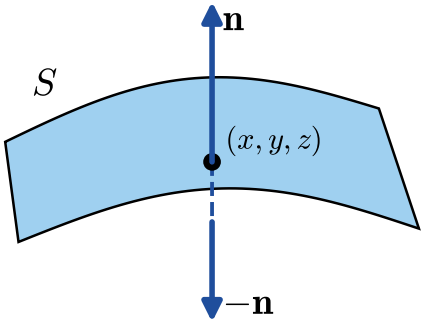
- Surface Integral

  Let G be a function of three variables defined over a region of space containing the surface S. Then the surface integral of G over S is given by

$$\begin{aligned}
\lim_{\|P\| \to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*, z_k^*)\, \Delta S_k &= \iint_S G(x, y, z)\, dS \\
&= \iint_R G(x, y, f(x, y))\, \sqrt{1 + \left[f_x(x, y)\right]^2 + \left[f_y(x, y)\right]^2}\, dA
\end{aligned}$$

- If a smooth surface S is $g(x, y, z) = 0$, a unit normal is

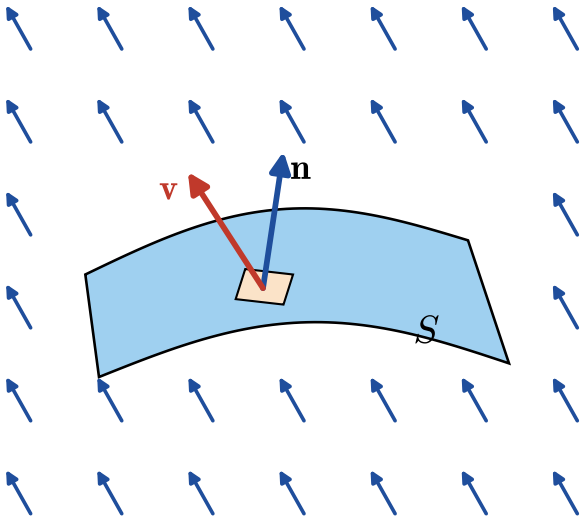
$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|}$$



- Integrals of Vector Fields

  The total volume of a fluid passing through S is called the **flux of v through** S and is given by

$$\text{flux} = \iint_S (\mathbf{v} \cdot \mathbf{n})\, dS$$



10.14 Stokes' Theorem

- Vector Form of Green's Theorem

  If $\mathbf{f}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a two-dimensional vector field, then

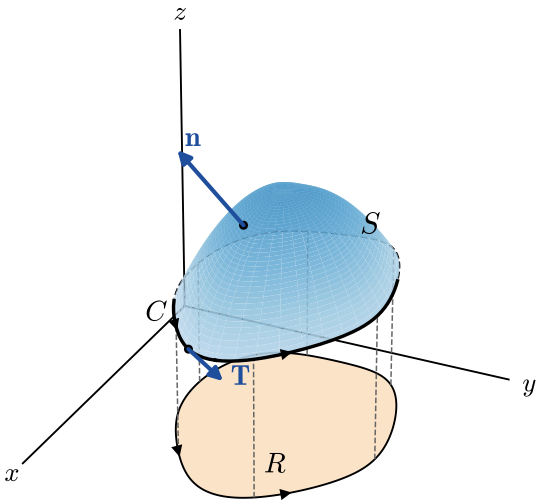
$$\operatorname{curl} \mathbf{f} = \nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

  Green's theorem can be written in vector notation as

$$\oint_C \mathbf{f} \cdot\, d\mathbf{r} = \oint_C \mathbf{f} \cdot \mathbf{T}\, ds = \iint_R (\nabla \times \mathbf{f}) \cdot \mathbf{k}\, dA$$

- **Stokes' Theorem - Green's Theorem in 3-Space**
  - Let S be a piecewise-smooth orientable surface bounded by a piecewise-smooth simple closed curve C
  - Let $\mathbf{f}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a vector field for which P, Q, and R are continuous and have continuous first partial derivatives in a region of 3-space containing S. If C is traversed in the positive direction, then

$$\oint_C \mathbf{f} \cdot\, d\mathbf{r} = \oint_C \mathbf{f} \cdot \mathbf{T}\, ds = \iint_S (\nabla \times \mathbf{f}) \cdot \mathbf{n}\, dS$$



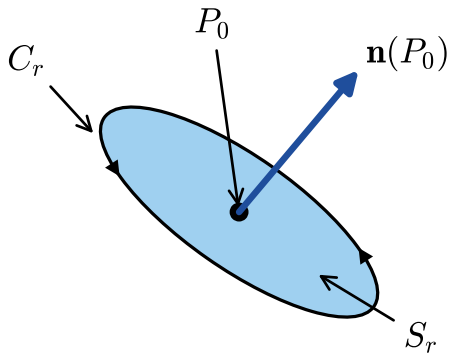
- Physical Interpretation of Curl

$$\oint_{C_r} \mathbf{f} \cdot d\mathbf{r} = \iint_{S_r} [\nabla \times \mathbf{f}(P_0)] \cdot \mathbf{n}(P_0)\, dS$$

$$= [\nabla \times \mathbf{f}(P_0)] \cdot \mathbf{n}(P_0) \iint_{S_r} dS = [\nabla \times \mathbf{f}(P_0)] \cdot \mathbf{n}(P_0)\, S_r$$

$$\Downarrow$$

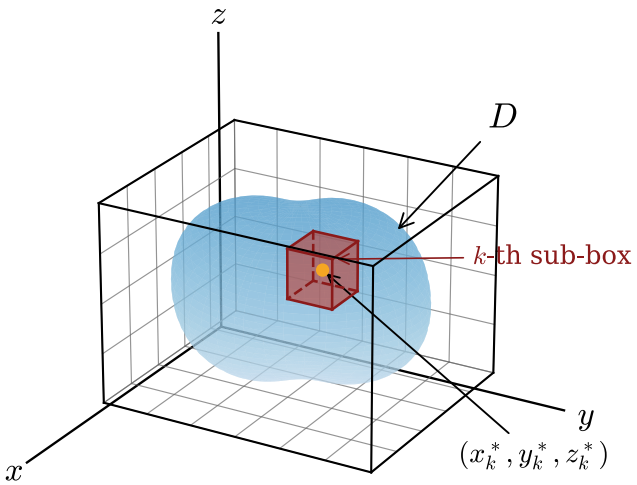
$$[\nabla \times \mathbf{f}(P_0)] \cdot \mathbf{n}(P_0) = \lim_{r \to 0} \frac{1}{S_r} \oint_{C_r} \mathbf{f} \cdot d\mathbf{r} \approx \frac{1}{S_r} \oint_{C_r} \mathbf{f} \cdot d\mathbf{r}$$



10.15 Triple Integrals

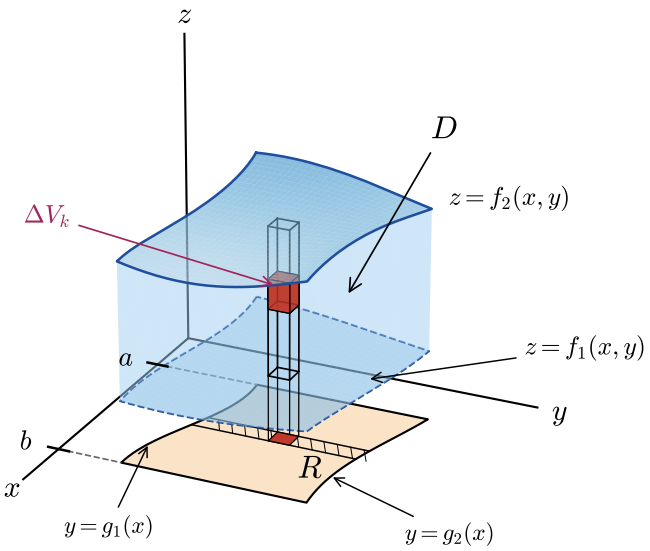
- Let F be a function of three variables defined over a closed region D of space. Then the triple integral of F over D is given by

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} F(x_k^*, y_k^*, z_k^*)\, \Delta V_k = \iiint_D F(x, y, z)\, dV$$

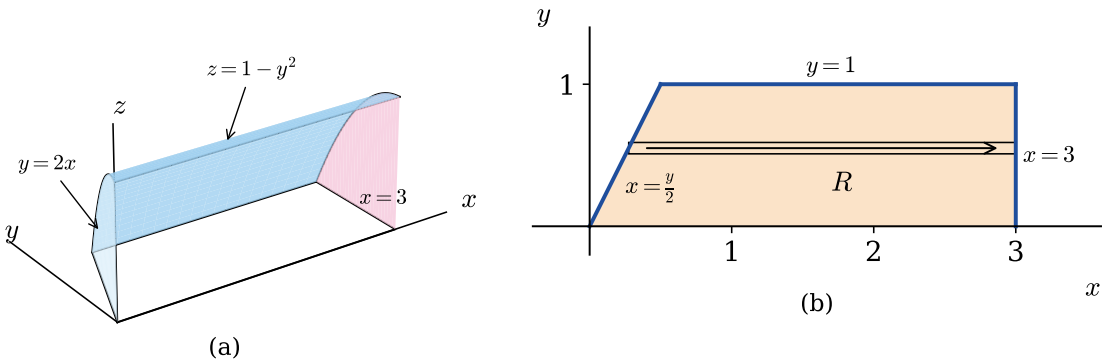


- Evaluation by Iterated Integrals

$$\begin{aligned}\iiint_D F(x, y, z)\, dV &= \iint_R \left[\int_{f_1(x,y)}^{f_2(x,y)} F(x, y, z)\, dz \right] dA \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} F(x, y, z)\, dz\, dy\, dx\end{aligned}$$



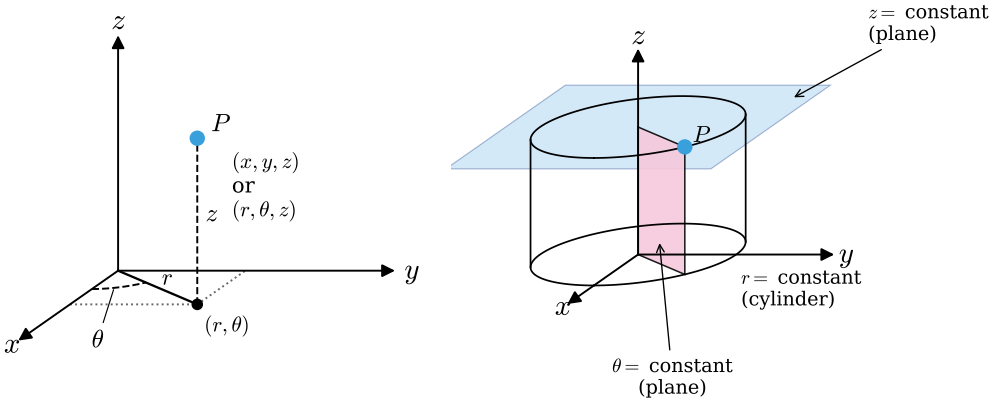
Example Find the volume of the solid in the first octant bounded by the graphs of $z = 1 - y^2$, $y = 2x$, and $x = 3$



Answer 3 (end of chapter) · Solution

- Cylindrical Coordinates

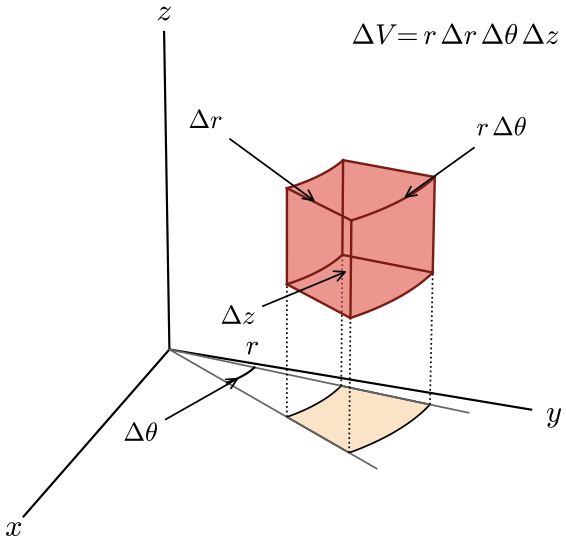
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$
$$\Downarrow$$
$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}, \quad z = z$$



(a) (b)

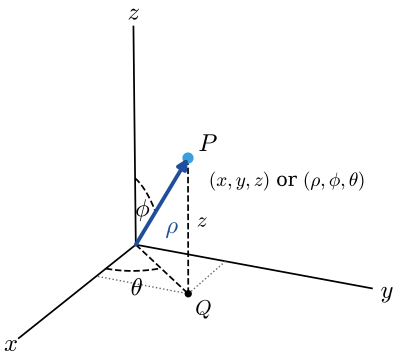
- Triple Integrals in Cylindrical Coordinates

$$\iiint_D F(r,\theta,z)\,dV = \iint_R \left[\int_{f_1(r,\theta)}^{f_2(r,\theta)} F(r,\theta,z)\,dz\right] dA$$
$$= \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(r,\theta)}^{f_2(r,\theta)} F(r,\theta,z)\,r\,dz\,dr\,d\theta$$

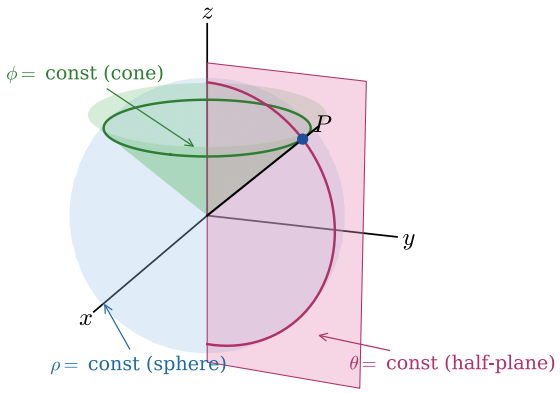


- Spherical Coordinates

$$x = \rho\sin\phi\cos\theta,\ \ y = \rho\sin\phi\sin\theta,\ \ z = \rho\cos\phi$$
$$\Downarrow$$
$$\rho^2 = x^2+y^2+z^2,\ \ \tan\theta = \frac{y}{x},\ \ \cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}}$$



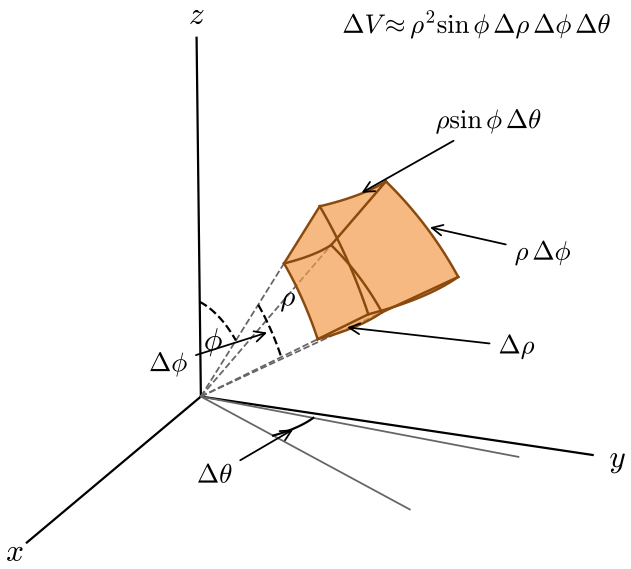
(a)



(b)

- Triple Integrals in Spherical Coordinates

$$\iiint_D F(\rho,\theta,\phi)\,dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(\phi,\theta)}^{f_2(\phi,\theta)} F(\rho,\theta,\phi)\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$



10.16 Divergence Theorem

- Another Vector Form of Green's Theorem

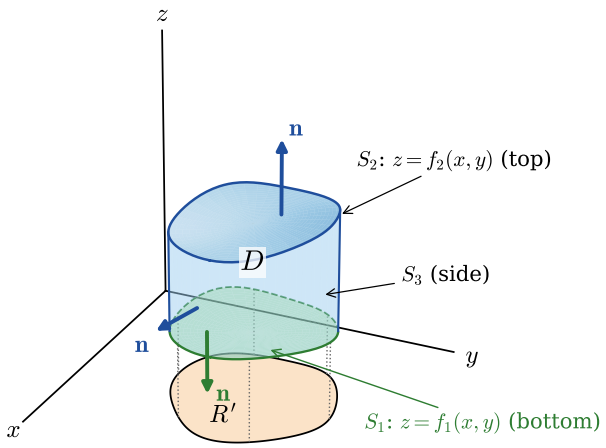
$$\oint_C (\mathbf{f}\cdot\mathbf{n})\,ds = \oint_C -Q\,dx + P\,dy$$
$$= \iint_R \left[\frac{\partial P}{\partial x} - \left(-\frac{\partial Q}{\partial y}\right)\right] dA = \iint_R \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right] dA$$
$$= \iint_R \nabla\cdot\mathbf{f}\,dA$$

where $\mathbf{n} = \dfrac{dy}{ds}\mathbf{i} - \dfrac{dx}{ds}\mathbf{j}$

- **Divergence Theorem**
  - Let D be a closed and bounded region in 3-space with a piecewise-smooth boundary S that is oriented outward
  - Let $\mathbf{f}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$ be a vector field for which P, Q, and R are continuous and have continuous first partial derivatives in a region of 3-space containing D. Then

$$\iint_S (\mathbf{f}\cdot\mathbf{n})\,dS = \iiint_D \nabla\cdot\mathbf{f}\,dV$$

- **Partial Proof**



$$\iiint_D \frac{\partial R}{\partial z}\,dV = \iint_{R'} \left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial R}{\partial z}\,dz\right] dA = \iint_{R'} \left[R(x,y,f_2(x,y)) - R(x,y,f_1(x,y))\right] dA$$

$$\iint_S R(\mathbf{k}\cdot\mathbf{n})\,dS = \iint_{S_1} R(\mathbf{k}\cdot\mathbf{n})\,dS + \iint_{S_2} R(\mathbf{k}\cdot\mathbf{n})\,dS + \iint_{S_3} R(\mathbf{k}\cdot\mathbf{n})\,dS$$

– On S_1: Since the outward normal points downward, $g(x,y,z) = f_1(x,y) - z = 0$. Thus

$$\mathbf{n} = \frac{\frac{\partial f_1}{\partial x}\mathbf{i} + \frac{\partial f_1}{\partial y}\mathbf{j} - \mathbf{k}}{\sqrt{1 + \left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2}} \quad \Rightarrow \quad \mathbf{k}\cdot\mathbf{n} = \frac{-1}{\sqrt{1 + \left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2}}$$

$$\Downarrow$$

$$\iint_{S_1} R(\mathbf{k}\cdot\mathbf{n})\, dS = -\iint_{R'} R(x,y,f_1(x,y))\, dA$$

– On S_2: Since the outward normal points upward, $g(x,y,z) = z - f_2(x,y) = 0$. Thus

$$\mathbf{n} = \frac{-\frac{\partial f_2}{\partial x}\mathbf{i} - \frac{\partial f_2}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial f_2}{\partial x}\right)^2 + \left(\frac{\partial f_2}{\partial y}\right)^2}} \quad \Rightarrow \quad \mathbf{k}\cdot\mathbf{n} = \frac{1}{\sqrt{1 + \left(\frac{\partial f_2}{\partial x}\right)^2 + \left(\frac{\partial f_2}{\partial y}\right)^2}}$$

$$\Downarrow$$

$$\iint_{S_2} R(\mathbf{k}\cdot\mathbf{n})\, dS = \iint_{R'} R(x,y,f_2(x,y))\, dA$$

– On S_3: $\mathbf{k}\cdot\mathbf{n} = 0$

$$\iint_{S_3} R(\mathbf{k}\cdot\mathbf{n})\, dS = 0$$

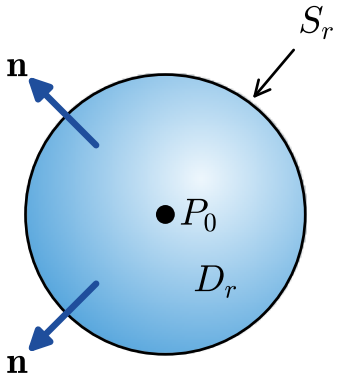
- Physical Interpretation of Divergence

$$\iint_{S_r} (\mathbf{f}\cdot\mathbf{n})\, dS = \iiint_{D_r} \nabla\cdot\mathbf{f}\, dV$$

$$\approx \iiint_{D_r} \nabla\cdot\mathbf{f}(P_0)\, dV = \nabla\cdot\mathbf{f}(P_0) \iiint_{D_r} dV = \nabla\cdot\mathbf{f}(P_0) D_r$$

$$\Downarrow$$

$$\nabla\cdot\mathbf{f}(P_0) = \lim_{r\to 0} \frac{1}{D_r} \iint_{S_r} (\mathbf{f}\cdot\mathbf{n})\, dS$$



- Continuity Equation

$$\begin{aligned}\frac{dm}{dt} &= \frac{d}{dt}\iiint_D \rho(x,y,z,t)\,dV = \iiint_D \frac{\partial \rho}{\partial t}\,dV \\ &= -\iint_S (\rho \mathbf{v}\cdot\mathbf{n})\,dS = -\iiint_D \nabla\cdot(\rho\mathbf{v})\,dV \\ &\Downarrow \\ \iiint_D \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right] dV &= 0 \\ &\Downarrow \\ \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) &= 0\end{aligned}$$

- Euler's Equation of Motion

  Allowing for the presence of a gravitational body force per unit mass $\mathbf{g}$, the basic equations of an ideal fluid are

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}$$

$$\nabla\cdot\mathbf{v} = 0$$

- Furthermore, it can be helpful to use $\mathbf{g} = -\nabla\chi$ and the identity

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} = (\nabla\times\mathbf{v})\times\mathbf{v} + \nabla\left(\frac{1}{2}\mathbf{v}^2\right)$$

  to cast the momentum equation into the form

$$\frac{\partial \mathbf{v}}{\partial t} + (\nabla\times\mathbf{v})\times\mathbf{v} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\mathbf{v}^2 + \chi\right) = -\nabla H$$

10.17 Change of Variables in Multiple Integrals

- To change the Variable in a Definite Integral

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

- Double Integrals

$$\begin{aligned}&\iint_R F(x,y)\,dA \\ &\quad = \iint_{R'} F(f(u,v),g(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA'\end{aligned}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \qquad dA = h_1 h_2\, dq_1 dq_2 \ \text{ (orthogonal curvilinear coordinates)}$$

- Triple Integrals

$$\iiint_D F(x,y,z)\,dV$$
$$= \iiint_{D'} F\left(f(u,v,w), g(u,v,w), h(u,v,w)\right) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV'$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}, \qquad dV = h_1 h_2 h_3\, dq_1 dq_2 dq_3$$

Summary

Topic	Formula
Motion	$\mathbf{v} = \mathbf{r}'$, $\mathbf{a} = \mathbf{r}''$; $\kappa = \dfrac{\|d\mathbf{T}/dt\|}{v}$, $\mathbf{a} = \dfrac{dv}{dt}\mathbf{T} + \kappa v^2 \mathbf{N}$
Directional derivative	$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$; largest, $\|\nabla f\|$, in the direction of ∇f
Tangent plane	$\nabla F(x_0, y_0, z_0)$ is normal to the level surface $F(x,y,z) = c$
Divergence and curl	$\nabla \cdot \mathbf{f}$, $\nabla \times \mathbf{f}$
Path independence	$\mathbf{f} = \nabla\phi$: $\displaystyle\int_C \mathbf{f} \cdot d\mathbf{r} = \phi(B) - \phi(A)$
Green's theorem	$\displaystyle\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$
Surface area	$\displaystyle\iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$, $\mathbf{n} = \dfrac{\nabla g}{\|\nabla g\|}$
Stokes' theorem	$\displaystyle\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{f}) \cdot \mathbf{n}\, dS$
Divergence theorem	$\displaystyle\iint_S (\mathbf{f} \cdot \mathbf{n})\, dS = \iiint_D \nabla \cdot \mathbf{f}\, dV$
Change of variables	$\displaystyle\iint_R f\, dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$

Worked Exercises

1. Find an equation of the plane containing the lines $x = t,\ \ y = 4t,\ \ z = -2t$, and $\ x = 1 + t,\ y = 1 + 4t,\ \ z = 3 - 2t$

Solution

Step 1: Write the parametric forms of both lines

First line:

$$L_1 : \begin{cases} x = t \\ y = 4t \\ z = -2t \end{cases} \quad \Rightarrow \begin{array}{l} \text{Point: } P_1 = (0, 0, 0) \\ \text{Direction vector: } \mathbf{d}_1 = \langle 1, 4, -2 \rangle \end{array}$$

Second line:

$$L_2 : \begin{cases} x = 1 + t \\ y = 1 + 4t \\ z = 3 - 2t \end{cases} \quad \Rightarrow \begin{array}{l} \text{Point: } P_2 = (1, 1, 3) \\ \text{Direction vector: } \mathbf{d}_2 = \langle 1, 4, -2 \rangle \end{array}$$

Step 2: Observe relationship between the lines

- Both lines share the same direction vector: $\mathbf{d}_1 = \mathbf{d}_2 = \langle 1, 4, -2 \rangle$
- But they pass through different points

So the lines are parallel and distinct, and therefore they lie in a common plane

Step 3: Find two vectors in the plane

The plane is determined by:

1. Direction vector of the lines:
$$\mathbf{v}_1 = \langle 1, 4, -2 \rangle$$
2. Vector connecting a point on one line to a point on the other line:

$$\mathbf{v}_2 = P_2 - P_1 = \langle 1, 1, 3 \rangle - \langle 0, 0, 0 \rangle = \langle 1, 1, 3 \rangle$$

Step 4: Find a normal vector to the plane

Take the cross product:

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -2 \\ 1 & 1 & 3 \end{vmatrix}$$

Compute:

$$\mathbf{n} = \mathbf{i}(4 \cdot 3 - (-2) \cdot 1) - \mathbf{j}(1 \cdot 3 - (-2) \cdot 1) + \mathbf{k}(1 \cdot 1 - 4 \cdot 1) \ \Rightarrow \ \mathbf{n} = \langle 14, -5, -3 \rangle$$

Step 5: Use the point-normal form of a plane

With point $P_1 = (0, 0, 0)$ and normal vector $\langle 14, -5, -3 \rangle$, the equation is:

$$14(x-0)-5(y-0)-3(z-0)=0 \Rightarrow 14x-5y-3z=0$$

2. Find an equation of the plane containing $(1,7,-1)$ that is perpendicular to the line of intersection of $-x+y-8z=4$ and $3x-y+2z=0$

Solution

Step 1: Find direction vector of the line of intersection

The line of intersection of two planes lies in both planes, and its direction vector is perpendicular to both normals of the planes

So, compute the cross product of the normal vectors of the two planes

- Normal to π_1: $\mathbf{n}_1 = \langle -1, 1, -8 \rangle$
- Normal to π_2: $\mathbf{n}_2 = \langle 3, -1, 2 \rangle$

Then:

$$\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -8 \\ 3 & -1 & 2 \end{vmatrix}$$

Compute the determinant:

$$\begin{aligned} \mathbf{d} &= \mathbf{i}(1 \cdot 2 - (-8) \cdot (-1)) - \mathbf{j}((-1) \cdot 2 - (-8) \cdot 3) + \mathbf{k}((-1) \cdot (-1) - 1 \cdot 3) \\ &= \langle -6, -22, -2 \rangle \end{aligned}$$

This is the direction vector of the line of intersection.

Step 2: Use the fact that the desired plane is perpendicular to this line

So the direction vector $\mathbf{d} = \langle -6, -22, -2 \rangle$ is the normal vector of the desired plane

We now have:

- Point on the plane: $(1, 7, -1)$
- Normal vector: $\mathbf{n} = \langle -6, -22, -2 \rangle$

Step 3: Use point-normal form of a plane

$$-6(x-1)-22(y-7)-2(z+1)=0$$

Expand:

$$-6x+6-22y+154-2z-2=0 \Rightarrow -6x-22y-2z+158=0$$

Multiply both sides by -1 to make the leading coefficient positive:

$$6x+22y+2z=158$$

3. Use Green's theorem to evaluate the given double integral by a line integral

$$\iint_R x^2\,dA$$

where R is the region bounded by the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$

Solution

Step 1: Green's Theorem

Green's Theorem relates a line integral over a positively oriented, simple closed curve C to a double integral over the region R it encloses:

$$\iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA=\oint_C P\,dx+Q\,dy$$

We need to rewrite $\iint_R x^2\,dA$ as the right-hand side of Green's Theorem.

Step 2: Choose $P(x,y)$ and $Q(x,y)$

We want:

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=x^2$$

Let's choose:

- $P=0 \Rightarrow \dfrac{\partial P}{\partial y}=0$
- $Q=\dfrac{1}{3}x^3 \Rightarrow \dfrac{\partial Q}{\partial x}=x^2$

So:

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=x^2$$

Step 3: Set up the line integral

Then by Green's Theorem:

$$\iint_R x^2\,dA=\oint_C P\,dx+Q\,dy=\oint_C \frac{1}{3}x^3\,dy$$

where C is the positively oriented boundary of the ellipse.

Step 4: Parametrize the ellipse

Given:

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

Use the parametric equations:

$$x = 3\cos\theta, \quad y = 2\sin\theta, \quad 0 \le \theta \le 2\pi$$

Compute:

- $x^3 = (3\cos\theta)^3 = 27\cos^3\theta$
- $dy = \dfrac{dy}{d\theta}\,d\theta = 2\cos\theta\,d\theta$

Now plug into the line integral:

$$\oint_C \frac{1}{3}x^3\,dy = \int_0^{2\pi} \frac{1}{3}\cdot 27\cos^3\theta\cdot 2\cos\theta\,d\theta = \int_0^{2\pi} 18\cos^4\theta\,d\theta$$

Step 5: Evaluate the integral

Use the identity:

$$\cos^4\theta = \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$$

So:

$$\int_0^{2\pi} 18\cos^4\theta\,d\theta = 18\int_0^{2\pi}\left(\frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)\right)d\theta$$

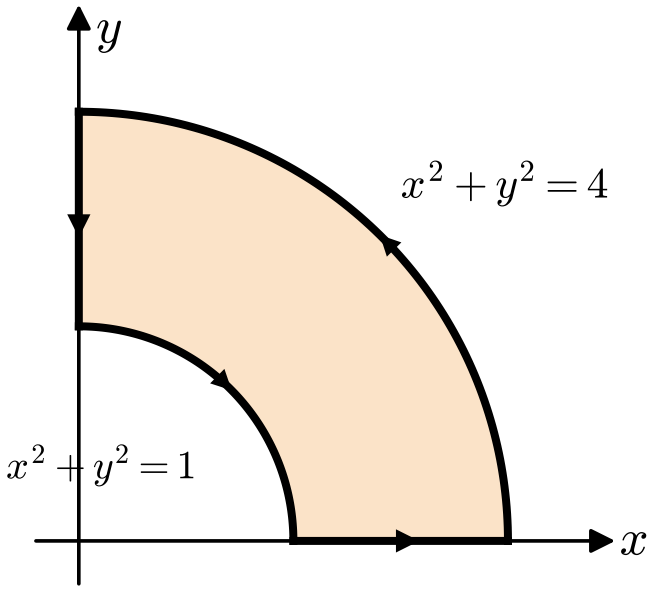
Now integrate:

- $\displaystyle\int_0^{2\pi} \frac{3}{8}\,d\theta = \frac{3}{8}\cdot 2\pi = \frac{3\pi}{4}$
- $\displaystyle\int_0^{2\pi} \cos(2\theta)\,d\theta = 0$
- $\displaystyle\int_0^{2\pi} \cos(4\theta)\,d\theta = 0$

So:

$$\iint_R x^2\,dA = 18\cdot\frac{3\pi}{4} = \frac{27\pi}{2}$$

4. Find the work done by the given force $\mathbf{f} = (x-y)\mathbf{i} + (x+y)\mathbf{j}$ around the closed curve in the following



Solution

The image shows a region bounded between two quarter circles:

- Outer curve: $x^2 + y^2 = 4$ (radius 2)
- Inner curve: $x^2 + y^2 = 1$ (radius 1)

The region is the quarter annulus in the first quadrant, swept from the positive x-axis to the positive y-axis, and the curve is oriented counterclockwise, forming a positively oriented closed curve C

Step 1: Use Green's Theorem

We are to compute:

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Where:

- $M = x - y \Rightarrow \frac{\partial M}{\partial y} = -1$
- $N = x + y \Rightarrow \frac{\partial N}{\partial x} = 1$

So:

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - (-1) = 2$$

Step 2: Area of the Region R

The region R is a quarter annulus between radius 1 and 2, in the first quadrant.

Area of an annulus between $r = 1$ and $r = 2$ is:

$$\text{Full area} = \pi(2^2) - \pi(1^2) = 4\pi - \pi = 3\pi$$

But we only have 1/4 of it:

$$\text{Area}(R) = \frac{1}{4} \cdot 3\pi = \frac{3\pi}{4}$$

Step 3: Apply Green's Theorem

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_R 2\, dA = 2 \cdot \frac{3\pi}{4} = \frac{3\pi}{2}$$

5. Evaluate the integral

$$\iint_R \frac{1}{\sqrt{(x-y)^2 + 2(x+y) + 1}}\, dA$$

where R is the region bounded by the graphs of $y = x$, $x = 2$, and $y = 0$ by means of the change of variables $x = u + uv$, $y = v + uv$

Solution

Step 1: Understand the transformation

We compute:

- $x - y = (u + uv) - (v + uv) = u - v$
- $x + y = (u + uv) + (v + uv) = u + v + 2uv$

Then,

$$(x-y)^2 + 2(x+y) + 1 = (u-v)^2 + 2(u+v+2uv) + 1 = (u+v+1)^2$$

Step 2: Jacobian of the transformation

We need the Jacobian determinant $J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$. Compute partial derivatives:

$$x = u + uv \Rightarrow \begin{cases} \frac{\partial x}{\partial u} = 1 + v, \\ \frac{\partial x}{\partial v} = u, \end{cases} \qquad y = v + uv \Rightarrow \begin{cases} \frac{\partial y}{\partial u} = v, \\ \frac{\partial y}{\partial v} = 1 + u. \end{cases}$$

So the Jacobian matrix is:

$$J = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = (1+v)(1+u) - uv = 1 + u + v$$

Step 3: Find the region S in the uv-plane

We translate the region R into the uv-coordinates

From the substitution:

$$x = u + uv = u(1+v), \quad y = v + uv = v(1+u)$$

We now rewrite the boundaries:

- $y = 0 \Rightarrow v(1+u) = 0 \Rightarrow v = 0$ (since $1 + u \neq 0$)
- $y = x \Rightarrow v(1+u) = u(1+v) \Rightarrow v = u$
- $x = 2 \Rightarrow u(1+v) = 2 \Rightarrow u = \frac{2}{1+v}$

So the region in the uv-plane is:

- lower bound: $v = 0$
- upper bound: $v = u$
- and $u \leq \frac{2}{1+v}$

Let's fix bounds:

Since $u = \frac{2}{1+v}$, and also $v = u$, we substitute:

$$v = \frac{2}{1+v} \Rightarrow v(1+v) = 2 \Rightarrow (v-1)(v+2) = 0$$

So $v = 1$ is the upper limit (since $v > 0$), and the region S is bounded by:

- $0 \leq v \leq 1$
- $v \leq u \leq \frac{2}{1+v}$

Step 4: Change of variables in the integral

We rewrite the integral in terms of u, v:

$$\iint_S \frac{1}{\sqrt{(u+v+1)^2}} \cdot |J| \, du \, dv$$

So the integrand becomes simply 1. Then the integral becomes the area of the region S:

$$\iint_S 1 \, du \, dv = \int_{v=0}^{1} \int_{u=v}^{\frac{2}{1+v}} 1 \, du \, dv = \int_0^1 \left(\frac{2}{1+v} - v \right) dv = 2 \ln 2 - \frac{1}{2}$$

6. Evaluate the integral

$$\iint_R (x^2 + y^2) \sqrt[3]{x^2 - y^2} \, dA$$

where R is the region bounded by the graphs of $x = 0$, $x = 1$, $y = 0$, and $y = 1$ by means of the change of variables $u = 2xy$, $v = x^2 - y^2$

Solution

Step 1: Compute the Jacobian

We need the Jacobian determinant $J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$, but we are given $u = 2xy$, $v = x^2 - y^2$, so we compute $\frac{\partial(u,v)}{\partial(x,y)}$ and then take the reciprocal for the change of variables

Compute partial derivatives:

- $u = 2xy \ \Rightarrow \dfrac{\partial u}{\partial x} = 2y, \ \dfrac{\partial u}{\partial y} = 2x$
- $v = x^2 - y^2 \ \Rightarrow \dfrac{\partial v}{\partial x} = 2x, \ \dfrac{\partial v}{\partial y} = -2y$

So the Jacobian matrix is:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} 2y & 2x \\ 2x & -2y \end{pmatrix}$$

Therefore,

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{|J|} = \frac{1}{|-4(x^2+y^2)|} = \frac{1}{4(x^2+y^2)}$$

Step 2: Substituting the integrand and Jacobian

The integrand is $(x^2+y^2)\sqrt[3]{x^2-y^2}$, and from the transformation:

- $v = x^2 - y^2 \Rightarrow \sqrt[3]{x^2-y^2} = \sqrt[3]{v}$
- x^2+y^2 appears in both the integrand and the Jacobian, so:

$$(x^2+y^2)\cdot\sqrt[3]{x^2-y^2}\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = (x^2+y^2)\cdot\sqrt[3]{v}\cdot\frac{1}{4(x^2+y^2)} = \frac{1}{4}\sqrt[3]{v}$$

So the transformed integral becomes:

$$\iint_S \frac{1}{4}\sqrt[3]{v}\,du\,dv$$

Step 3: Find the region S in uv-coordinates

We are given the transformation:

$$u = 2xy, \quad v = x^2 - y^2$$

and the original region R is the unit square:

$$0 \le x \le 1, \quad 0 \le y \le 1$$

We compute the image of the four corners of the square under the transformation:

(x,y)	$u = 2xy$	$v = x^2 - y^2$	(u,v)
$(0,0)$	0	0	$(0,\ \ 0)$
$(1,0)$	0	1	$(0,\ \ 1)$
$(0,1)$	0	-1	$(0,-1)$
$(1,1)$	2	0	$(2,\ \ 0)$

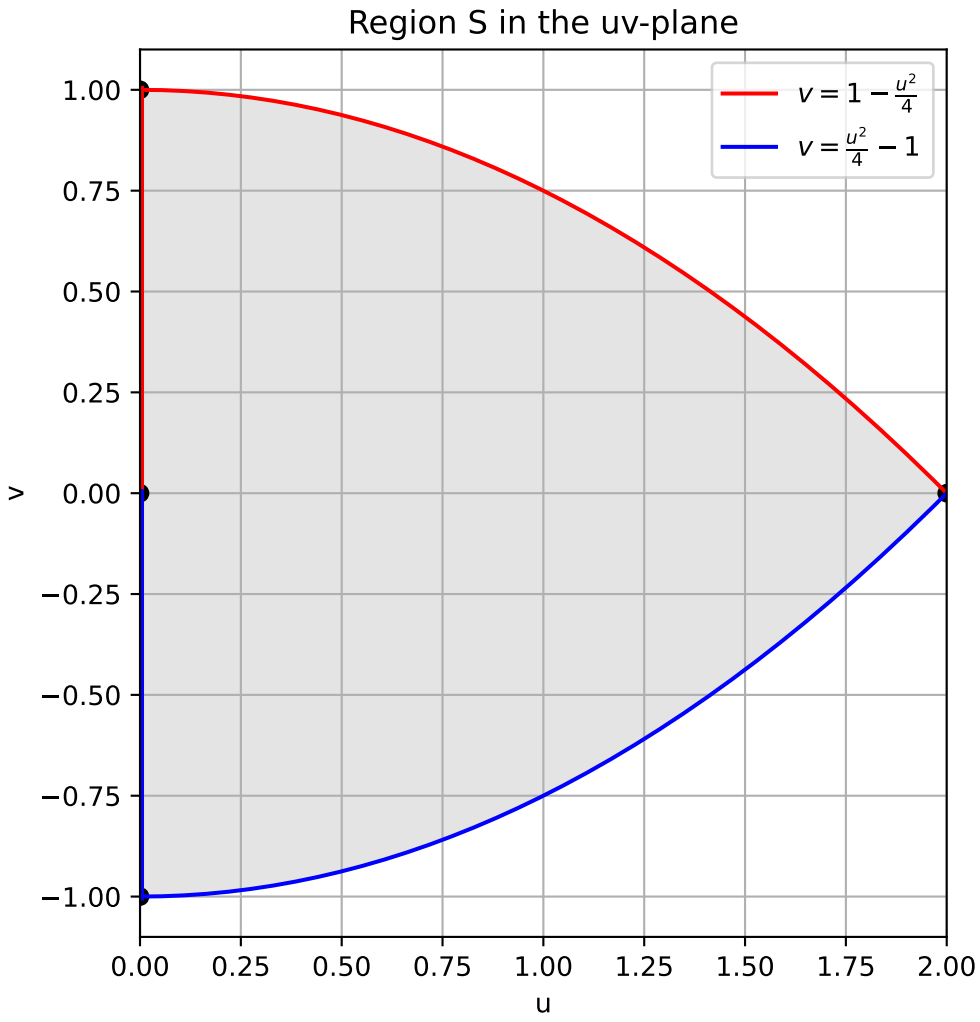
We examine how each edge of the square transforms under the change of variables:

- Edge $x = 0$: $u = 0, v = -y^2 \in [-1,0] \Rightarrow$ Vertical segment on the $u = 0$
- Edge $y = 0$: $u = 0, v = x^2 \in [0,1] \Rightarrow$ Another vertical segment on the $u = 0$
- Edge $x = 1$: $u = 2y \in [0,2], v = 1 - y^2 = 1 - \frac{u^2}{4} \Rightarrow$ Upper parabola
- Edge $y = 1$: $u = 2x \in [0,2], v = x^2 - 1 = \frac{u^2}{4} - 1 \Rightarrow$ Lower parabola

So, the transformed region is:

$$S = \left\{ (u, v) \;\middle|\; 0 \le u \le 2, \quad \frac{u^2}{4} - 1 \le v \le 1 - \frac{u^2}{4} \right\}$$

```
import numpy as np
import matplotlib.pyplot as plt

# Create u values
u = np.linspace(0, 2, 400)

# Define the upper and lower boundaries of v
v_upper = 1 -(u**2)/4
v_lower = (u**2)/4 -1

# Plot the region S
plt.figure(figsize=(6, 6))
plt.plot((0, 0, 0, 2), (0, 1, -1, 0), 'ko')
plt.plot(u, v_upper, 'r', label=r'$v = 1 -\frac{u^2}{4}$')
plt.plot((0, 0), (0, 1), 'r', lw=3)
plt.plot(u, v_lower, 'b', label=r'$v = \frac{u^2}{4} -1$')
plt.plot((0, 0), (0, -1), 'b', lw=3)
plt.fill_between(u, v_lower, v_upper, color='lightgray', alpha=0.6)

# Formatting
plt.title("Region S in the uv-plane")
plt.xlabel("u")
plt.ylabel("v")
plt.legend()
plt.grid(True)
plt.xlim(0, 2)
plt.ylim(-1.1, 1.1)
plt.gca().set_aspect('equal')

plt.show()
```



Step 4: Evaluate the integral

Since the integrand $v^{1/3}$ is an odd function, and the bounds of u are symmetric in v, the integral is:

$$\iint_S \frac{1}{4}\sqrt[3]{v}\,du\,dv = 0$$

7. Find the tangential and normal components of acceleration for the path:

$$\mathbf{x}(t) = (2t, e^{2t})$$

Solution

Step 1: Compute the velocity vector

$$\mathbf{v}(t) = \frac{d\mathbf{x}}{dt} = (2, 2e^{2t})$$

Step 2: Compute the acceleration vector

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = (0, 4e^{2t})$$

Step 3: Compute the speed

$$|\mathbf{v}(t)| = \sqrt{2^2 + (2e^{2t})^2} = \sqrt{4 + 4e^{4t}} = 2\sqrt{1 + e^{4t}}$$

Step 4: Tangential component of acceleration

The tangential component of acceleration is given by the derivative of speed:

$$a_T = \frac{d}{dt}|\mathbf{v}(t)| = \frac{d}{dt}\left(2\sqrt{1 + e^{4t}}\right) = \frac{2 \cdot \frac{1}{2} \cdot 4e^{4t}}{\sqrt{1 + e^{4t}}} = \frac{4e^{4t}}{\sqrt{1 + e^{4t}}}$$

Step 5: Normal component of acceleration

The normal component of acceleration is given by:

$$a_N = \sqrt{|\mathbf{a}(t)|^2 - a_T^2}$$

We compute:

- $|\mathbf{a}(t)| = \sqrt{0^2 + (4e^{2t})^2} = 4e^{2t}$
- So $|\mathbf{a}(t)|^2 = 16e^{4t}$
- And $a_T^2 = \left(\frac{4e^{4t}}{\sqrt{1+e^{4t}}}\right)^2 = \frac{16e^{8t}}{1+e^{4t}}$

Now plug into the normal acceleration formula:

$$\begin{aligned} a_N &= \sqrt{16e^{4t} - \frac{16e^{8t}}{1 + e^{4t}}} \\ &= \sqrt{\frac{16e^{4t}(1 + e^{4t}) - 16e^{8t}}{1 + e^{4t}}} \\ &= \sqrt{\frac{16e^{4t}}{1 + e^{4t}}} \\ &= \frac{4e^{2t}}{\sqrt{1 + e^{4t}}} \end{aligned}$$

8. Find the tangential and normal components of acceleration for the path:

$$\mathbf{x}(t) = (t, t, t^2)$$

Solution

Step 1: Compute velocity and acceleration

$$\mathbf{v}(t) = \frac{d\mathbf{x}}{dt} = (1, 1, 2t)$$
$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = (0, 0, 2)$$

Step 2: Compute speed

$$|\mathbf{v}(t)| = \sqrt{1^2 + 1^2 + (2t)^2} = \sqrt{2 + 4t^2}$$

Step 3: Tangential component of acceleration

The tangential component is:

$$a_T = \frac{d}{dt}|\mathbf{v}(t)| = \frac{d}{dt}\sqrt{2 + 4t^2} = \frac{1}{2\sqrt{2 + 4t^2}} \cdot 8t = \frac{4t}{\sqrt{2 + 4t^2}}$$

Step 4: Normal component of acceleration

The normal component is given by:

$$a_N = \sqrt{|\mathbf{a}(t)|^2 - a_T^2}$$

We have:

- $|\mathbf{a}(t)| = \sqrt{0^2 + 0^2 + 2^2} = 2$
- $|\mathbf{a}(t)|^2 = 4$
- $a_T^2 = \left(\frac{4t}{\sqrt{2+4t^2}}\right)^2 = \frac{16t^2}{2+4t^2}$

Now compute:

$$\begin{aligned} a_N &= \sqrt{4 - \frac{16t^2}{2 + 4t^2}} = \sqrt{\frac{4(2 + 4t^2) - 16t^2}{2 + 4t^2}} \\ &= \sqrt{\frac{8 + 16t^2 - 16t^2}{2 + 4t^2}} = \sqrt{\frac{8}{2 + 4t^2}} \\ &= \frac{2\sqrt{2}}{\sqrt{2 + 4t^2}} \end{aligned}$$

9. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and r denote $\|\mathbf{r}\|$. Evaluate the following:

$$\nabla \cdot (r^n \mathbf{r})$$

Solution

Step 1: Use product rule for divergence

Let's use the identity:

$$\nabla \cdot (f\mathbf{v}) = (\nabla f) \cdot \mathbf{v} + f(\nabla \cdot \mathbf{v})$$

Set:

- $f = r^n$
- $\mathbf{v} = \mathbf{r}$

Then:

$$\nabla \cdot (r^n \mathbf{r}) = (\nabla r^n) \cdot \mathbf{r} + r^n (\nabla \cdot \mathbf{r})$$

Step 2: Compute the terms

1. Gradient of r^n:

We use the chain rule:

$$\nabla r^n = n r^{n-1} \nabla r$$

and since $r = \sqrt{x^2 + y^2 + z^2}$, we know:

$$\nabla r = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) = \frac{\mathbf{r}}{r}$$

So:

$$\nabla r^n = n r^{n-1} \cdot \frac{\mathbf{r}}{r} = n r^{n-2} \mathbf{r}$$

Therefore,

$$(\nabla r^n) \cdot \mathbf{r} = n r^{n-2} \mathbf{r} \cdot \mathbf{r} = n r^{n-2} r^2 = n r^n$$

2. Divergence of $\mathbf{r}$:

$$\nabla \cdot \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

Step 3: Combine everything

$$\nabla \cdot (r^n \mathbf{r}) = n r^n + 3 r^n = (n + 3) r^n$$

10. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and r denote $\|\mathbf{r}\|$. Evaluate the following:

$$\nabla \times (r^n \mathbf{r})$$

Solution

Step 1: Use Vector Identity

We apply the identity:

$$\nabla \times (f\mathbf{v}) = (\nabla f) \times \mathbf{v} + f(\nabla \times \mathbf{v})$$

Let:

- $f = r^n$
- $\mathbf{v} = \mathbf{r}$

So:

$$\nabla \times (r^n \mathbf{r}) = (\nabla r^n) \times \mathbf{r} + r^n (\nabla \times \mathbf{r})$$

Step 2: Compute Each Term

(1) $\nabla \times \mathbf{r}$

$$\nabla \times \mathbf{r} = \nabla \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \mathbf{0}$$

(2) ∇r^n

We compute:

$$\nabla r^n = n r^{n-1} \nabla r$$

and since:

$$\nabla r = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \frac{\mathbf{r}}{r} \Rightarrow \nabla r^n = n r^{n-1} \cdot \frac{\mathbf{r}}{r} = n r^{n-2} \mathbf{r}$$

(3) Compute the cross product

$$(\nabla r^n) \times \mathbf{r} = (n r^{n-2} \mathbf{r}) \times \mathbf{r} = n r^{n-2} (\mathbf{r} \times \mathbf{r}) = n r^{n-2} \cdot \mathbf{0} = \mathbf{0}$$

11. Let $\mathbf{v} = u(x, y)\mathbf{i} - v(x, y)\mathbf{j}$ be an **incompressible**, **irrotational** velocity field

(a) Show that the functions u and v (which determine the component functions of $\mathbf{v}$) satisfy the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(b) Show that u and v are **harmonic**, that is, that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Solution

We are given a 2D velocity field and told that this flow is:

- Incompressible (i.e., $\nabla \cdot \mathbf{v} = 0$)
- Irrotational (i.e., $\nabla \times \mathbf{v} = 0$)

(a)

Step 1: Incompressibility condition

The divergence of a 2D vector field $\mathbf{v} = (u, -v)$ is:

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial(-v)}{\partial y} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

Incompressibility implies:

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$$

Step 2: Irrotationality condition

In 2D, the scalar curl of a vector field $\mathbf{v} = (u, -v)$ is:

$$\nabla \times \mathbf{v} = \frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Irrotationality implies:

$$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

(b)

Step 1: Differentiate the Cauchy-Riemann equations

Start with the first:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Differentiate both sides with respect to x:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad (1)$$

Now differentiate the second Cauchy-Riemann equation:

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Differentiate both sides with respect to y:

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad (2)$$

Step 2: Add the second derivatives of u

From equations (1) and (2):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

(because mixed partial derivatives are equal under continuity)

So,

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Step 3: Do the same for v

Start again from:

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \Rightarrow \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} \quad (3)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \quad (4)$$

Now add:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} = 0$$

Thus:

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

12. Let a particle of mass m travel along a differentiable path $\mathbf{x}$ in a Newtonian vector field $\mathbf{F}$ (i.e., one that satisfies Newton's second law $\mathbf{F} = m\mathbf{a}$, where $\mathbf{a}$ is the acceleration of $\mathbf{x}$). We define the **angular momentum** $\mathbf{l}(t)$ of the particle to be the cross product of the position vector and the linear momentum $m\mathbf{v}$, i.e.,

$$\mathbf{l}(t) = \mathbf{x}(t) \times m\mathbf{v}(t)$$

(Here $\mathbf{v}$ denotes the velocity of $\mathbf{x}$.) The **torque** about the origin of the coordinate system due to the force $\mathbf{F}$ is the cross product of position and force:

$$\mathbf{M}(t) = \mathbf{x}(t) \times \mathbf{F}(t)$$

(a) Show that

$$\frac{d\mathbf{l}}{dt} = \mathbf{M}$$

Thus, we see that the rate of change of angular momentum is equal to the torque imparted to the particle by the vector field $\mathbf{F}$

(*b*) Suppose that $\mathbf{F}$ is a central force (i.e., a force that always points directly towards or away from the origin). Show that in this case the angular momentum is **conserved**, that is, that it must remain constant

Solution

(*a*)

Start from the definition of angular momentum:

$$\mathbf{l}(t) = \mathbf{x}(t) \times m\mathbf{v}(t)$$

Differentiate using the product rule for cross products:

$$\frac{d\mathbf{l}}{dt} = \frac{d}{dt}\left[\mathbf{x}(t) \times m\mathbf{v}(t)\right] = \dot{\mathbf{x}}(t) \times m\mathbf{v}(t) + \mathbf{x}(t) \times m\dot{\mathbf{v}}(t)$$

Since $\dot{\mathbf{x}}(t) = \mathbf{v}(t)$, and $\dot{\mathbf{v}}(t) = \mathbf{a}(t)$, this becomes:

$$\frac{d\mathbf{l}}{dt} = \mathbf{v}(t) \times m\mathbf{v}(t) + \mathbf{x}(t) \times m\mathbf{a}(t)$$

Now note:

- $\mathbf{v}(t) \times \mathbf{v}(t) = \mathbf{0}$ (any vector crossed with itself is zero)
- So the first term vanishes

Thus:

$$\frac{d\mathbf{l}}{dt} = \mathbf{x}(t) \times m\mathbf{a}(t) = \mathbf{x}(t) \times \mathbf{F}(t) = \mathbf{M}(t)$$

(*b*)

What is a central force?

A central force is a vector field where the force always points directly toward or away from the origin. That is:

$$\mathbf{F}(t) = f(r)\,\hat{\mathbf{r}} = f(r)\frac{\mathbf{x}(t)}{r} \quad \text{where} \quad r = \|\mathbf{x}(t)\|$$

In other words, $\mathbf{F}(t)$ is parallel to $\mathbf{x}(t)$, so:

$$\mathbf{x}(t) \times \mathbf{F}(t) = \mathbf{0}$$

Recall from part (a):

We showed that:

$$\frac{d\mathbf{l}}{dt} = \mathbf{x}(t) \times \mathbf{F}(t) = \mathbf{M}(t)$$

Now if $\mathbf{F}$ is a central force, then $\mathbf{x}(t) \times \mathbf{F}(t) = \mathbf{0}$, so:

$$\frac{d\mathbf{l}}{dt} = \mathbf{0}$$

This implies that:

$$\boxed{\mathbf{l}(t) = \text{constant vector}}$$

13. Find the curvature at $t = \pi$ of the cycloid that is described by

$$\mathbf{r}(t) = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j}, \quad a > 0$$

Solution

Compute derivatives and use the planar curvature formula

$$\kappa(t) = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}$$

For $\mathbf{r}(t) = \langle a(t - \sin t),\, a(1 - \cos t)\rangle$ we have

$$x' = a(1 - \cos t), \quad x'' = a\sin t, \qquad y' = a\sin t, \quad y'' = a\cos t$$

So the numerator is

$$\begin{aligned} |x'y'' - y'x''| &= a^2\left|(1 - \cos t)\cos t - \sin^2 t\right| \\ &= a^2|\ \cos t - (\cos^2 t + \sin^2 t)\ | = a^2(1 - \cos t) \end{aligned}$$

The denominator inside the 3/2 power is

$$\begin{aligned} x'^2 + y'^2 &= a^2((1 - \cos t)^2 + \sin^2 t) \\ &= a^2(2 - 2\cos t) = 2a^2(1 - \cos t) \end{aligned}$$

hence

$$\left(x'^2 + y'^2\right)^{3/2} = a^3\big(2(1 - \cos t)\big)^{3/2} = a^3 2^{3/2}(1 - \cos t)^{3/2}$$

Thus

$$\kappa(t) = \frac{a^2(1 - \cos t)}{a^3 2^{3/2}(1 - \cos t)^{3/2}} = \frac{1}{a\,2^{3/2}\sqrt{1 - \cos t}}$$

Using $1 - \cos t = 2\sin^2(t/2)$, this becomes the simpler form

$$\kappa(t) = \frac{1}{4a\,|\sin(t/2)|}$$

At $t = \pi$, $\sin(\pi/2) = 1$, so

$$\boxed{\kappa(\pi) = \frac{1}{4a}}$$

14. Find the osculating plane to the following circular helix at $t = \pi/4$

$$\mathbf{r}(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + 3t\mathbf{k}$$

Solution

For $\mathbf{r}(t) = \langle -2\sin t,\, 2\cos t,\, 3t\rangle$, the osculating plane at t_0 is the plane through $\mathbf{r}(t_0)$ whose normal is the binormal vector

$$\mathbf{B}(t_0) = \frac{\mathbf{r}'(t_0) \times \mathbf{r}''(t_0)}{\|\mathbf{r}'(t_0) \times \mathbf{r}''(t_0)\|}$$

Compute derivatives:

$$\mathbf{r}'(t) = (-2\cos t,\, -2\sin t,\, 3), \qquad \mathbf{r}''(t) = (2\sin t,\, -2\cos t,\, 0)$$

At $t = \pi/4$:

$$\mathbf{r}(\tfrac{\pi}{4}) = (\,-\sqrt{2},\ \sqrt{2},\ \frac{3\pi}{4})$$

and

$$\mathbf{r}'(\tfrac{\pi}{4}) \times \mathbf{r}''(\tfrac{\pi}{4}) = \begin{pmatrix} 3\sqrt{2} \\ 3\sqrt{2} \\ 4 \end{pmatrix}$$

Use this cross product as the plane normal. The osculating plane equation is

$$(3\sqrt{2},\, 3\sqrt{2},\, ,\, 4) \cdot ((x, y, z) - (-\sqrt{2}, \sqrt{2}, \frac{3\pi}{4})) = 0$$

Expanding and simplifying gives the tidy form

$$\boxed{3\sqrt{2}(x + y) + 4z = 3\pi}$$

which is the osculating plane at $t = \pi/4$

15. Let $\omega = \nabla \times \mathbf{u}$ represent the vorticity of an inviscid, barotropic fluid governed by the Euler equation

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{f}$$

where $\frac{D}{Dt}$ denotes the material derivative $(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla)$. Under the barotropic assumption, the pressure is a function of density alone, $p = p(\rho)$

(a) Take the curl of the Euler equation to obtain the evolution equation for the vorticity field

(b) Show that for an inviscid, barotropic fluid with *conservative* body forces,

$$\Gamma(t) = \oint_{C(t)} \mathbf{u} \cdot d\mathbf{r}$$

remains constant in time if $C(t)$ moves with the fluid. **Hint** Differentiate $\Gamma(t)$:

$$\frac{d\Gamma}{dt} = \oint_{C(t)} \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{r}$$

(c) Evaluate $\frac{D\omega_z}{Dt}$ for a *two-dimensional incompressible* flow, and discuss the resulting consequences for the evolution of vortex patches and vortex sheets

Solution

(a) Curl the Euler equation → vorticity evolution

Euler (compressible) in material-derivative form:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{f}$$

Define vorticity $\omega = \nabla \times \mathbf{u}$. Take $\nabla \times (\cdot)$ of both sides. Use the identity

$$(u \cdot \nabla)u = \nabla\Big(\tfrac{1}{2}|u|^2\Big) + \omega \times \mathbf{u}$$

so $\nabla \times \big((u \cdot \nabla)u\big) = \nabla \times (\omega \times \mathbf{u})$.

Another identity gives

$$\nabla \times (\omega \times \mathbf{u}) = (\mathbf{u} \cdot \nabla)\omega - (\omega \cdot \nabla)\mathbf{u} + \omega(\nabla \cdot \mathbf{u}) - \mathbf{u}(\nabla \cdot \omega)$$

and $\nabla \cdot \omega = \nabla \cdot (\nabla \times \mathbf{u}) = 0$. Thus

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = (\omega \cdot \nabla)\mathbf{u} - \omega(\nabla \cdot \mathbf{u}) + \nabla \times \mathbf{f} - \nabla \times \Big(\frac{1}{\rho}\nabla p\Big)$$

Under the barotropic assumption $p = p(\rho)$

$$\nabla \times \Big(\frac{1}{\rho}\nabla p\Big) = \nabla\Big(\frac{1}{\rho}\Big) \times \nabla p = \frac{1}{\rho^2}(\nabla\rho \times \nabla p) = 0$$

because ∇p is parallel to $\nabla\rho$

Therefore the vorticity equation becomes

$$\boxed{\frac{D\omega}{Dt} = (\omega \cdot \nabla)\mathbf{u} - \omega(\nabla \cdot \mathbf{u}) + \nabla \times \mathbf{f}}$$

For an incompressible flow ($\nabla \cdot \mathbf{u} = 0$) this reduces to the familiar

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)\mathbf{u} + \nabla \times \mathbf{f}$$

The term $(\omega \cdot \nabla)\mathbf{u}$ is the vortex-stretching term that can amplify vorticity in 3D

(b) Kelvin's circulation theorem (conservation of circulation)

Circulation around a material loop $C(t)$ is

$$\Gamma(t) = \oint_{C(t)} \mathbf{u} \cdot d\mathbf{r}$$

Differentiate following the moving loop (hint given):

$$\frac{d\Gamma}{dt} = \oint_{C(t)} \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{r} = \oint_{C(t)} \Big(-\frac{1}{\rho}\nabla p + \mathbf{f} \Big) \cdot d\mathbf{r}$$

Under barotropic $p = p(\rho)$ there exists a scalar $H(\rho)$ (specific enthalpy) with $\nabla H = \frac{1}{\rho}\nabla p$. If the body force is conservative, $\mathbf{f} = -\nabla\Phi$ for some potential Φ. Then the integrand is a gradient:

$$-\frac{1}{\rho}\nabla p + \mathbf{f} = -\nabla H - \nabla\Phi = -\nabla(H + \Phi)$$

and the line integral of a gradient around a closed curve vanishes. Hence

$$\frac{d\Gamma}{dt} = 0$$

So circulation around any material loop $C(t)$ is constant in time — this is Kelvin's circulation theorem (for inviscid, barotropic fluid with conservative body forces)

(c) Two-dimensional incompressible flow: $\dfrac{D\omega_z}{Dt}$ and consequences

For 2D flow in the xy-plane, let $\mathbf{u} = (u(x, y, t), v(x, y, t), 0)$ and assume no z-dependence. The vorticity has only a z-component:

$$\omega = (0, 0, \omega_z), \qquad \omega_z = \partial_x v - \partial_y u$$

For incompressible flow $\nabla \cdot \mathbf{u} = 0$. Also $(\omega \cdot \nabla)\mathbf{u}$ vanishes because ω points in z and there is no z-variation:

$$(\omega \cdot \nabla)\mathbf{u} = \omega_z \partial_z \mathbf{u} = 0$$

If body forces are irrotational so $\nabla \times \mathbf{f} = 0$, the vorticity equation reduces to

$$\frac{D\omega_z}{Dt} = 0$$

So in 2D incompressible Euler flow the scalar vorticity ω_z is materially conserved along particle paths

Consequences:

- **Vortex patches.** A vortex patch is a region where ω_z is nonzero (often constant) and zero outside. Because ω_z is advected with the flow, every fluid particle keeps its initial vorticity value. Thus a patch's vorticity value is preserved and the patch is merely carried/deformed by the flow. Incompressibility preserves area, so the area of the patch remains constant (though its shape can become complex — e.g. filamentation)
- **Vortex sheets.** A vortex sheet is a curve across which the tangential velocity is discontinuous (equivalently a delta-distribution vorticity concentrated on a curve). In 2D inviscid flow the sheet strength (the jump in tangential velocity, or equivalently circulation per unit length) is transported with the sheet — the sheet is convected by the flow. However, vortex sheets in inviscid flow are nonlinearly unstable (Kelvin–Helmholtz instability) and tend to roll up into spirals (the sheet can develop finer and finer structure). In the ideal Euler limit this leads to complicated, potentially singular behavior; with even small viscosity the sheet diffuses and rolls up more smoothly
- **No vortex stretching in 2D.** Because there is no $(\omega \cdot \nabla)\mathbf{u}$ term, vorticity cannot be amplified by stretching as it can in 3D. This is the fundamental reason 2D turbulence and vortex dynamics differ qualitatively from 3D: vorticity values are conserved along particle paths, and inverse energy cascade / persistence of coherent vortices are characteristic features

16. Let $\mathbf{u}(x, y, z) = \langle -yf(r, z),\ xf(r, z),\ g(r, z)\rangle\,, \quad r = \sqrt{x^2 + y^2}$

(a) In cylindrical coordinates (r, θ, z), find the condition from $\nabla \cdot \mathbf{u} = 0$

(b) Assuming no θ-dependence, show that there exists a streamfunction $\Psi(r, z)$ such that

$$u_r = \frac{1}{r}\frac{\partial \Psi}{\partial z}, \qquad u_z = -\frac{1}{r}\frac{\partial \Psi}{\partial r}$$

(c) Write the formal decomposition $\mathbf{u} = \nabla\phi + \nabla \times \mathbf{A}$. Discuss what ϕ (and boundary conditions on $\mathbf{A}$, this part is optional) become for incompressible flow

(d) For $f(r, z) = \dfrac{\alpha}{(r^2 + z^2)^{3/2}}$, $g = 0$, compute $\omega = \nabla \times \mathbf{u}$ and describe its physical meaning

Solution

Throughout use cylindrical coordinates (r, θ, z) with

$$x = r\cos\theta, \qquad y = r\sin\theta$$

and unit vectors

$$\mathbf{e}_r = (\cos\theta, \sin\theta, 0),\ \mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0),\ \mathbf{e}_z = (0, 0, 1)$$

A quick and useful observation from the given Cartesian form

$$\mathbf{u} = (-yf(r, z),\ xf(r, z),\ g(r, z))$$

is that

$$u_r = \mathbf{u} \cdot \mathbf{e}_r = 0, \qquad u_\theta = \mathbf{u} \cdot \mathbf{e}_\theta = r\, f(r, z), \qquad u_z = g(r, z)$$

(a) Divergence-free condition in cylindrical coordinates

The divergence in cylindrical coordinates is

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

Substitute $u_r = 0,\ u_\theta = rf(r,z),\ u_z = g(r,z)$. Then

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial \theta}(rf(r,z)) + \frac{\partial g}{\partial z}$$

So the condition $\nabla \cdot \mathbf{u} = 0$ is

$$\boxed{\frac{1}{r}\frac{\partial}{\partial \theta}(rf(r,z)) + \frac{\partial g}{\partial z} = 0}$$

If the flow is axisymmetric (no θ-dependence) this reduces to

$$\boxed{\frac{\partial g}{\partial z} = 0 \quad \Longrightarrow \quad g = g(r)}$$

(b) Existence of an axisymmetric stream function $\Psi(r,z)$

Assume no θ-dependence (axisymmetric). For general axisymmetric flows (not necessarily the special form above) the divergence-free condition becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0$$

Multiply by r:

$$\frac{\partial}{\partial r}(ru_r) + \frac{\partial}{\partial z}(ru_z) = 0$$

This is the 2D divergence-free condition in the (r,z)-plane for the vector $(ru_r,\ ru_z)$; hence there exists a stream function $\Psi(r,z)$ (unique up to additive constant) such

$$ru_r = \frac{\partial \Psi}{\partial z}, \qquad ru_z = -\frac{\partial \Psi}{\partial r}$$

Dividing by r gives the standard axisymmetric Stokes stream function relations

$$\boxed{u_r = \frac{1}{r}\frac{\partial \Psi}{\partial z}, \qquad u_z = -\frac{1}{r}\frac{\partial \Psi}{\partial r}}$$

(These formulae automatically satisfy $\nabla \cdot \mathbf{u} = 0$)

(c) Helmholtz decomposition $\mathbf{u} = \nabla\phi + \nabla \times \mathbf{A}$ and consequences for incompressible flow

The Helmholtz decomposition states any sufficiently smooth vector field on $\mathbb{R}^3$ (with suitable decay or boundary conditions) can be written as

$$\mathbf{u} = \nabla\phi + \nabla \times \mathbf{A}$$

where ϕ is a scalar potential and $\mathbf{A}$ a vector potential. Taking divergence gives

$$\nabla \cdot \mathbf{u} = \nabla^2 \phi + \nabla \cdot (\nabla \times \mathbf{A}) = \nabla^2 \phi$$

since $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. Thus for an incompressible flow $\nabla \cdot \mathbf{u} = 0$ we have

$$\boxed{\nabla^2 \phi = 0}$$

so ϕ is harmonic. With common boundary/decay conditions (e.g. $\mathbf{u}$ vanishes at infinity or normal velocity specified on a closed boundary), the only physically allowable harmonic ϕ compatible with zero net flux is typically a constant (so $\nabla \phi = 0$), and all motion is described by the solenoidal part $\nabla \times \mathbf{A}$. In short:

- For incompressible flow, ϕ is harmonic; if boundary/decay conditions force ϕ to be constant, the flow is purely solenoidal $\mathbf{u} = \nabla \times \mathbf{A}$
- A convenient gauge for $\mathbf{A}$ is the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. Then

  $$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} = \mathbf{u} - \nabla \phi$$

  For incompressible flows with ϕ chosen zero, $\mathbf{A}$ solves the vector Poisson equation

  $$-\nabla^2 \mathbf{A} = \mathbf{u}$$

  with boundary conditions on $\mathbf{A}$ chosen to make the solution unique, e.g. $\mathbf{A} \to 0$ at infinity and $\nabla \cdot \mathbf{A} = 0$

So physically: for incompressible flows the irrotational part is absent (or trivial) under typical no-flux/decay conditions, and the velocity is given entirely by the curl of a vector potential which encodes the vorticity

(d) Vorticity for $f(r,z) = \dfrac{\alpha}{(r^2+z^2)^{3/2}}$, $g = 0$. Physical interpretation

With the specified f and $g = 0$ the flow is purely azimuthal:

$$u_r = 0, \qquad u_\theta = r f(r,z) = \frac{\alpha r}{(r^2+z^2)^{3/2}}, \qquad u_z = 0$$

Compute the vorticity $\omega = \nabla \times \mathbf{u}$. For axisymmetric, θ-independent fields the cylindrical-component formulas simplify to

$$\omega_r = -\frac{\partial u_\theta}{\partial z}, \qquad \omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = 0, \qquad \omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)$$

Compute derivatives. Put $s = r^2 + z^2$. Then

$$f = \alpha s^{-3/2}, \qquad f_z = -\frac{3\alpha z}{s^{5/2}}$$

Thus

$$\omega_r = -\frac{\partial}{\partial z}(rf) = -r f_z = -r\Big(-\frac{3\alpha z}{s^{5/2}}\Big) = \frac{3\alpha r z}{s^{5/2}}$$

For ω_z:

$$ru_\theta = r^2 f = \alpha \frac{r^2}{s^{3/2}}$$

Differentiate in r:

$$\frac{\partial}{\partial r}(r^2 f) = \alpha\Big(2rs^{-3/2} + r^2 \cdot (-\tfrac{3}{2})s^{-5/2} \cdot 2r\Big) = \alpha r s^{-5/2}(2s - 3r^2)$$

Therefore

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r^2 f) = \alpha s^{-5/2}(2s - 3r^2) = \alpha \frac{-r^2 + 2z^2}{(r^2 + z^2)^{5/2}}$$

So the vorticity vector is

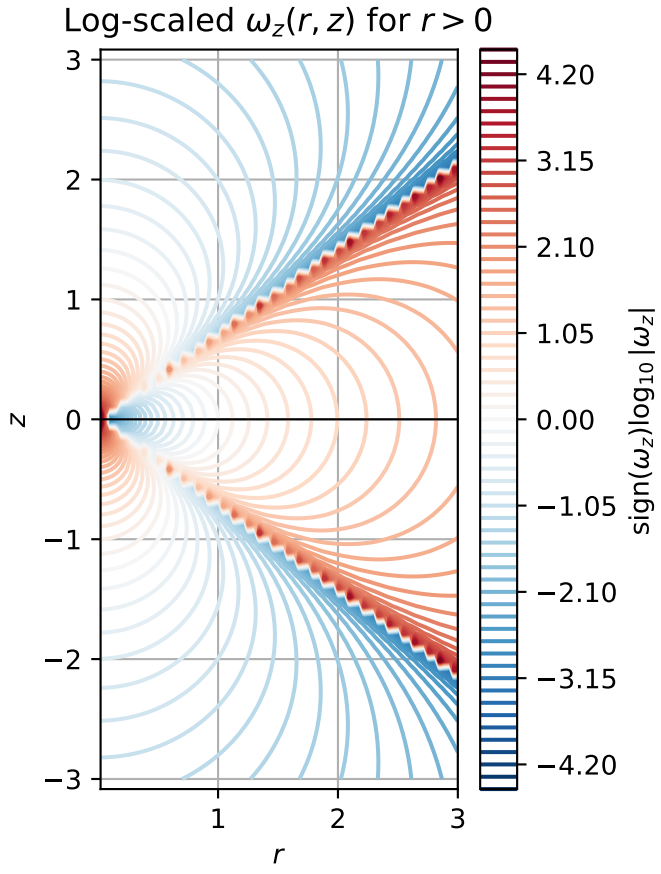
$$\boxed{\omega(r, z) = \omega_r \, \mathbf{e}_r + \omega_z \, \mathbf{e}_z = \frac{3\alpha r z}{(r^2 + z^2)^{5/2}} \, \mathbf{e}_r + \frac{\alpha(-r^2 + 2z^2)}{(r^2 + z^2)^{5/2}} \, \mathbf{e}_z}$$

Notes / physical interpretation

- The vorticity has no azimuthal component ($\omega_\theta = 0$); it is in the meridional (poloidal) plane (r, z). Thus swirling motion u_θ produces poloidal vorticity — typical of axisymmetric swirling flows
- The vorticity field is singular at the origin (behaves like a power of distance) and decays rapidly for large distance: the denominator $(r^2 + z^2)^{5/2}$ gives a far-field decay like $O(\text{distance}^{-5})$ for components (so the induced velocity decays more slowly)
- The structure of ω resembles a dipole-like poloidal vorticity: ω_r is odd in z (changes sign across the $z = 0$ plane), while ω_z has a quadrupolar character (combination of r^2 and z^2). Vorticity lines (integral curves of ω) lie in meridional planes and encircle/loop around the symmetry axis — qualitatively similar to a vortex ring/poloidal-dipole structure concentrated near the origin
- Physically this flow can be seen as an idealized concentrated swirl (azimuthal velocity decaying like $(\text{distance})^{-2}$ for large distance because $u_\theta \sim r/(r^2+z^2)^{3/2}$), and the computed ω describes the local twisting/rotation of fluid elements produced by that swirl. The sign pattern indicates opposite vorticity above and below the midplane $z = 0$ in the radial component, etc

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.colors as colors

alpha = 1.0

# Grid for r > 0 region
r = np.linspace(0.02, 3, 100)
z = np.linspace(-3, 3, 80)
R, Z = np.meshgrid(r, z)

# Define vorticity component ω_z
```

```
s = R**2 +Z**2
omega_z = alpha *(-R**2 +2 *Z**2) /s**(2.5)

# Compute log10 of magnitude and preserve sign
log_omega = np.sign(omega_z) * np.log10(np.abs(omega_z) + 1e-10)

# --- Plot log-scaled ω_z ---
plt.figure(figsize=(3, 5))
cont = plt.contour(R, Z, log_omega, levels=60, cmap="RdBu_r")
plt.colorbar(cont, label=r"$\mathrm{sign}(\omega_z)\log_{10}|\omega_z|$")
plt.title(r"Log-scaled $\omega_z(r,z)$ for $r > 0$")
plt.xlabel("$r$")
plt.ylabel("$z$")
plt.axhline(0, color='k', lw=0.8)
plt.grid(True)
plt.axis("equal")
plt.show()
```



Answers

Answer 1 (Section 10.12)

$$\oint_{C_1 \cup C_2} \frac{-y\,dx + x\,dy}{x^2 + y^2} = 0 \quad \text{(Green's theorem on } R\text{, which excludes the origin)}$$

Answer 2 (Section 10.12)

$$\oint_{C_1} \frac{-y\,dx + x\,dy}{x^2 + y^2} = 2\pi$$

Answer 3 (Section 10.15)

$$V = \frac{15}{8} = 1.875$$

11 Orthogonal Functions and Fourier Series

Learning objectives After this chapter you should be able to

- test functions for orthogonality and expand a function in an orthogonal set
- compute Fourier series, half-range cosine and sine series, and complex Fourier series, and describe their convergence and the Gibbs phenomenon
- formulate regular and singular Sturm-Liouville problems and use the orthogonality of their eigenfunctions
- expand functions in Fourier-Bessel and Fourier-Legendre series

11.1 Orthogonal Functions

- Inner Product

  If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ are two vectors in $\mathbb{R}^3$, then inner product is defined:

$$(\mathbf{u}, \mathbf{v}) = u_1v_1 + u_2v_2 + u_3v_3 = u_iv_i$$

- The inner product $(\mathbf{u}, \mathbf{v})$ possesses the following properties:
  - $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$
  - $(k\mathbf{u}, \mathbf{v}) = k(\mathbf{u}, \mathbf{v})$, $k:$ a scalar
  - $(\mathbf{u}, \mathbf{u}) = 0$ if $\mathbf{u} = \mathbf{0}$ and $(\mathbf{u}, \mathbf{u}) > 0$ if $\mathbf{u} \neq \mathbf{0}$
  - $(\mathbf{u} + \mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$
- Inner Product of Functions

  The inner product of two functions f_1 and f_2 on an interval $[a, b]$ is the number

$$(f_1, f_2) = \int_a^b f_1(x)\, f_2(x)\, dx$$

- Orthogonal Functions

  Two functions f_1 and f_2 are said to be **orthogonal** on $[a, b]$ if

$$(f_1, f_2) = \int_a^b f_1(x)\, f_2(x)\, dx = 0$$

- Orthogonal Sets

  A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \cdots\}$ is said to be **orthogonal** on $[a, b]$ if

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\, \phi_n(x)\, dx = 0, \quad m \neq n$$

  With the property that $\|\phi_n(x)\| = 1$, then $\{\phi_n(x)\}$ is said to be an **orthonormal set** on the interval

Example $\left\{ \dfrac{1}{\sqrt{2\pi}}, \dfrac{\cos x}{\sqrt{\pi}}, \dfrac{\cos 2x}{\sqrt{\pi}}, \cdots \right\}, \quad -\pi \leq x \leq \pi$

Answer 1 (end of chapter) · Solution

- Orthogonal Series Expansion

  Suppose $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on $[a, b]$. If $y = f(x)$ is a function defined on $[a, b]$,

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots$$

$$\Downarrow$$

$$\int_a^b f(x)\phi_n(x)\, dx = c_0 \int_a^b \phi_0(x)\phi_n(x)\, dx + c_1 \int_a^b \phi_1(x)\phi_n(x)\, dx + \cdots + c_n \int_a^b \phi_n^2(x)\, dx + \cdots$$

$$= c_n \int_a^b \phi_n^2(x)\, dx$$

$$\Downarrow$$

$$c_n = \frac{\displaystyle\int_a^b f(x)\, \phi_n(x)\, dx}{\displaystyle\int_a^b \phi_n^2(x)\, dx}, \quad n = 0, 1, 2, \cdots$$

- Orthogonal Set/Weight Function

  A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \cdots\}$ is said to be **orthogonal with respect to a weight function** $w(x)$ on $[a, b]$ if

$$(\phi_m, \phi_n) = \int_a^b w(x)\phi_m(x)\phi_n(x)\, dx = 0, \quad m \neq n$$

  With this inner product notation,

$$f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x)$$

  where $c_n = \dfrac{\displaystyle\int_a^b f(x)w(x)\phi_n(x)\, dx}{\displaystyle\int_a^b w(x)\phi_n^2(x)\, dx}$

11.2 Fourier Series

- Trigonometric Series

$$\left\{1,\ \cos\frac{\pi}{p}x,\ \cos\frac{2\pi}{p}x,\ \cos\frac{3\pi}{p}x,\cdots,\ \sin\frac{\pi}{p}x,\ \sin\frac{2\pi}{p}x,\ \sin\frac{3\pi}{p}x,\ \cdots\right\}$$

- Fourier Series

  The **Fourier series** of a function f defined on the interval $(-p,p)$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\right)$$

  where

$$a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\frac{n\pi}{p}x\,dx,\quad a_0=\frac{1}{p}\int_{-p}^{p}f(x)\,dx$$
$$b_n=\frac{1}{p}\int_{-p}^{p}f(x)\sin\frac{n\pi}{p}x\,dx$$

  At a point of discontinuity, the Fourier series converges to the average $\dfrac{f(x^+)+f(x^-)}{2}$

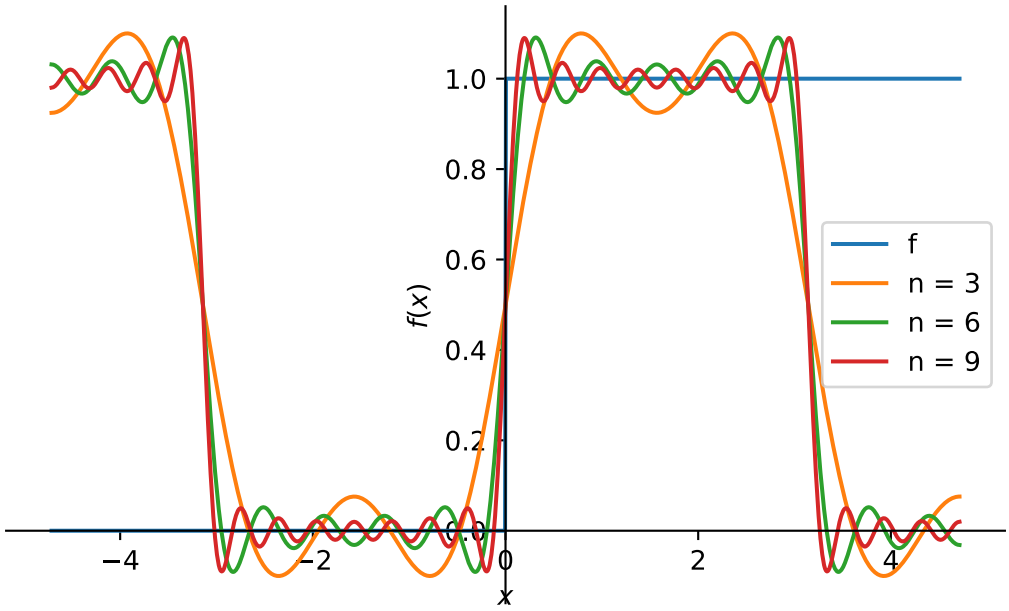
Example

$$f(x)=\begin{cases}0, & -\pi\le x\le 0\\ 1, & 0<x\le\pi\end{cases}$$
$$\Downarrow$$
$$f(x)=\frac{1}{2}+\sum_{m=1}^{\infty}\frac{2}{(2m-1)\pi}\sin(2m-1)x$$

```
from sympy import fourier_series, Piecewise, pi, plot
from sympy.abc import x

f = Piecewise((0, x < 0), (1, True))
s = fourier_series(f, (x, -pi, pi))
s.truncate(5)
```

$$\frac{2\sin(x)}{\pi}+\frac{2\sin(3x)}{3\pi}+\frac{2\sin(5x)}{5\pi}+\frac{2\sin(7x)}{7\pi}+\frac{1}{2}$$

```
s3 = s.truncate(n = 3)
s6 = s.truncate(n = 6)
s9 = s.truncate(n = 9)

p = plot(f, s3, s6, s9, (x, -1.5 *pi, 1.5 *pi), show=False, legend=True)

p[0].label = 'f'
p[1].label = 'n = 3'
p[2].label = 'n = 6'
p[3].label = 'n = 9'

p.show()
```




```
import numpy as np
import matplotlib.pyplot as plt
plt.rcParams['font.size'] = 10

x_t1 = [-2*np.pi, -np.pi, -np.pi, 0, 0, np.pi, np.pi, 2*np.pi]
y_t1 = [1, 1, 0, 0, 1, 1, 0, 0]

x_t2 = [-np.pi, 0, 0, np.pi]
y_t2 = [0, 0, 1, 1]

x = np.linspace(-2 *np.pi, 2 *np.pi, 1000)

def example_plot(func, ns):

    fig, axes = plt.subplots(2, 2, figsize=(7.5, 7))
    i = 0
    for n in ns:
        ax = axes[i//2, i%2]
        ax.plot(x_t1, y_t1, 'b:')
        ax.plot(x_t2, y_t2, 'g-', lw=3)
```

```
        ax.plot(x, func(x, n), c='darkorange')
        ax.set_xticks([-2.0 *np.pi, -np.pi, 0, np.pi, 2.0 *np.pi])
        ax.set_xticklabels([r'$-2\pi$', r'$-\pi$', '0',
          r'$\pi$', r'$2\pi$'])
        ax.set_xlim(-2.0 *np.pi, 2.0 *np.pi)
        ax.set_ylim(-0.2, 1.2)
        ax.set_title(f"$n=${n}")

        i += 1
```

```
def rect_wave(x, n):
    y = 0.5
    for m in range(1, n +1):
        y = y +2.0 /((2.0*m -1) *np.pi) *np.sin((2.0*m -1)*x)
    return y

example_plot(rect_wave, ns=[1, 3, 6, 9])
```

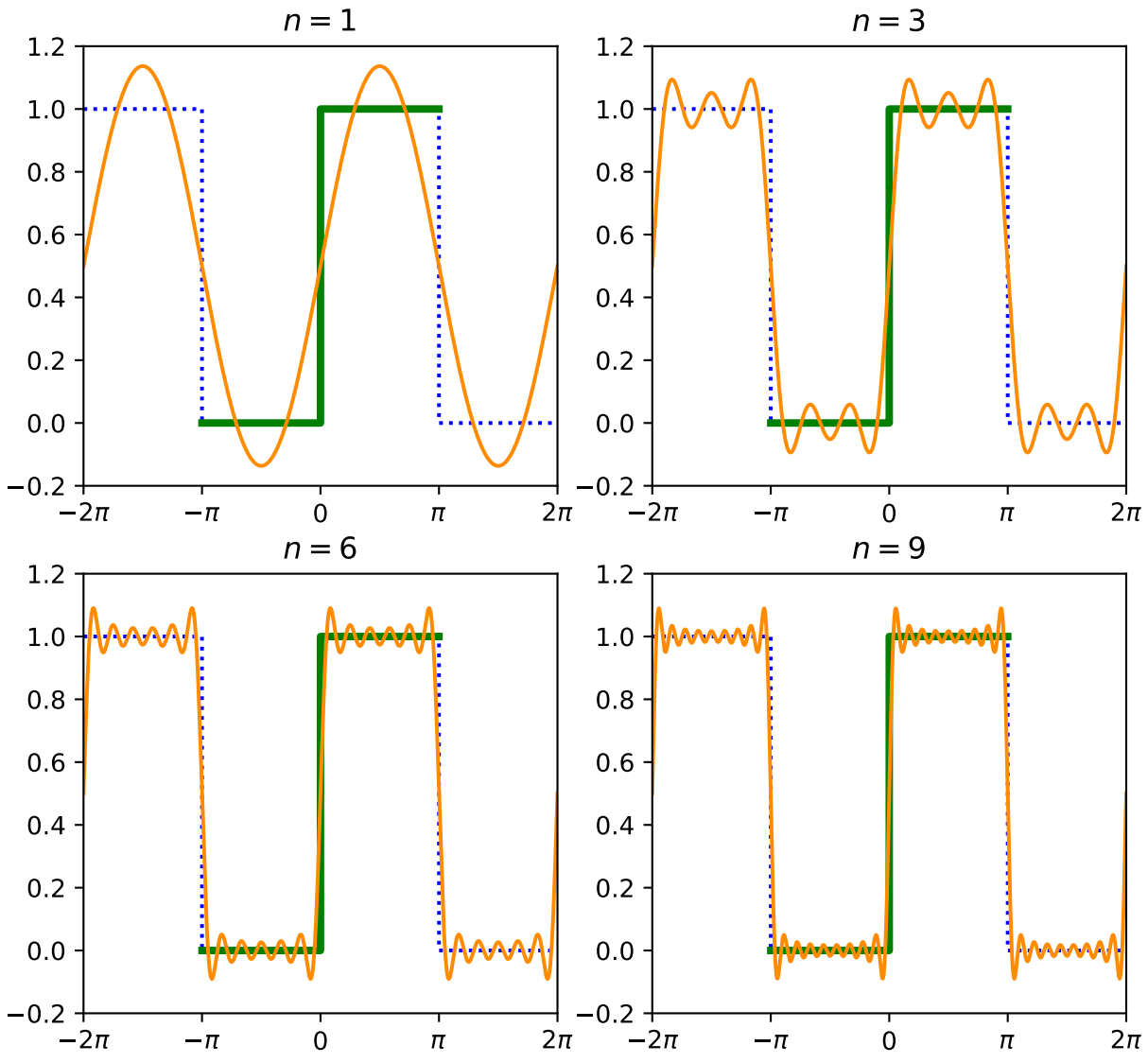


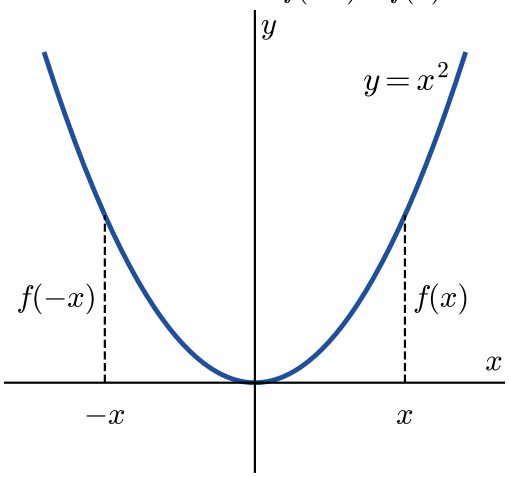
Figure 11.1: Fourier Series

- The *overshooting* by the partial sums from the functional values near a point of discontinuity does not smooth out but remains fairly constant. This behavior of a Fourier series near a point at which f is discontinuous is known as the *Gibbs phenomenon*

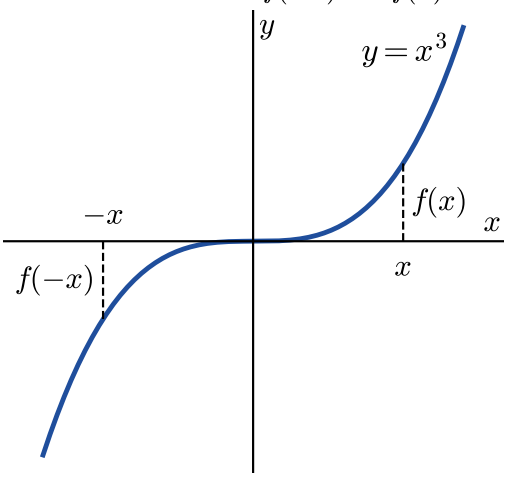
Try it in Python: Computing Fourier coefficients by numerical integration (`scipy.integrate.quad`): Numerical integration with Scipy; or exactly with `sympy`: Integrals

11.3 Fourier Cosine and Sine Series

even function: $f(-x) = f(x)$



odd function: $f(-x) = -f(x)$



- If f is even, then
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$
- If f is odd, then
$$\int_{-a}^{a} f(x)\,dx = 0$$
- The Fourier series of an even function on the interval $(-p, p)$ is the cosine series
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$$
where $a_0 = \dfrac{2}{p}\displaystyle\int_0^p f(x)\,dx$, $a_n = \dfrac{2}{p}\displaystyle\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$
- The Fourier series of an odd function on the interval $(-p, p)$ is the sine series
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x \quad \text{where } b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$

Example

$$f(x) = x, \quad 0 < x < 2$$

$$\Downarrow$$

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2} x$$

or

$$f(x) = 1 - \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} \cos \frac{(2m-1)\pi}{2} x$$

```
x_t1 = [-4,-2,-2, 2, 2, 4]
y_t1 = [ 0, 2,-2, 2,-2, 0]

x_t2 = [0, 2]
y_t2 = [0, 2]

x = np.linspace(-4, 4, 1000)

def example_plot2(func, ns):
    fig, axes = plt.subplots(2, 2, figsize=(7.5, 7))
    i = 0
    for n in ns:
        ax = axes[i//2, i%2]
        ax.plot(x_t1, y_t1, 'b:')
        ax.plot(x_t2, y_t2, 'g-', lw=3)
        ax.plot(x, func(x, n), c='darkorange')
        ax.set_xlim(-4, 4)
        ax.set_title(f"$n=${n}")

        i += 1
```

```
def ramp_sin(x, n):
    y = 0
    for m in range(1, n +1):
        y = y +4.0 /np.pi *(-1)**(m +1)/m *np.sin(m *np.pi /2 *x)
    return y

example_plot2(ramp_sin, ns=[1, 3, 6, 9])
```

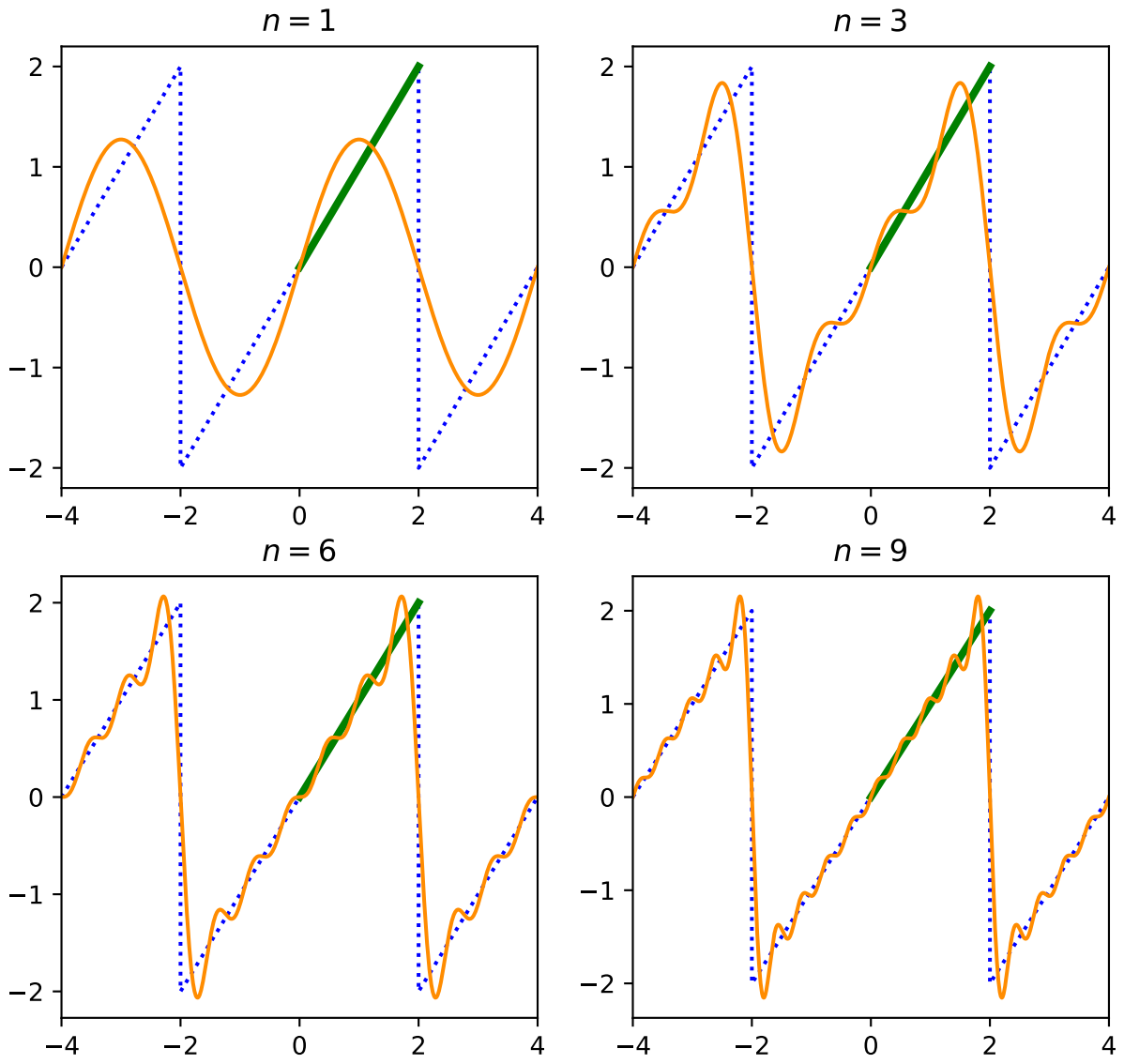


Figure 11.2: Sine Series

```
x_t1 = [-4,-2, 0, 2, 4]
y_t1 = [ 0, 2, 0, 2, 0]

def ramp_cos(x, n):
    y = 1
    for m in range(1, n +1):
        y = y -8.0 /(np.pi *(2*m -1))**2.0 *np.cos((2*m -1) *np.pi /2 *x)
    return y

example_plot2(ramp_cos, ns=[1, 3, 6, 9])
```

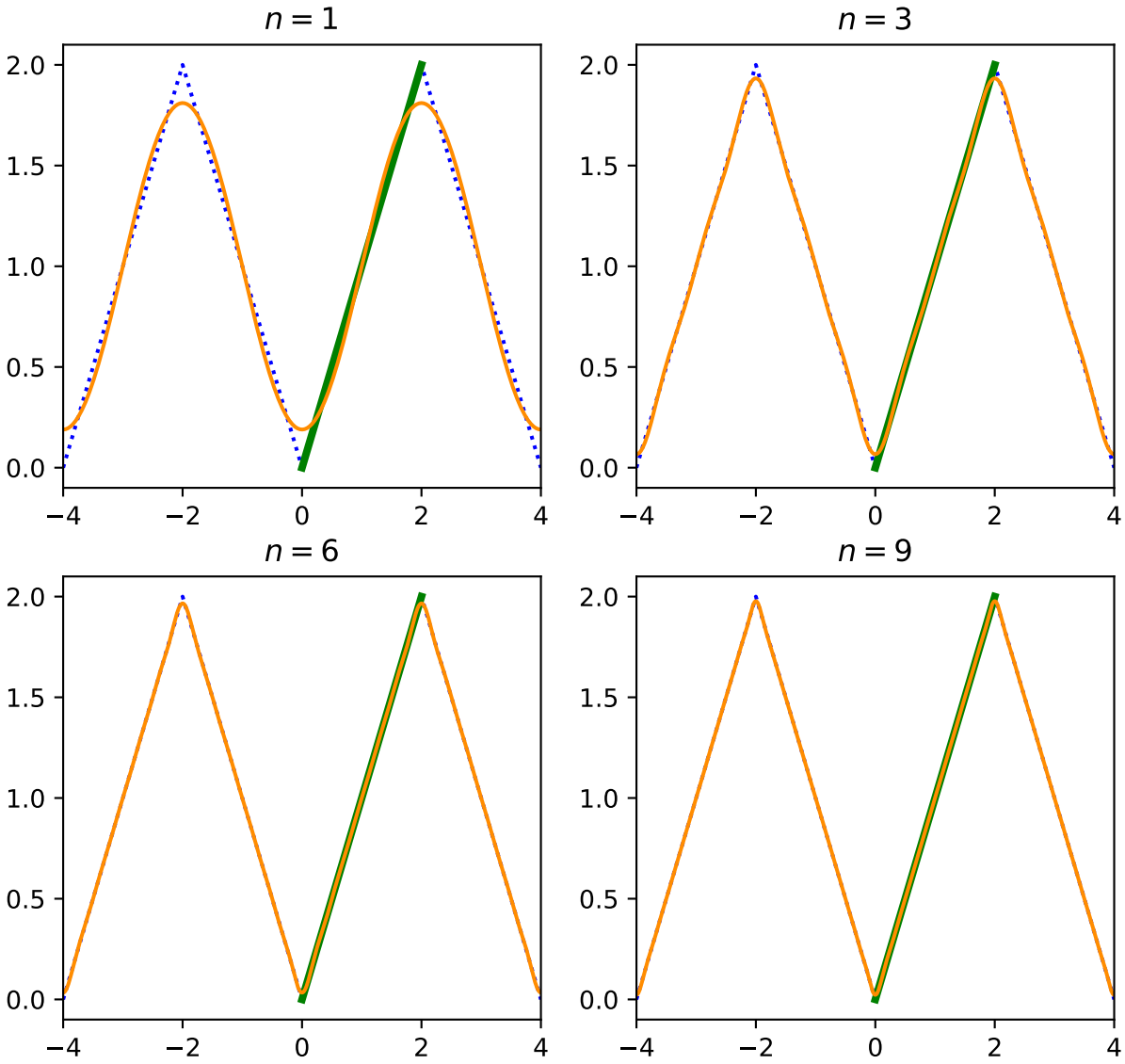
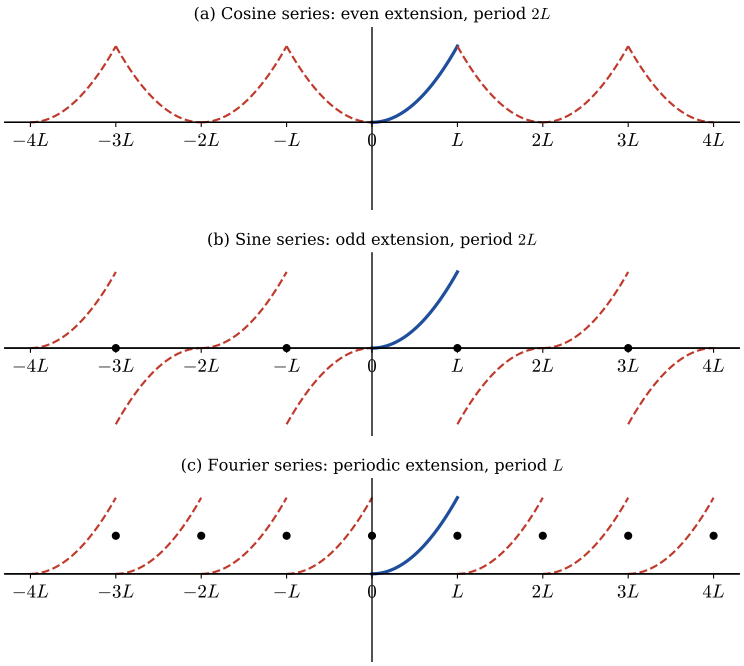


Figure 11.3: Cosine Series

- Half-Range Expansion
  - Throughout the preceding discussion, it was understood that a function f was defined on an interval with the origin as midpoint
  - However, in many instances, we are interested in representing a function that is defined only for $0 < x < L$ by a trigonometric series
- This can be done in many different ways by supplying an arbitrary definition of the function on the interval $-L < x < 0$. For brevity we consider the three most important cases:
  - Reflect the graph of the function about the y-axis onto $-L < x < 0$
  - Reflect the graph of the function through the origin onto $-L < x < 0$
  - Define f on $-L < x < 0$ by $f(x) = f(x + L)$



11.4 Complex Fourier Series

- Complex Fourier Series

  The **complex Fourier series** of functions f defined on an interval $(-p, p)$ is given by

  $$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{p}x}$$

  where

  $$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-\frac{in\pi}{p}x}\,dx$$

- Fundamental Frequency

  The Fourier series defines a periodic function and the fundamental period of that function is $T = 2p$. Since $p = T/2$,

  $$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega x + b_n \sin n\omega x)$$

  and

  $$\sum_{n=-\infty}^{\infty} c_n e^{in\omega x}$$

  where $\omega = 2\pi/T$ is called the **fundamental angular frequency**

- Frequency Spectrum

  If f is periodic and has fundamental period T, the plot of the points $(n\omega, |c_n|)$ is called the frequency spectrum of f

Example Expand $f(x) = e^{-x}$, $-\pi < x < \pi$ in a complex Fourier series and find the frequency spectrum

Answer 2 (end of chapter) · Solution

11.5 Sturm-Liouville Problem

- Self-Adjoint Form

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + [\lambda c(x) + d(x)]\, y = 0, \quad a(x) \neq 0$$

$$\Downarrow$$

$$\frac{d}{dx}\left[\exp\left(\int \frac{b(x)}{a(x)}dx\right)\frac{dy}{dx}\right] + \left\{\lambda\frac{c(x)}{a(x)}\exp\left(\int \frac{b(x)}{a(x)}dx\right) + \frac{d(x)}{a(x)}\exp\left(\int \frac{b(x)}{a(x)}dx\right)\right\} y = 0$$

$$\Downarrow$$

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + [\lambda p(x) + q(x)]\, y = 0$$

- Eigenvalues and Eigenfunctions

  An orthogonal set of functions can be generated by solving a two-point boundary-value problem involving a linear second-order differential equation containing a parameter λ

- The boundary-value problem

$$y'' + \lambda y = 0,\ y(0) = 0,\ y(L) = 0$$

  possessed nontrivial solutions only when the parameter λ took on the values

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \cdots$$

  called **eigenvalues**. The corresponding nontrivial solutions

$$y_n = c_n \sin\frac{n\pi x}{L}$$

  are called the **eigenfunctions**

- **Regular Sturm-Liouville Problem**

  Let p, q, r, and r' be real-valued functions continuous on an interval $[a, b]$, and let $r(x) > 0$ and $p(x) > 0$ for every x in the interval. Then

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + [\lambda p(x) + q(x)]\, y = 0$$

  subject to

$$\begin{aligned} A_1 y(a) + B_1 y'(a) &= 0 \\ A_2 y(b) + B_2 y'(b) &= 0 \end{aligned}$$

  is said to be a **regular Sturm-Liouville problem**

- Properties of the Regular Sturm-Liouville problem

  1. There are an infinite number of real eigenvalues that can be arranged in increasing order $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$ such that $\lambda_n \to \infty$ as $n \to \infty$
  2. For each eigenvalue λ_i, there is only one eigenfunction (except for nonzero constant multiples)
  3. Eigenfunctions corresponding to different eigenvalues are linearly independent
  4. The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on the interval $[a, b]$

- **Proof of 4**

$$\begin{aligned} \frac{d}{dx}\left[r(x)\frac{dy_m}{dx}\right] + (\lambda_m p(x) + q(x))\, y_m &= 0 \\ \frac{d}{dx}\left[r(x)\frac{dy_n}{dx}\right] + (\lambda_n p(x) + q(x))\, y_n &= 0 \\ \Downarrow & \\ (\lambda_m - \lambda_n)\, p(x) y_m y_n &= y_m \frac{d}{dx}\left[r(x)\frac{dy_n}{dx}\right] - y_n \frac{d}{dx}\left[r(x)\frac{dy_m}{dx}\right] \\ \Downarrow & \\ (\lambda_m - \lambda_n)\int_a^b p(x) y_m y_n\, dx &= y_m r(x)\frac{dy_n}{dx}\Big|_a^b - \int_a^b \frac{dy_m}{dx} r(x)\frac{dy_n}{dx}\, dx - y_n r(x)\frac{dy_m}{dx}\Big|_a^b + \int_a^b \frac{dy_n}{dx} r(x)\frac{dy_m}{dx} \\ &= r(b)\,[y_m(b)y_n'(b) - y_n(b)y_m'(b)] - r(a)\,[y_m(a)y_n'(a) - y_n(a)y_m'(a)] \\ &= 0 \\ \Downarrow & \\ \int_a^b p(x) y_m y_n\, dx &= 0, \quad \lambda_m \neq \lambda_n \end{aligned}$$

Example Solve the boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0$$

Solution

$$y = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$$

$$\Downarrow\ y(0) = 0,\ y(1) + y'(1) = 0$$

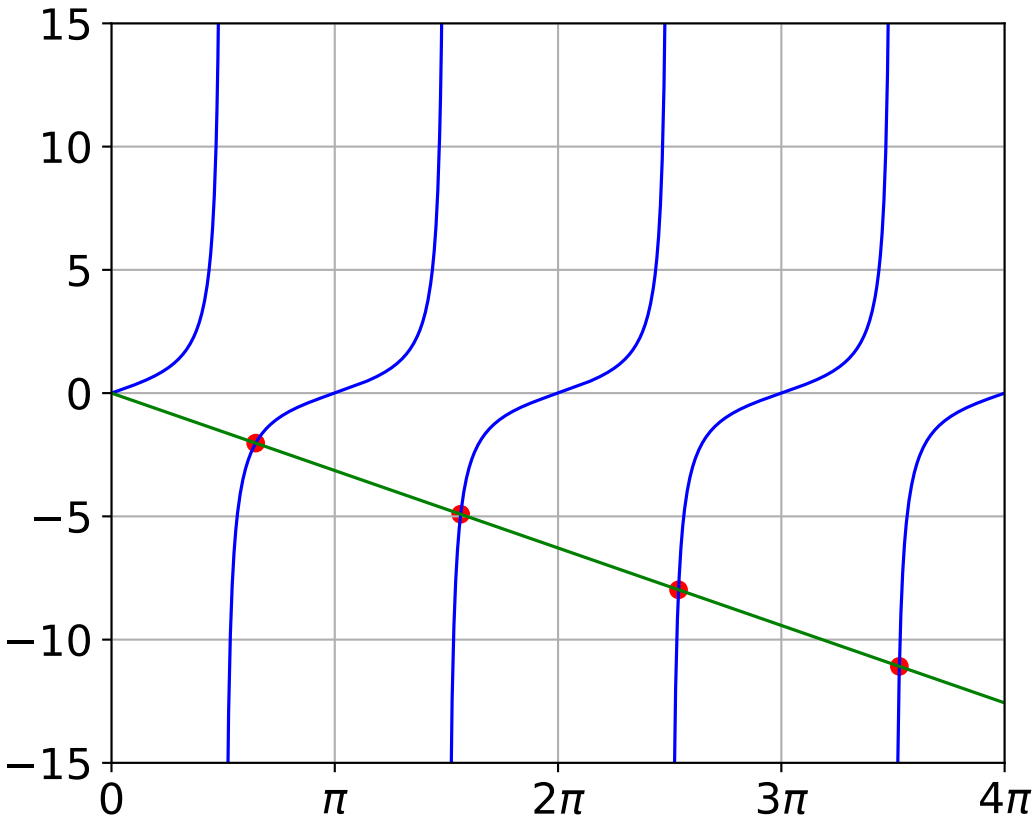
$$c_1 \cos 0 + c_2 \sin 0 = 0\ \rightarrow\ c_1 = 0$$
$$c_2 \sin\sqrt{\lambda} + c_2\sqrt{\lambda}\cos\sqrt{\lambda} = 0\ \rightarrow\ c_2 \neq 0,\ \tan\sqrt{\lambda} = -\sqrt{\lambda}$$

```
x = np.linspace(0, 4*np.pi, 1000)
y = np.tan(x)

threshold = 100
y[y > threshold] = np.inf
y[y <-threshold] = np.inf

def example_plot3(eigenvalues):
    plt.figure(figsize=(6, 5))
    plt.plot(x, y, linewidth=1.2, color="blue")
    plt.plot(x,-x, linewidth=1.2, color="green")
    plt.scatter(eigenvalues,-eigenvalues, color='red')
    plt.xticks([0, np.pi, 2.0*np.pi, 3.0*np.pi, 4.0*np.pi],
        ['0', r'$\pi$', r'$2\pi$', r'$3\pi$', r'$4\pi$'],
        fontsize=16)
    plt.yticks(fontsize=16)
    plt.xlim(0, 4.0*np.pi)
    plt.ylim(-15, 15)
    plt.grid()
    plt.show()
```

```
from scipy import optimize

sqrt_eigenvalues = np.zeros(4)
for i in range(4):
    sqrt_eigenvalues[i] = np.round(optimize.brentq(
      lambda x : np.tan(x) +x,
      (i +0.5001) *np.pi,
      (i +0.9999) *np.pi), 4)

print(f'{sqrt_eigenvalues =}')
```

```
sqrt_eigenvalues =array([ 2.0288,  4.9132,  7.9787, 11.0855])
```

```
example_plot3(sqrt_eigenvalues)
```




```
def example_plot3_(x, n_eig, sqrt_eig, eigfunc):

    plt.figure(figsize=(6, 5))
    for n in range(n_eig):
        plt.plot(x, eigfunc(x, sqrt_eig[n]), label='$X_%d(x)$' % (n +1))

    plt.xlim(0, 1)
    plt.ylim(-1.2, 1.2)
    plt.grid(ls=':')
    plt.legend(loc='upper left', bbox_to_anchor=(1, 1.03))
    plt.xlabel('$x$')
    plt.ylabel('$X_n(x)$')
    plt.title(r'$X_n(x)=\sin\sqrt{\lambda_n} x$')
    plt.show()

def eigfunc(x, sqrt_eig):
    return np.sin(sqrt_eig *x)

x = np.linspace(0, 1, 150)
example_plot3_(x, len(sqrt_eigenvalues), sqrt_eigenvalues, eigfunc)
```

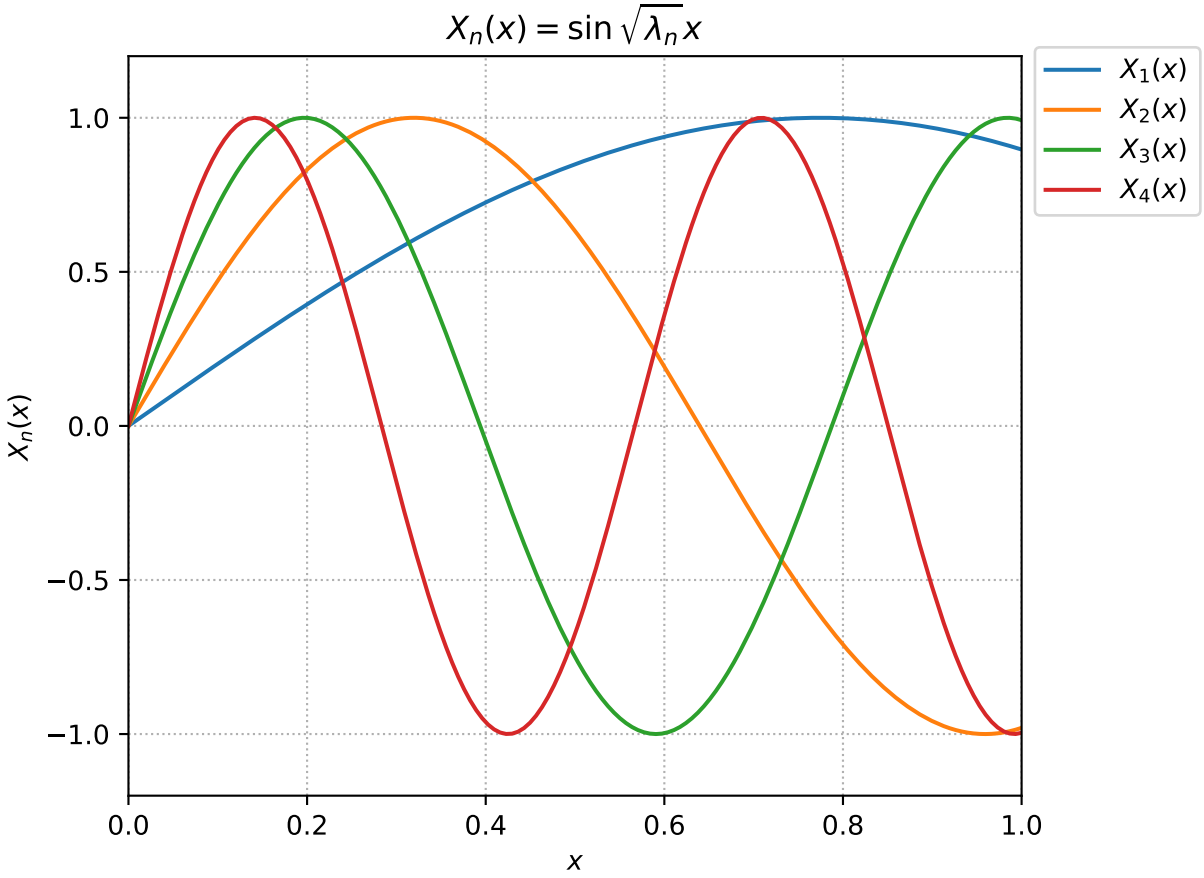


Figure 11.4: Eigenfunctions

- **Singular Sturm-Liouville Problem**

  There are several important conditions under which we seek nontrivial solutions

  1. $r(a) = 0$ and $A_2y(b) + B_2y'(b) = 0$
  2. $A_1y(a) + B_1y'(a) = 0$ and $r(b) = 0$
  3. $r(a) = r(b) = 0$ and no boundary condition is specified at either $x = a$ or $x = b$
  4. $r(a) = r(b)$ and $y(a) = y(b),\ y'(a) = y'(b)$

$$\begin{aligned}(\lambda_m - \lambda_n)\int_a^b p(x)y_m y_n\,dx &= r(b)\left[y_m(b)y_n'(b) - y_n(b)y_m'(b)\right] \\ &\quad - r(a)\left[y_m(a)y_n'(a) - y_n(a)y_m'(a)\right] = 0 \\ &\Downarrow \\ \int_a^b p(x)y_m y_n\,dx &= 0,\ \ \lambda_m \neq \lambda_n\end{aligned}$$

- Parametric Bessel Equation:

$$x^2y'' + xy' + (\alpha^2x^2 - n^2)y = 0,\ \ n = 0, 1, 2, \cdots$$

$$\frac{d}{dx}\left[xy'\right] + \left(\alpha^2 x - \frac{n^2}{x}\right) y = 0$$

- Legendre Equation:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \quad n = 0, 1, 2, \cdots$$

$$\frac{d}{dx}\left[(1-x^2)y'\right] + n(n+1)y = 0$$

- Consider $y'' + \lambda y = 0$ subject to the periodic boundary condition $y(-L) = y(L),\ y'(-L) = y'(L)$. Show that the eigenfunctions are

$$\left\{1, \cos\frac{\pi}{L}x, \cos\frac{2\pi}{L}x, \cdots, \sin\frac{\pi}{L}x, \sin\frac{2\pi}{L}x, \cdots\right\}$$

  This set, which is orthogonal on $[-L, L]$, is the basis for the Fourier Series

Answer 3 (end of chapter) · Solution

11.6 Bessel and Legendre Series

11.6.1 Fourier-Bessel Series

- The parametric Bessel differential equation is

$$\frac{d}{dx}\left[xy'\right] + \left(\alpha^2 x - \frac{n^2}{x}\right) y = 0$$

  in which $r(x) = x,\ p(x) = x,\ q(x) = -n^2/x$, and $\lambda = \alpha^2$. The general solution of this equation is

$$y = c_1 J_n(\alpha x) + c_2 Y_n(\alpha x)$$

- Now $r(0) = 0$, and of two solutions $J_n(\alpha x)$ and $Y_n(\alpha x)$, only $J_n(\alpha x)$ is bounded at $x = 0$
- The eigenvalues $\lambda_i = \alpha_i^2,\ i = 1, 2, 3, \cdots$, are defined by means of a boundary condition at $x = b$:

$$A_2 J_n(\alpha b) + B_2 \alpha J_n'(\alpha b) = 0$$

- For any choice of A_2 and B_2, not both zero, it is known that the above boundary condition gives an infinite number of roots $x_i = \alpha_i b$. The eigenvalues are then $\lambda_i = \alpha_i^2 = (x_i/b)^2$

- Differential Recurrence Relations

$$
\begin{aligned}
xJ_n'(x) &= xJ_{n-1}(x) - nJ_n(x) \\
xJ_n'(x) &= nJ_n(x) - xJ_{n+1}(x) \\
&\Downarrow \\
\frac{d}{dx}\left[x^n J_n(x)\right] &= x^n J_{n-1}(x) \\
\frac{d}{dx}\left[x^{-n} J_n(x)\right] &= -x^{-n} J_{n+1}(x)
\end{aligned}
$$

- Square Norm

$$
\begin{aligned}
&\frac{d}{dx}\left[xy'\right] + \left(\alpha^2 x - \frac{n^2}{x}\right) y = 0 \\
&\Downarrow \ \times 2xy' \\
&\frac{d}{dx}\left[xy'\right]^2 + \left(\alpha^2 x^2 - n^2\right) \frac{d}{dx}[y]^2 = 0 \\
&\Downarrow \text{ integrating by parts on } [0, b] \\
&\left[[xy']^2 + (\alpha^2 x^2 - n^2) y^2\right]_0^b = 2\alpha^2 \int_0^b xy^2 \, dx \\
&\Downarrow \ y = J_n(\alpha x), \ y' = \alpha J_n'(\alpha x), \ J_n(0) = 0 \text{ for } n > 0 \\
&2\alpha^2 \int_0^b x J_n^2(\alpha x) \, dx = \alpha^2 b^2 \left[J_n'(\alpha b)\right]^2 + \left(\alpha^2 b^2 - n^2\right) \left[J_n(\alpha b)\right]^2
\end{aligned}
$$

- **Consider three cases of the boundary condition**

$$
A_2 J_n(\alpha b) + B_2 \alpha J_n'(\alpha b) = 0
$$

- **Case I :** $A_2 = 1, \ B_2 = 0 \Rightarrow J_n(\alpha b) = 0 \ \Rightarrow \ \lambda_i = \alpha_i^2 = (x_i/b)^2 > 0$

$$
\begin{aligned}
&2\alpha_i^2 \int_0^b x J_n^2(\alpha_i x) \, dx = \alpha_i^2 b^2 \left[J_n'(\alpha_i b)\right]^2 + \left(\alpha_i^2 b^2 - n^2\right) \left[J_n(\alpha_i b)\right]^2 \\
&\Downarrow \ xJ_n'(x) = n \underbrace{J_n(x)}_{=0} - xJ_{n+1}(x) \\
&\|J_n(\alpha_i x)\|^2 = \int_0^b x J_n^2(\alpha_i x) \, dx \ = \frac{b^2}{2} J_{n+1}^2(\alpha_i b)
\end{aligned}
$$

```
from scipy.special import jv, jn_zeros
from mpl_toolkits.axisartist.axislines import SubplotZero

def make_plot_form():

    fig = plt.figure(figsize=(7, 8))
    ax1 = SubplotZero(fig, 211)
    ax2 = SubplotZero(fig, 212)
```

```
    fig.add_subplot(ax1)
    ax1.axis["left", "right", "bottom", "top"].set_visible(False)
    ax1.axis["xzero", "yzero"].set_visible(True)
    ax1.axis["xzero", "yzero"].set_axisline_style("-|>")

    ax1.set_xticks([2, 4, 6, 8, 10])
    ax1.text(12, -0.03, r'$\alpha$', fontsize=14)

    fig.add_subplot(ax2)
    ax2.axis["left", "right", "bottom", "top"].set_visible(False)
    ax2.axis["xzero", "yzero"].set_visible(True)
    ax2.axis["xzero", "yzero"].set_axisline_style("-|>")

    ax2.set_xticks([0.2, 0.4, 0.6, 0.8, 1.0])
    ax2.text(1.1, -0.03, r'$x$', fontsize=14)

    r1 = np.linspace(0, 11, 100)
    r2 = np.linspace(0, 1, 100)

    return ax1, ax2, r1, r2

def make_plot(case_no, ax1, ax2, r1, r2, mm, bc_txt, eigfn_txt):

    alpha_0m = eval_alpha(mm)

    y1 = eval_bc(r1)
    ax1.plot(r1, y1, label=bc_txt)
    ax1.plot(alpha_0m, np.zeros(mm), 'ro')

    for m in range(mm):
        ax1.text(alpha_0m[m] -0.2, 0.1, rf'$\alpha_{m +1}$',
            fontsize=12)
    ax1.legend(bbox_to_anchor=(1, 1), loc="upper left")

    x_zeros = eval_x_zeros(mm)
    for m in range(mm):
        y2 = eval_eigfn(alpha_0m[m]*r2)
        ax2.plot(r2, y2, label=eigfn_txt +rf'$(\alpha_{m +1}x)$')
        if case_no == 1:
          ax2.plot(x_zeros[:m +1] /alpha_0m[m], np.zeros(m +1), 'x')
        elif case_no == 2:
          ax2.plot(x_zeros[:m] /alpha_0m[m], np.zeros(m), 'x')
        elif case_no == 3:
          if m != 0:
            ax2.plot(x_zeros[:m] /alpha_0m[m], np.zeros(m), 'x')

    ax2.legend(bbox_to_anchor=(1, 1), loc="upper left")
```

```
def eval_eigfn(x):
    return jv(0, x)
```

```
def eval_bc(alpha):
    return eval_eigfn(alpha)

def eval_alpha(mm):
    return jn_zeros(0, mm)

def eval_x_zeros(mm):
    return eval_alpha(mm)

def example_plot4():

    ax1, ax2, r1, r2 = make_plot_form()

    ax1.set_ylim(-0.5, 1.1)
    ax1.set_yticks([-0.5, 0.5, 1.0])

    ax2.set_ylim(-0.5, 1.1)
    ax2.set_yticks([-0.5, 0.5, 1.0])

    mm = 3

    make_plot(1, ax1, ax2, r1, r2, mm, r'$J_0(\alpha)$', '$J_0$')

    plt.show()

example_plot4()
```

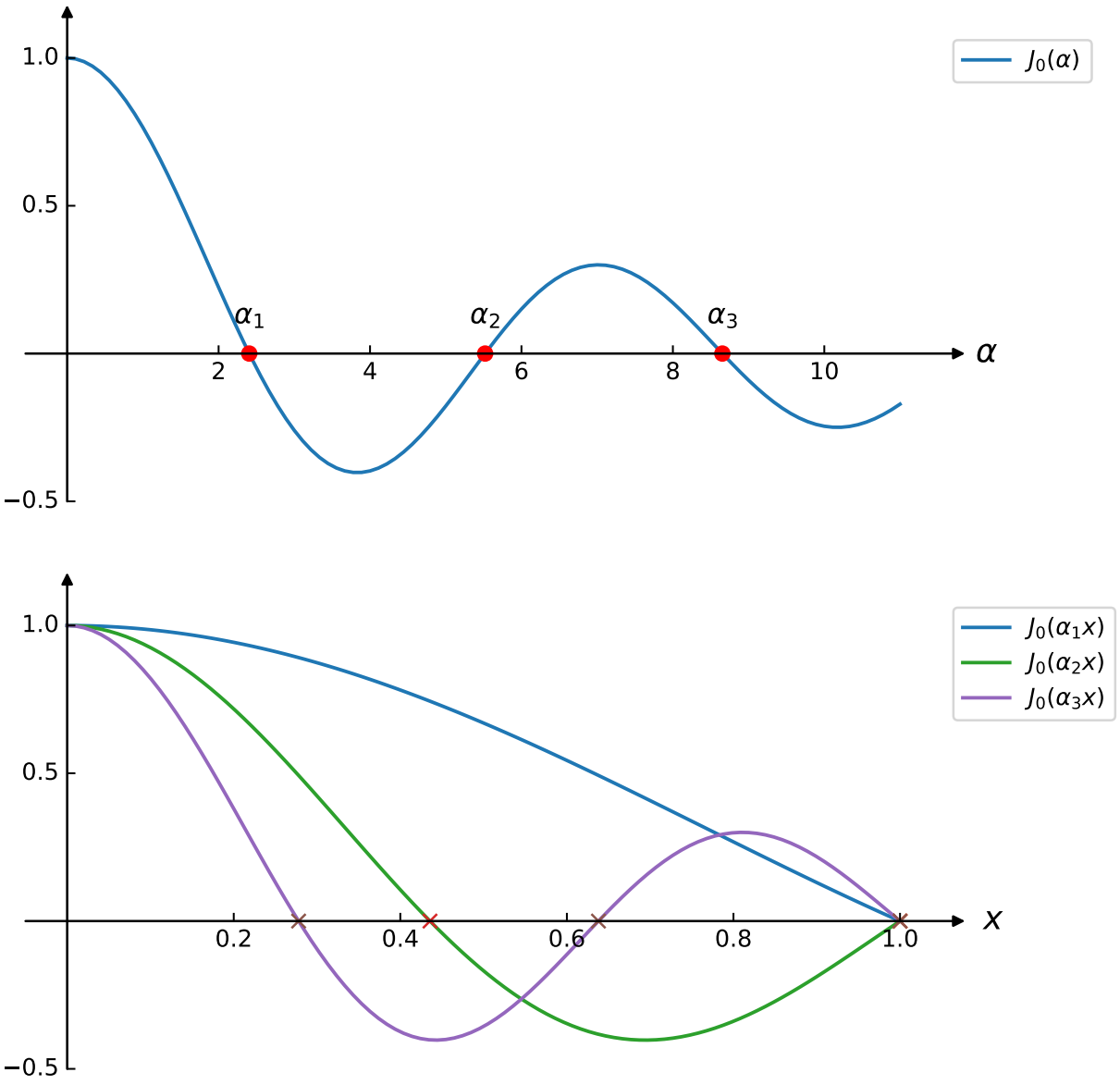


Figure 11.5: Case I: $n = 0, A_2 = 1, B_2 = 0, b = 1$

```
def eval_eigfn(x):
    return jv(1, x)

def eval_bc(alpha):
    return eval_eigfn(alpha)

def eval_alpha(mm):
    return jn_zeros(1, mm)

def eval_x_zeros(mm):
    return eval_alpha(mm)

def example_plot4_():

    ax1, ax2, r1, r2 = make_plot_form()
```

```
    ax1.set_ylim(-0.5, 1.1)
    ax1.set_yticks([-0.5, 0.5, 1.0])

    ax2.set_ylim(-0.5, 1.1)
    ax2.set_yticks([-0.5, 0.5, 1.0])

    mm = 3

    make_plot(1, ax1, ax2, r1, r2, mm, r'$J_1(\alpha)$', '$J_1$')

    plt.show()

example_plot4_()
```

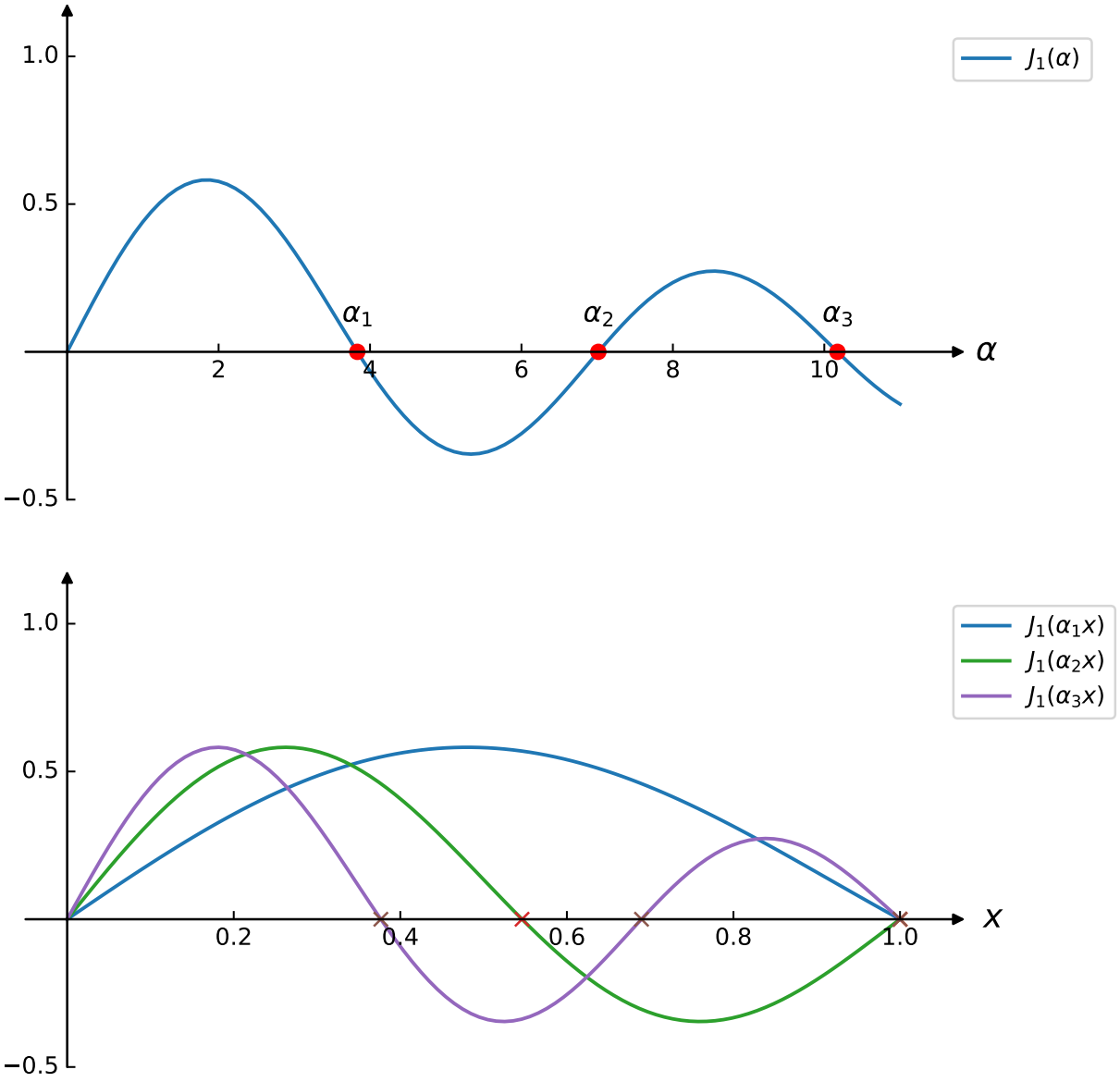


Figure 11.6: Case I: $n = 1, A_2 = 1, B_2 = 0, b = 1$

- **Case II :** $(A_2 = h,\, B_2 = b \Rightarrow h \geq 0)$

$$hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0 \;\Rightarrow\; \lambda_i = \alpha_i^2 = \left(\frac{x_i}{b}\right)^2 \quad \text{except for} \quad h = 0 \quad \text{and} \quad n = 0$$

$$2\alpha_i^2 \int_0^b x J_n^2(\alpha_i x)\, dx = \alpha_i^2 b^2 \left[J_n'(\alpha_i b)\right]^2 + (\alpha_i^2 b^2 - n^2)\left[J_n(\alpha_i b)\right]^2$$

$$\Downarrow\; \alpha_i b J_n'(\alpha_i b) = -h J_n(\alpha_i b)$$

$$\|J_n(\alpha_i x)\|^2 = \int_0^b x J_n^2(\alpha_i x)\, dx = \frac{\alpha_i^2 b^2 - n^2 + h^2}{2\alpha_i^2} J_n^2(\alpha_i b)$$

```
from scipy.special import j0, j1
from scipy.optimize import fsolve

def eval_eigfn(x):
    return j0(x)

def eval_bc(alpha):
    return j0(alpha) -alpha *j1(alpha)

def eval_alpha(mm):
    alpha_0m = np.zeros(mm)
    for i in range(mm):
      alpha_0m[i] = fsolve(eval_bc, 1 +3*i)[0]

    return alpha_0m

def eval_x_zeros(mm):
    return jn_zeros(0, mm -1)

def example_plot4__():

    ax1, ax2, r1, r2 = make_plot_form()

    ax1.set_ylim(-4, 4)
    ax1.set_yticks([-3, -2, -1, 1, 2, 3])

    ax2.set_ylim(-0.5, 1.2)
    ax2.set_yticks([-0.5, 0.5, 1.0])

    mm = 4

    make_plot(2, ax1, ax2, r1, r2, mm,
        r"$J_0(\alpha) +\alpha J_0'(\alpha)$", '$J_0$')

    plt.show()
```

```
example_plot4__()
```

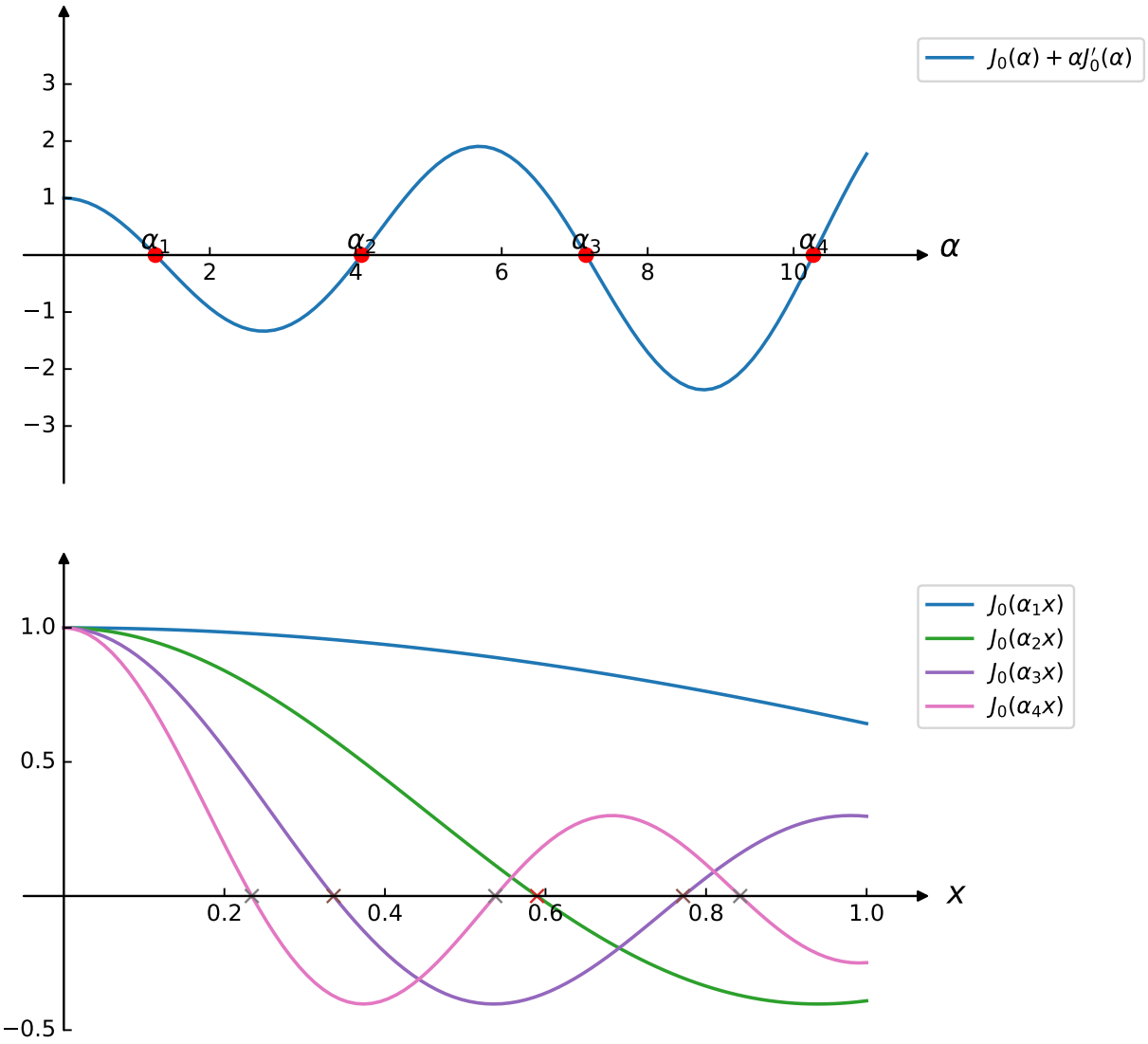


Figure 11.7: Case II: $n = 0, h = 1, b = 1,\ J_0(\alpha) + \alpha J_0'(\alpha) = 0$

- **Case III :** ($h = 0$ and $n = 0$) : $\quad J_0'(\alpha b) = 0$

  It is the only situation for which $\lambda = 0$ is an eigenvalue

$$\alpha_1 = 0 \to \|J_0(\alpha_1 x)\|^2 = \int_0^b x\,dx = \frac{b^2}{2}$$

$$2\alpha_i^2 \int_0^b x J_0^2(\alpha_i x)\,dx = \alpha_i^2 b^2 \left[J_0'(\alpha_i b)\right]^2 + (\alpha_i^2 b^2 - 0^2)\left[J_0(\alpha_i b)\right]^2$$

$$\Downarrow$$

$$\alpha_i,\ i = 2, 3, 4, \cdots \to \|J_0(\alpha_i x)\|^2 = \int_0^b x J_0^2(\alpha_i x)\,dx = \frac{b^2}{2} J_0^2(\alpha_i b)$$

```
from scipy.special import jnp_zeros

def eval_eigfn(x):
    return j0(x)

def eval_bc(alpha):
    return -j1(alpha)

def eval_alpha(mm):
    return np.append([0], jnp_zeros(0, mm -1))

def eval_x_zeros(mm):
    return jn_zeros(0, mm -1)

def example_plot4___():

    ax1, ax2, r1, r2 = make_plot_form()

    ax1.set_ylim(-0.8, 0.5)
    ax1.set_yticks([-0.5, 0.5])

    ax2.set_ylim(-0.5, 1.2)
    ax2.set_yticks([-0.5, 0.5, 1.0])

    mm = 4

    make_plot(3, ax1, ax2, r1, r2, mm, r"$J_0'(\alpha)$", '$J_0$')

    plt.show()

example_plot4___()
```

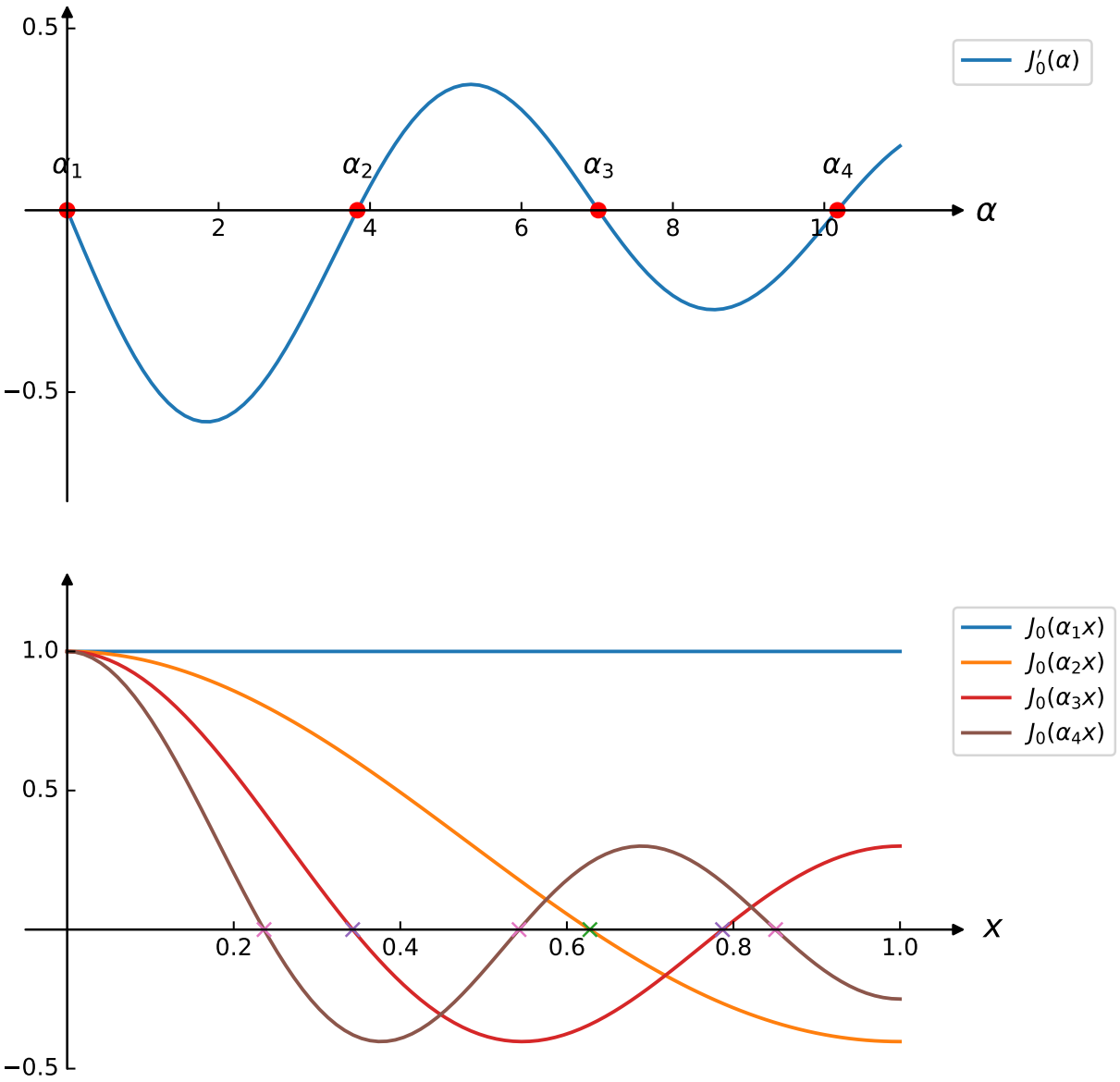


Figure 11.8: Case III: $n = 0, h = 0, b = 1,\ J_0'(\alpha) = 0$

- The set $\{J_n(\alpha_i x),\ i = 1, 2, 3, \cdots\}$ is orthogonal with respect to the weight function $p(x) = x$ on $[0, b]$

$$\int_0^b x\, J_n(\alpha_i x)\, J_n(\alpha_j x)\, dx = 0,\ i \neq j$$

- The Fourier-Bessel series of a function f defined on the interval $(0, b)$ is given by

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$

where

$$c_i = \frac{\displaystyle\int_0^b x\, f(x)\, J_n(\alpha_i x)\, dx}{\displaystyle\int_0^b x\, J_n^2(\alpha_i x)\, dx} = \frac{1}{\|J_n(\alpha_i x)\|^2} \int_0^b x\, f(x)\, J_n(\alpha_i x)\, dx$$

11.6.2 Fourier-Legendre Series

- Legendre's differential equation is

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

or

$$\frac{d}{dx}\left[(1 - x^2)y'\right] + n(n+1)y = 0$$

in which $r(x) = 1 - x^2$, $p(x) = 1$, $\lambda = n(n+1)$, and $q(x) = 0$

- The Legendre polynomials $P_n(x)$ are the only solutions that are bounded on the interval [-1, 1]

```
from scipy.special import eval_legendre

fig = plt.figure(figsize=(5, 5))
ax = SubplotZero(fig, 111)

fig.add_subplot(ax)
ax.axis["left", "right", "bottom", "top"].set_visible(False)
ax.axis["xzero", "yzero"].set_visible(True)
ax.axis["xzero", "yzero"].set_axisline_style("-|>")

ax.set_xticks([-1, -0.5, 0.5, 1], ['-1', '', '', '1'])
ax.set_yticks([-1, -0.5, 0.5, 1], ['-1', '', '', '1'])
ax.grid(ls=':')
ax.text(-0.1, 1.3, r'$P_n(x)$', fontsize=14)
ax.text(1.3, -0.05, r'$x$', fontsize=14)
ax.set_xlim(-1.2, 1.2)
ax.set_ylim(-1.2, 1.2)

x = np.linspace(-1, 1, 100)
for n in range(5):
    y = eval_legendre(n, x)
    ax.plot(x, y, label=f'$P_{n}(x)$')
```

```
ax.legend(loc='lower left', bbox_to_anchor=(1, 0))

plt.show()
```

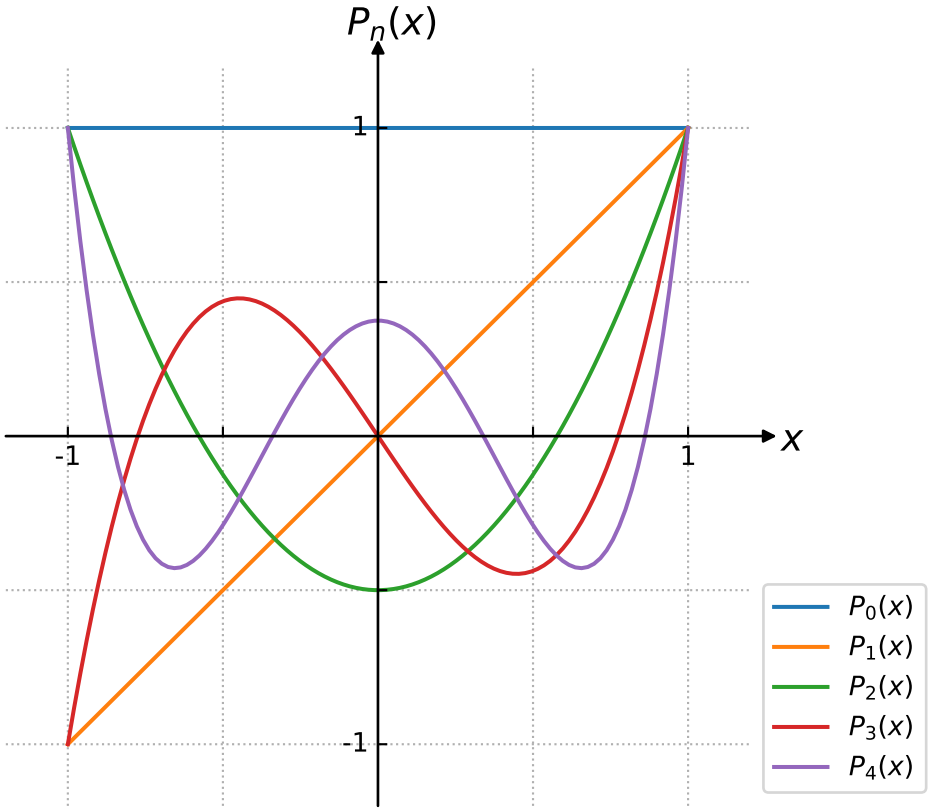


Figure 11.9: Legendre Polynomials

- The set $\{P_n(x),\ n = 0, 1, 2, \cdots\}$ is orthogonal with respect to the weight function $p(x) = 1$ on [-1, 1]

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0,\ m \neq n$$

- The Fourier-Legendre series of a function f defined on the interval (-1, 1) is given by

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

where

$$c_n = \frac{\int_{-1}^{1} f(x) P_n(x)\, dx}{\int_{-1}^{1} P_n^2(x)\, dx} = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx$$

11.6.3 Other Important Equations

- Hermite

$$y'' - 2xy' + 2ny = 0$$

- Laguerre

$$xy'' + (1-x)y' + ny = 0$$

- Chebyshev

$$(1-x^2)y'' - xy' + n^2 y = 0$$

- Hypergeometric

$$x(1-x)y'' + [c - (a+b+1)x]\,y' - aby = 0$$

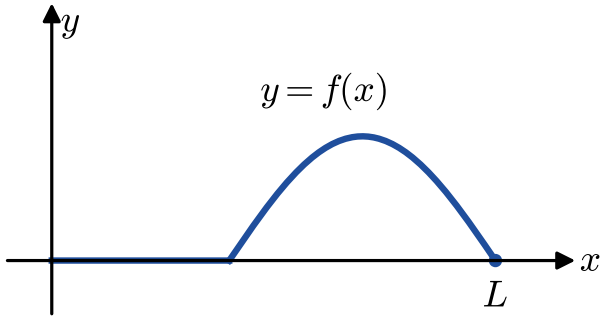
Summary

Topic	Formula
Orthogonal expansion	$f = \sum c_n \phi_n, \quad c_n = \dfrac{\int_a^b f\phi_n\,dx}{\int_a^b \phi_n^2\,dx}$
Fourier series on $(-p,p)$	$f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$, $a_n = \dfrac{1}{p}\displaystyle\int_{-p}^{p} f\cos\frac{n\pi}{p}x\,dx$, $b_n = \dfrac{1}{p}\displaystyle\int_{-p}^{p} f\sin\frac{n\pi}{p}x\,dx$
Convergence	at a jump the series converges to $\dfrac{f(x^+)+f(x^-)}{2}$
Half-range series on $(0,p)$	cosine: $a_n = \dfrac{2}{p}\displaystyle\int_0^p f\cos\frac{n\pi}{p}x\,dx$; sine: $b_n = \dfrac{2}{p}\displaystyle\int_0^p f\sin\frac{n\pi}{p}x\,dx$
Complex Fourier series	$f(x) = \displaystyle\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$, $c_n = \dfrac{1}{2p}\displaystyle\int_{-p}^{p} f(x)e^{-in\pi x/p}\,dx$
Sturm-Liouville	$\dfrac{d}{dx}\left[r(x)y'\right] + \left[\lambda p(x) + q(x)\right]y = 0$: eigenfunctions are orthogonal with weight $p(x)$

Topic	Formula
Fourier-Bessel	$f(x) = \sum c_i J_n(\alpha_i x),$ $c_i = \dfrac{\int_0^b x\, f(x) J_n(\alpha_i x)\, dx}{\int_0^b x\, J_n^2(\alpha_i x)\, dx}$
Fourier-Legendre	$f(x) = \sum c_n P_n(x),$ $c_n = \dfrac{2n+1}{2} \displaystyle\int_{-1}^{1} f(x) P_n(x)\, dx$

Worked Exercises

1. Suppose the function $y = f(x)$, $0 < x < L$, given in figure is expanded in a cosine series, in a sine series, and in a Fourier series. Sketch the periodic extension to which each series converges



$$f(x) = \begin{cases} 0 & 0 \le x < \dfrac{L}{2} \\ -\sin \dfrac{2\pi x}{L} & \dfrac{L}{2} \le x \le L \end{cases}$$

Solution

1. Cosine Series (Even Extension)

- A cosine series is the Fourier cosine series, which assumes that the function is extended evenly about $x = 0$
- The extended function is even, meaning $f(-x) = f(x)$
- The domain is extended to $[-L, L]$ and periodically repeated with period $2L$

So the full even extension on $[-L, L]$:

$$f_{\text{even}}(x) = \begin{cases} \sin\left(\frac{2\pi x}{L}\right) & -L \le x < -\frac{L}{2} \\ 0 & -\frac{L}{2} \le x < \frac{L}{2} \\ -\sin\left(\frac{2\pi x}{L}\right) & \frac{L}{2} \le x \le L \end{cases}$$

Then repeat with period $2L$

2. Sine Series (Odd Extension)

- A sine series extends the function oddly about $x = 0$
- The extended function is odd, so $f(-x) = -f(x)$
- Defined on $[-L, L]$ and repeated every $2L$

So the odd extension is:

$$f_{\text{odd}}(x) = \begin{cases} \sin\left(\frac{2\pi x}{L}\right) & -L \le x < -\frac{L}{2} \\ 0 & -\frac{L}{2} \le x < \frac{L}{2} \\ -\sin\left(\frac{2\pi x}{L}\right) & \frac{L}{2} \le x \le L \end{cases}$$

Then repeated every $2L$

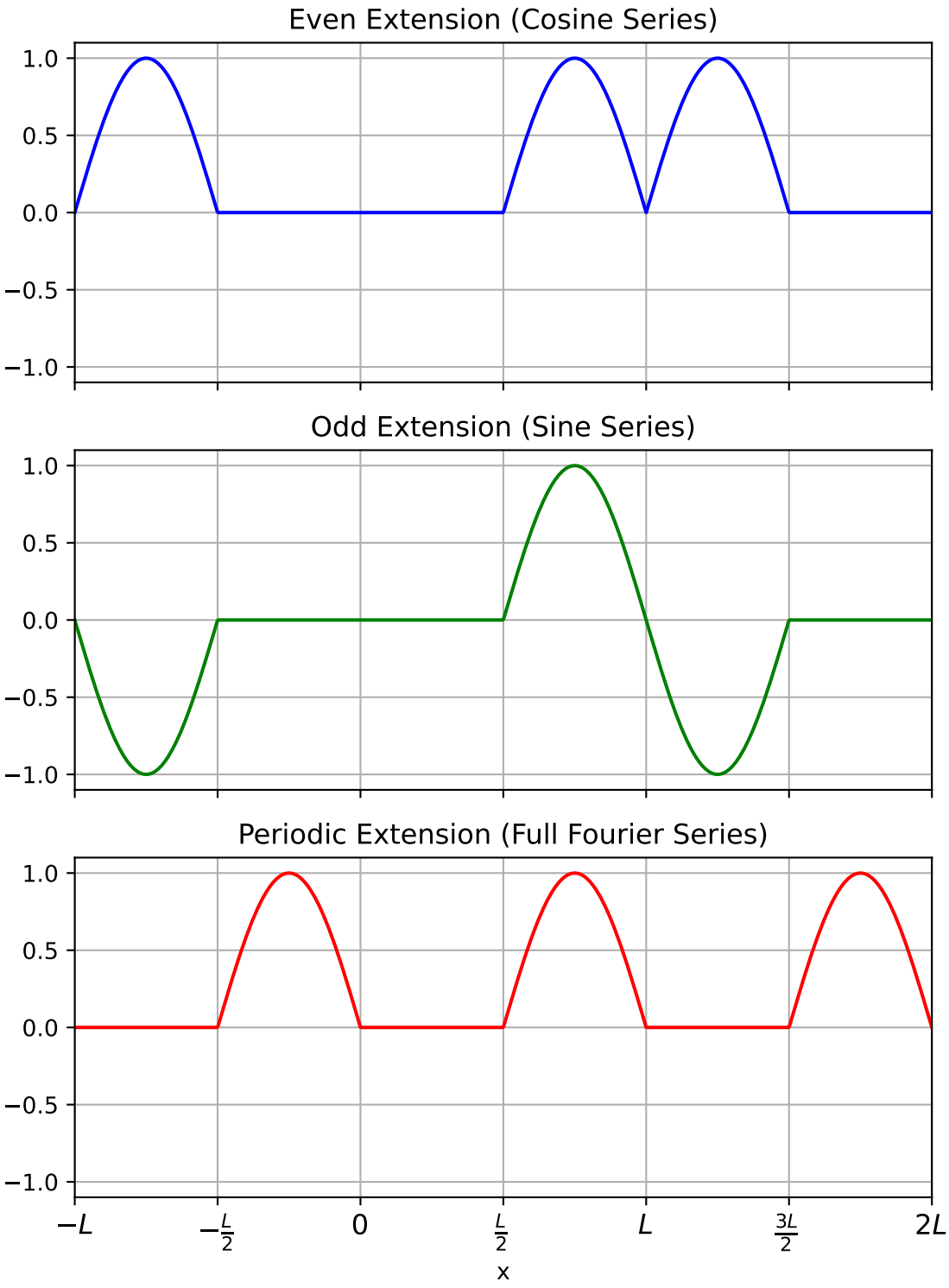
3. Full Fourier Series (Periodic Extension)

- The full Fourier series uses both sine and cosine terms
- The function is extended periodically with period L

This shape repeats every L units

```
import numpy as np
import matplotlib.pyplot as plt

# Define function f(x) on [0, L]
L = 2 * np.pi  # for easy visualization, let L = 2π
x = np.linspace(-2*L, 2*L, 2000)

# Define the base function f(x) on [0, L]
def f_base(x):
    x_mod = x % L
    return np.where(
        x_mod < L/2,
        0,
        -np.sin(2 * np.pi * x_mod / L)
    )

# Even extension (cosine series)

def f_even(x):
    # Reflect across x = 0 and repeat with period 2L
    x_mod = np.mod(x, 2*L)
    reflected = np.where(x_mod < L, x_mod, 2*L - x_mod)
    return f_base(reflected)

# Odd extension (sine series)

def f_odd(x):
    # Reflect and invert across x = 0, repeat with period 2L
    x_mod = np.mod(x, 2*L)
    reflected = np.where(x_mod < L, x_mod, 2*L - x_mod)
    sign = np.where(x_mod < L, 1, -1)
    return sign * f_base(reflected)

# Full periodic extension (Fourier series)
```

```
def f_periodic(x):
    return f_base(x)

# Redefine x-axis from -L to 2L
x_zoom = np.linspace(-L, 2*L, 1000)

xticks = [-L, -L/2, 0, L/2, L, 3*L/2, 2*L]
xtick_labels = [r"$-L$", r"$-\frac{L}{2}$",
  r"$0$", r"$\frac{L}{2}$", r"$L$", r"$\frac{3L}{2}$", r"$2L$"]

# Plotting with labeled ticks
fig, axs = plt.subplots(3, 1, figsize=(6, 8), sharex=True)

axs[0].plot(x_zoom, f_even(x_zoom), color='blue')
axs[0].set_title('Even Extension (Cosine Series)')
axs[0].grid(True)
axs[0].set_xlim([-L, 2*L])
axs[0].set_ylim([-1.1, 1.1])

axs[1].plot(x_zoom, f_odd(x_zoom), color='green')
axs[1].set_title('Odd Extension (Sine Series)')
axs[1].grid(True)
axs[1].set_xlim([-L, 2*L])
axs[1].set_ylim([-1.1, 1.1])

axs[2].plot(x_zoom, f_periodic(x_zoom), color='red')
axs[2].set_title('Periodic Extension (Full Fourier Series)')
axs[2].grid(True)
axs[2].set_xlim([-L, 2*L])
axs[2].set_ylim([-1.1, 1.1])

# Set common x-axis ticks and labels
axs[2].set_xticks(xticks)
axs[2].set_xticklabels(xtick_labels, fontsize=12)
plt.xlabel('x')
plt.tight_layout()
plt.show()
```



2. Find the half-range cosine and sine expansions of the given function:

$$f(x) = x(2 - x), \quad 0 < x < 2$$

Solution

1. Half-Range Cosine Series (Even Extension)

We write $f(x)$ on $0 < x < 2$ as a cosine series:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where $L = 2$, and the coefficients are:

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{2}\int_0^2 x(2-x)\,dx$$

$$a_n = \frac{1}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx = \frac{1}{2}\int_0^2 x(2-x)\cos\left(\frac{n\pi x}{2}\right)\,dx$$

- Compute a_0:

$$a_0 = \frac{1}{2}\int_0^2 (2x - x^2)\,dx = \frac{1}{2}\left[x^2 - \frac{x^3}{3}\right]_0^2 = \frac{1}{2}\left(4 - \frac{8}{3}\right) = \frac{2}{3}$$

- Compute a_n:

We split the integral:

$$\begin{aligned} a_n &= \frac{1}{2}\left[2\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx - \int_0^2 x^2\cos\left(\frac{n\pi x}{2}\right)dx\right] \\ &= \frac{-4\left[(-1)^n + 1\right]}{\pi^2 n^2} \end{aligned}$$

Therefore, the final simplified form is:

$$a_n = \begin{cases} \dfrac{-8}{\pi^2 n^2}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases} \quad \text{for } n \geq 1$$

2. Half-Range Sine Series (Odd Extension)

We write $f(x)$ on $0 < x < 2$ as a sine series:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

where $L = 2$, and the coefficients are:

$$b_n = \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx = \frac{1}{2}\int_0^2 x(2-x)\sin\left(\frac{n\pi x}{2}\right)\,dx$$

The exact expression for the sine series coefficients b_n is:

$$b_n = \begin{cases} \dfrac{-4\pi n\sin(n\pi) - 8\cos(n\pi) + 8}{\pi^3 n^3}, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

So the final simplified form is:

$$b_n = \begin{cases} \dfrac{16}{\pi^3 n^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases} \quad \text{for } n \geq 1$$

3. (a) Find the eigenvalues and eigenfunctions of the boundary value problem

$$y'' + 2y' + (\lambda + 1)y = 0, \;\; y(0) = 0, \; y(1) = 0$$

(b) Put the differential equation in self-adjoint form, (c) Give an orthogonality relation

Solution (a)

Step 1: Solve the ODE

Given:

$$y'' + 2y' + (\lambda + 1)y = 0$$

Let's reduce it to a homogeneous constant-coefficient equation.

The characteristic equation is:

$$p^2 + 2p + (\lambda + 1) = 0 \;\Rightarrow\; p = -1 \pm \sqrt{1 - \lambda}$$

We'll split into cases based on the sign of $1 - \lambda$:

Step 2.1: $\lambda > 1$

Let $\lambda = 1 + \mu^2$ with $\mu > 0$

Then:

$$p = -1 \pm i\mu$$

The general solution is:

$$y(x) = e^{-x} \left[A \cos(\mu x) + B \sin(\mu x) \right]$$

Apply boundary conditions:

- $y(0) = e^0 \, A = A = 0 \Rightarrow A = 0$
- $y(1) = e^{-1} B \sin(\mu) = 0$

Since $e^{-1} \neq 0$ and $B \neq 0$ (nontrivial solution), we require:

$$\sin(\mu) = 0 \;\Rightarrow\; \mu = n\pi, \; n = 1, 2, 3, ...$$

So:

$$\lambda_n = 1 + \mu^2 = 1 + n^2 \pi^2$$

And the corresponding eigenfunctions:

$$y_n(x) = e^{-x}\sin(n\pi x)$$

Step 2.2: $\lambda < 1$

The characteristic equation is:

$$p^2 + 2p + (\lambda + 1) = 0 \Rightarrow p = -1 \pm \sqrt{1-\lambda}$$

Since $\lambda < 1$, the discriminant $\sqrt{1-\lambda}$ is real and positive, so the roots are real and distinct. Let:

$$\alpha = \sqrt{1-\lambda} > 0$$

Then the general solution is:

$$y(x) = Ae^{(-1+\alpha)x} + Be^{(-1-\alpha)x}$$

Apply boundary conditions:

1. $y(0) = A + B = 0 \Rightarrow B = -A$

So:

$$y(x) = A\left(e^{(-1+\alpha)x} - e^{(-1-\alpha)x}\right)$$

2. Apply $y(1) = 0$:

$$y(1) = A\left(e^{-1+\alpha} - e^{-1-\alpha}\right) = 0 \Rightarrow e^{-1+\alpha} = e^{-1-\alpha} \Rightarrow \alpha = 0$$

But we assumed $\alpha > 0$, so this is a contradiction. For $\lambda < 1$, the only solution satisfying the boundary conditions is the trivial solution $y(x) = 0$. Therefore, no eigenvalue exists for $\lambda < 1$

Step 2.3: $\lambda = 1$

In this case:

$$p = -1 \pm \sqrt{1-\lambda} = -1 \pm 0 = -1 \quad \text{(repeated root)}$$

The general solution becomes:

$$y(x) = (A + Bx)e^{-x}$$

Apply boundary conditions:

- $y(0) = A = 0 \Rightarrow y(x) = Bxe^{-x}$
- $y(1) = Be^{-1} = 0 \Rightarrow B = 0$

So again, the only solution is the trivial one ⇒ no eigenvalue at $\lambda = 1$

Conclusion

Eigenvalues only exist for $\lambda > 1$, specifically $\lambda_n = 1 + n^2\pi^2,\ n = 1, 2, 3, ...$ The corresponding eigenfunctions are $y_n(x) = e^{-x}\sin(n\pi x)$

Solution (b)

We are given:

$$y'' + 2y' + (\lambda + 1)y = 0$$

We want to rewrite this in self-adjoint form, that is, in the general form:

$$\frac{d}{dx}\left[p(x)y'\right] + [\lambda w(x) + q(x)]y = 0$$

Or more specifically:

$$\frac{d}{dx}\left[p(x)y'\right] + \lambda w(x)y = 0$$

Step 1: Multiply by an integrating factor

We remove the first derivative term by multiplying the equation by an integrating factor $\mu(x)$ such that:

$$\mu(x)y'' + 2\mu(x)y' + \mu(x)(\lambda + 1)y = 0$$

We want:

$$\mu(x)y'' + 2\mu(x)y' = \frac{d}{dx}\left[\mu(x)y'\right]$$

So the condition is:

$$\frac{d}{dx}\mu(x) = 2\mu(x) \Rightarrow \mu(x) = e^{2x}$$

Step 2: Multiply the ODE by $\mu(x) = e^{2x}$:

$$e^{2x}y'' + 2e^{2x}y' + (\lambda + 1)e^{2x}y = 0 \Rightarrow \frac{d}{dx}\left(e^{2x}y'\right) + (\lambda + 1)e^{2x}y = 0$$

Self-adjoint form:

$$\frac{d}{dx}\left(e^{2x}y'\right) + (\lambda + 1)e^{2x}y = 0$$

Solution (c)

From part (a), the eigenfunctions were:

$$y_n(x) = e^{-x}\sin(n\pi x)$$

And from part (b), the weight function (from the self-adjoint form) is:

$$w(x) = e^{2x}$$

Then, the eigenfunctions $y_n(x)$ are orthogonal with respect to the weight function $w(x) = e^{2x}$ on the interval $[0, 1]$

Orthogonality relation:

$$\int_0^1 e^{2x} y_m(x) y_n(x)\, dx = 0 \quad \text{for } m \neq n$$

or equivalently:

$$\int_0^1 e^{2x} e^{-x}\sin(m\pi x)\cdot e^{-x}\sin(n\pi x)\, dx = 0, \quad m \neq n$$

4. (a) Find the Fourier series of f on the given interval

$$f(x) = |x|, \ -\pi \leq x \leq \pi$$

(b) Use the result of (a) and Parseval's Theorem to evaluate the following series:

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots$$

Parseval's Theorem If f is piecewise continuous, of period 2π, then

$$\frac{1}{\pi}\int_0^{2\pi} [f(x)]^2\, dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

where a_0, a_n, and b_n are Fourier coefficients

Solution

(a)

Step 1: Determine symmetry

Since

$$f(-x) = |-x| = |x| = f(x)$$

the function is even

So the Fourier series of $f(x)$ will have the form:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

(No sine terms appear, because $\sin(nx)$ is odd)

Step 2: Compute the coefficients

Constant term:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\cdot\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

Cosine coefficients:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

Now evaluate:

$$\int_0^{\pi} x\cos(nx)\,dx = \left.\frac{x\sin(nx)}{n}\right|_0^{\pi} + \left.\frac{\cos(nx)}{n^2}\right|_0^{\pi} = 0 + \frac{(-1)^n - 1}{n^2}$$

Then:

$$a_n = \frac{2}{\pi}\cdot\frac{(-1)^n - 1}{n^2} = \begin{cases} 0 & \text{if } n \text{ even} \\ -\dfrac{4}{\pi n^2} & \text{if } n \text{ odd} \end{cases}$$

Final Answer: Fourier series of $f(x) = |x|$

$$\begin{aligned} f(x) &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2}\cos(nx) \\ &= \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) \end{aligned}$$

(b)

Step 1: Recall the Fourier Series for $f(x) = |x|$

From part (a), the Fourier series is:

$$f(x) = |x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{\cos(nx)}{n^2}$$

So the coefficients are:

- $a_0 = \pi \Rightarrow \frac{a_0^2}{2} = \frac{\pi^2}{2}$
- $a_n = -\frac{4}{\pi n^2}$ for odd n, and $a_n = 0$ for even n
- $b_n = 0$ for all n (because $f(x)$ is even)

Step 2: Apply Parseval's Theorem

Recall Parseval's identity for f with period 2π:

$$\frac{1}{\pi}\int_0^{2\pi} f(x)^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

- Over $[0, \pi]$, $f(x) = x$
- Over $[\pi, 2\pi]$, $f(x) = 2\pi - x$

So:

$$\frac{1}{\pi}\int_0^{2\pi} f(x)^2 dx = \frac{1}{\pi}\left(\int_0^{\pi} x^2 dx + \int_{\pi}^{2\pi} (2\pi - x)^2 dx\right)$$

Change variable $u = 2\pi - x$ in second integral, $du = -dx$, it becomes:

$$\int_{\pi}^{2\pi} (2\pi - x)^2 dx = \int_0^{\pi} x^2 dx$$

So:

$$\frac{1}{\pi}\int_0^{2\pi} f(x)^2 dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

Step 3: Compute RHS from Fourier coefficients

Recall:

- $a_0 = \pi \Rightarrow \frac{a_0^2}{2} = \frac{\pi^2}{2}$
- For odd n, $a_n = -\frac{4}{\pi n^2}$, so $a_n^2 = \frac{16}{\pi^2 n^4}$

So:

$$\sum_{n=1}^{\infty} a_n^2 = \frac{16}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^4}$$

Plug into Parseval's identity:

$$\frac{2\pi^2}{3} = \frac{\pi^2}{2} + \frac{16}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^4}$$

Subtract $\frac{\pi^2}{2}$ from both sides:

$$\frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{\pi^2}{6} = \frac{16}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^4}$$

Solve for the sum:

$$\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}$$

5. The **root-mean-square value** of a function $f(x)$ defined over an interval (a, b) is given by

$$\mathrm{RMS}(f) = \sqrt{\frac{\int_a^b f^2(x)\,dx}{b-a}}$$

Show that the RMS value of f over the interval $(-p, p)$ is given by

$$\text{RMS}(f) = \sqrt{\frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$

where a_0, a_n, and b_n are the Fourier coefficients, respectively

Solution

Step 1: Fourier Series on $(-p, p)$

Assume $f(x)$ has a Fourier series expansion on $(-p, p)$, with period $2p$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right]$$

Step 2: Parseval's Theorem (general interval)

Parseval's identity for a function with period $2p$ is:

$$\frac{1}{p}\int_{-p}^{p} f^2(x)\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Now multiply both sides by $\frac{1}{2}$ to match the RMS definition, which has $\frac{1}{2p}$:

$$\frac{1}{2p}\int_{-p}^{p} f^2(x)\,dx = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Take square root of both sides:

$$\text{RMS}(f) = \sqrt{\frac{1}{2p}\int_{-p}^{p} f^2(x)\,dx} = \sqrt{\frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$

6. (a) Find the eigenvalues and eigenfunctions of the boundary-value problem

$$y'' + y' + \lambda y = 0,\ y(0) = 0,\ y(2) = 0$$

(b) Put the differential equation in self-adjoint form

(c) Give an orthogonality condition

Solution

(a)

Step 1: Solve the ODE

Consider the characteristic equation:

$$r^2 + r + \lambda = 0 \Rightarrow r = \frac{-1 \pm \sqrt{1 - 4\lambda}}{2}$$

We consider three cases depending on the discriminant $D = 1 - 4\lambda$:

Case 1: $\lambda < \frac{1}{4} \Rightarrow D > 0$

Real and distinct roots:

$$r_{1,2} = \frac{-1 \pm \sqrt{1 - 4\lambda}}{2}$$

General solution:

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Apply boundary conditions:

- $y(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$
- $y(2) = c_1(e^{r_1 \cdot 2} - e^{r_2 \cdot 2}) = 0$

This implies $e^{2r_1} = e^{2r_2} \Rightarrow r_1 = r_2$, but they are distinct $\Rightarrow$ only trivial solution. So, no nontrivial solution for $\lambda < \frac{1}{4}$

Case 2: $\lambda = \frac{1}{4} \Rightarrow D = 0$

Double root:

$$r = -\frac{1}{2} \Rightarrow y(x) = (c_1 + c_2 x)e^{-x/2}$$

Apply BCs:

- $y(0) = c_1 = 0$
- $y(2) = c_2 \cdot 2e^{-1} = 0 \Rightarrow c_2 = 0$

So again, only trivial solution $\Rightarrow$ no eigenvalue at $\lambda = \frac{1}{4}$

Case 3: $\lambda > \frac{1}{4} \Rightarrow D < 0$

Now roots are complex:

$$r = -\frac{1}{2} \pm i\mu, \quad \text{where } \mu = \frac{\sqrt{4\lambda - 1}}{2}$$

Then the general solution becomes:

$$y(x) = e^{-x/2}\left(c_1 \cos(\mu x) + c_2 \sin(\mu x)\right)$$

Apply BCs:

- $y(0) = e^0(c_1) = 0 \Rightarrow c_1 = 0$
- $y(2) = e^{-1} \cdot c_2 \sin(2\mu) = 0$

So $\sin(2\mu) = 0 \Rightarrow 2\mu = n\pi \Rightarrow \mu = \frac{n\pi}{2},\ n = 1, 2, 3, ...$

Then:

$$\mu = \frac{\sqrt{4\lambda - 1}}{2} = \frac{n\pi}{2} \Rightarrow \lambda_n = \frac{1 + n^2\pi^2}{4}$$

Eigenvalues and Eigenfunctions:

- Eigenvalues:

$$\boxed{\lambda_n = \frac{1+n^2\pi^2}{4}, \quad n = 1, 2, 3, ...}$$

- Eigenfunctions:

$$\boxed{y_n(x) = e^{-x/2} \sin\left(\frac{n\pi x}{2}\right)}$$

(b)

We want to rewrite the equation:

$$y'' + y' + \lambda y = 0$$

in the form:

$$\frac{d}{dx}\left[p(x)y'\right] + q(x)y + \lambda w(x)y = 0$$

We do this by finding an integrating factor $\mu(x)$ such that:

$$\mu(x)y'' + \mu(x)y' = \frac{d}{dx}\left[\mu(x)y'\right]$$

We want:

$$\frac{d}{dx}[\mu(x)y'] = \mu(x)y'' + \mu'(x)y' = \mu(x)y'' + \mu(x)y'$$
$$\Rightarrow \mu'(x) = \mu(x) \Rightarrow \mu(x) = e^x$$

Multiply the original equation by e^x:

$$e^x y'' + e^x y' + \lambda e^x y = 0 \Rightarrow \frac{d}{dx}\left(e^x y'\right) + \lambda e^x y = 0$$

So the self-adjoint form is:

$$\boxed{\frac{d}{dx}\left(e^x y'\right) + \lambda e^x y = 0}$$

(c) Orthogonality Condition

In a Sturm-Liouville problem, eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function $w(x)$

From (b), we see the weight function is:

$$w(x) = e^x$$

So, the orthogonality condition is:

$$\boxed{\int_0^2 e^x y_m(x) y_n(x)\, dx = 0 \quad \text{if } m \neq n}$$

7. Find the eigenfunctions and the equation that defines the eigenvalues for the given boundary-value problem:

$$y'' + \lambda y = 0,\ y(0) + y'(0) = 0,\ y(1) = 0$$

Solution

Step 1: General Solution of the ODE

We consider $\lambda > 0$, since we typically expect nontrivial eigenfunctions. Let $\lambda = \mu^2, \quad$ with $\mu > 0$. Then the general solution to the differential equation

$$y'' + \mu^2 y = 0$$

is:

$$y(x) = A\cos(\mu x) + B\sin(\mu x)$$

and

$$y'(x) = -A\mu\sin(\mu x) + B\mu\cos(\mu x)$$

Step 2: Apply the Boundary Conditions

- Condition 1: $y(0) + y'(0) = 0$

$$A + B\mu = 0 \quad \Rightarrow \quad A = -B\mu$$

- Condition 2: $y(1) = 0$

$$y(1) = A\cos(\mu) + B\sin(\mu) = B\left(\sin(\mu) - \mu\cos(\mu)\right)$$

We seek nontrivial solutions $B \neq 0$, so we must have:

$$\boxed{\sin(\mu) = \mu\cos(\mu)}$$

This is the equation that defines the eigenvalues $\lambda = \mu^2$

Step 3: Eigenfunctions

So the eigenfunctions are (up to constant multiples):

$$\boxed{y_\mu(x) = \sin(\mu x) - \mu\cos(\mu x)}$$

where μ satisfies $\sin(\mu) = \mu\cos(\mu)$, and the eigenvalue is $\lambda = \mu^2$

Final Answer

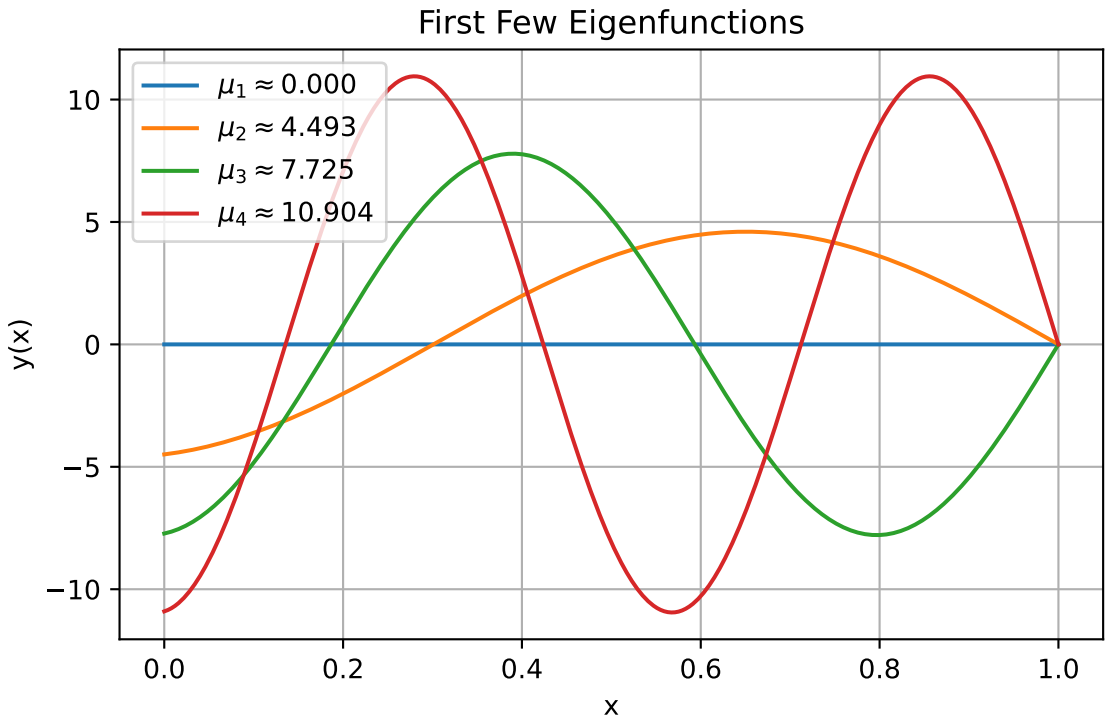
- Eigenvalue equation:

$$\boxed{\sin(\mu) = \mu\cos(\mu) \quad \text{with } \lambda = \mu^2}$$

- Eigenfunctions:

$$\boxed{y(x) = \sin(\mu x) - \mu\cos(\mu x)}$$

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import fsolve

# Define the function whose roots we want to find:
# f(mu) = sin(mu) - mu*cos(mu)
def f(mu):
  return np.sin(mu) - mu *np.cos(mu)

# Find the first few roots numerically using initial guesses
initial_guesses = [0.5, 4.5, 7.5, 11]
mu_roots = [fsolve(f, guess)[0] for guess in initial_guesses]
lambda_roots = [mu**2 for mu in mu_roots]

# Define eigenfunction for given mu
def eigenfunction(mu, x):
  return np.sin(mu *x) - mu *np.cos(mu *x)

# Plot the first few eigenfunctions on [0, 1]
x_vals = np.linspace(0, 1, 500)
plt.figure(figsize=(6, 4))

for i, mu in enumerate(mu_roots):
    y_vals = eigenfunction(mu, x_vals)
    plt.plot(x_vals, y_vals, label=fr"$\mu_{i +1} \approx {mu:6.3f}$")

plt.title("First Few Eigenfunctions")
plt.xlabel("x")
plt.ylabel("y(x)")
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()
```



8. Expand $f(x) = x^4$, $-1 < x < 1$ in a Fourier-Legendre series

Solution

Step 1: Legendre Polynomials

The first few Legendre polynomials are:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \end{aligned}$$

Since $f(x) = x^4$ is a polynomial of degree 4, the Legendre expansion will have at most 5 nonzero terms:

$$f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x) + a_4 P_4(x)$$

But since $f(x) = x^4$ is even, and odd Legendre polynomials are odd functions (so their integrals with even functions vanish), we expect:

$$a_1 = a_3 = 0$$

Step 2: Compute coefficients

We compute:

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} x^4 P_n(x)\, dx$$

- a_0

$$a_0 = \frac{1}{2} \int_{-1}^{1} x^4 \cdot 1\, dx = \frac{1}{2} \cdot \int_{-1}^{1} x^4 dx = \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5}$$

- a_2

$$a_2 = \frac{5}{4} \cdot \int_{-1}^{1} (3x^6 - x^4) dx = \frac{4}{7}$$

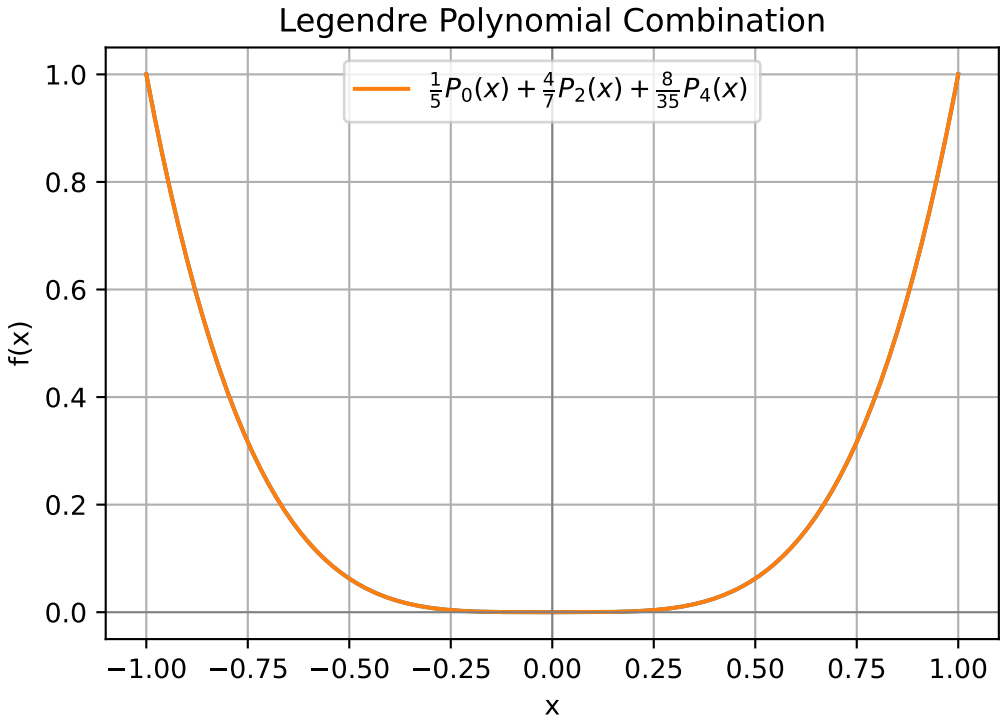
- a_4

$$a_4 = \frac{9}{16} \cdot \int_{-1}^{1} (35x^8 - 30x^6 + 3x^4) dx = \frac{8}{35}$$

Final Answer: Fourier–Legendre Expansion of $f(x) = x^4$

$$\boxed{x^4 = \frac{1}{5} P_0(x) + \frac{4}{7} P_2(x) + \frac{8}{35} P_4(x)}$$

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-1, 1, 400)
P0 = np.ones_like(x)
P2 = 0.5 *(3*x**2 -1)
P4 = (1/8) *(35*x**4 -30*x**2 +3)

f = (1/5)*P0 +(4/7)*P2 +(8/35)*P4

plt.figure(figsize=(6, 4))
plt.plot(x, x**4)
plt.plot(x, f,
  label=r'$ \frac{1}{5} P_0(x) +\frac{4}{7} P_2(x) +\frac{8}{35} P_4(x)$')
plt.axhline(0, color='gray', lw=0.7)
plt.axvline(0, color='gray', lw=0.7)
plt.title("Legendre Polynomial Combination")
plt.xlabel("x")
plt.ylabel("f(x)")
plt.legend()
plt.grid(True)
plt.show()
```



9. Expand the given function in a Fourier-Bessel series using Bessel functions of the same order as in the indicated boundary condition:

$$f(x) = x^2, \quad 0 < x < 3, \quad J_0'(3\alpha) = 0$$

Solution

Step 1: Understand the eigenfunctions and the orthogonality

This is a Fourier-Bessel series of order 0 with a Neumann boundary condition J_0'$(3\alpha_n) = 0$, so the eigenfunctions are:

$$\phi_n(x) = J_0(\alpha_n x)$$

where α_n are such that J_0'$(3\alpha_n) = 0$, or equivalently $\alpha_n = \frac{\beta_n}{3}$, with β_n being the n-th positive zero of J_0'(z)

These $\phi_n(x)$ are orthogonal with respect to the weight x on $(0, 3)$:

$$\int_0^3 x J_0(\alpha_m x) J_0(\alpha_n x) dx = 0 \quad \text{for } m \neq n$$

Step 2: General Fourier-Bessel expansion formula

The coefficients a_n are given by:

$$a_n = \frac{\displaystyle\int_0^3 x f(x) J_0(\alpha_n x) dx}{\displaystyle\int_0^3 x J_0^2(\alpha_n x) dx}$$

Here, $f(x) = x^2$, so the numerator becomes:

$$\int_0^3 x \cdot x^2 \cdot J_0(\alpha_n x)dx = \int_0^3 x^3 J_0(\alpha_n x)dx$$

Step 3: Define the coefficients

Let's write:

- $\alpha_n = \frac{\beta_n}{3}$, where β_n is the n-th zero of J_0'
- Then the expansion is:

$$f(x) = x^2 = \sum_{n=1}^{\infty} a_n J_0\left(\frac{\beta_n}{3}x\right)$$

with

$$a_n = \frac{\int_0^3 x^3 J_0\left(\frac{\beta_n}{3}x\right)dx}{\int_0^3 x J_0^2\left(\frac{\beta_n}{3}x\right)dx}$$

Now, do a substitution to simplify:

Let $u = \frac{\beta_n}{3}x \Rightarrow x = \frac{3}{\beta_n}u$, and when $x = 0$, $u = 0$; when $x = 3$, $u = \beta_n$. Then:

- $dx = \frac{3}{\beta_n}du$
- $x^3 = \left(\frac{3}{\beta_n}\right)^3 u^3$

So numerator becomes:

$$\int_0^3 x^3 J_0\left(\frac{\beta_n}{3}x\right)dx = \left(\frac{3}{\beta_n}\right)^4 \int_0^{\beta_n} u^3 J_0(u)du$$

Similarly, denominator becomes:

$$\int_0^3 x J_0^2\left(\frac{\beta_n}{3}x\right)dx = \left(\frac{3}{\beta_n}\right)^2 \int_0^{\beta_n} u J_0^2(u)du$$

So the coefficient is:

$$a_n = \frac{\left(\frac{3}{\beta_n}\right)^4 \int_0^{\beta_n} u^3 J_0(u)du}{\left(\frac{3}{\beta_n}\right)^2 \int_0^{\beta_n} u J_0^2(u)du} = \left(\frac{3}{\beta_n}\right)^2 \cdot \frac{\int_0^{\beta_n} u^3 J_0(u)du}{\int_0^{\beta_n} u J_0^2(u)du}$$

10. Suppose the function $y = f(x)$, $0 < x < L$, is expanded in a cosine series, in a sine series, and in a Fourier series. Sketch the periodic extension to which each series converges

Case A

$$f(x) = \begin{cases} 0, & 0 < x < \dfrac{L}{2} \\ L - x, & \dfrac{L}{2} < x < L \end{cases}$$

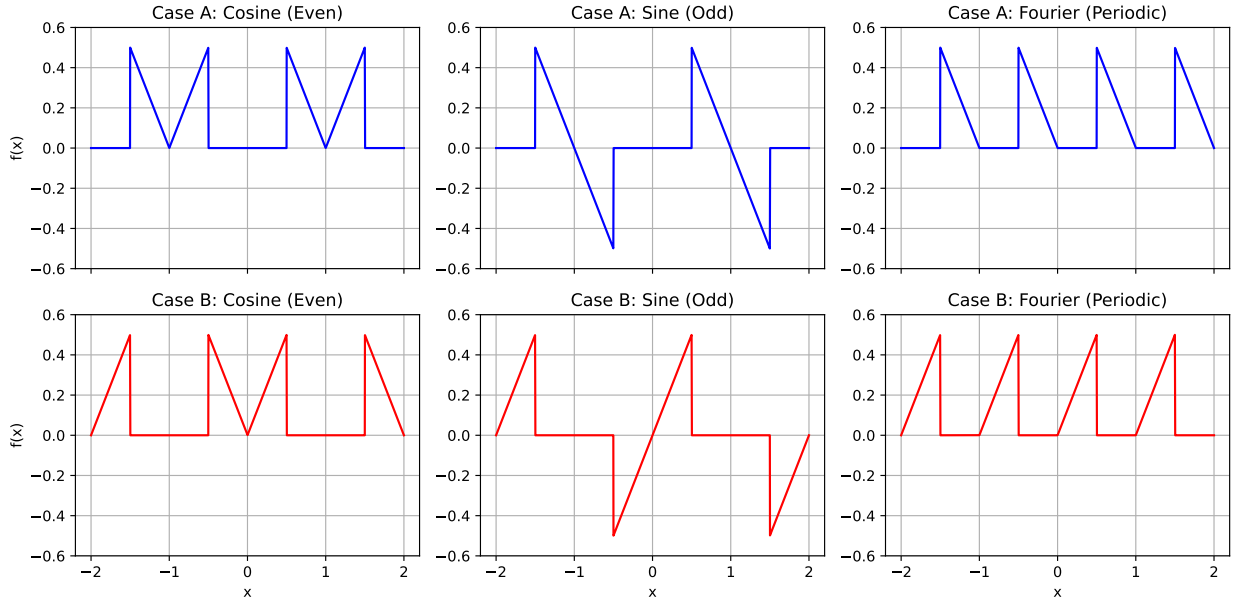
Case B

$$f(x) = \begin{cases} x, & 0 < x < \dfrac{1}{2} \\ 0, & \dfrac{1}{2} < x < 1 \end{cases}$$

Solution

```
import numpy as np
import matplotlib.pyplot as plt

# ----------- Original f1, f2 definitions -----------
def f1(x, L):
    f = np.zeros_like(x)
    mask1 = (x > L/2) & (x < L)
    f[mask1] = L - x[mask1]
    return f

def f2(x, L):
    f = np.zeros_like(x)
    mask1 = (x > 0) & (x < L/2)
    f[mask1] = x[mask1]
    return f

# ----------- Correct periodic extension helpers -----------
def even_extension(f, L, x):
    # Bring x into [-L, L] first
    x_mod = ((x + L) % (2*L)) - L
    f_base = np.where(x_mod >= 0, f(x_mod, L), f(-x_mod, L))
    return f_base

def odd_extension(f, L, x):
    x_mod = ((x + L) % (2*L)) - L
    f_base = np.where(x_mod >= 0, f(x_mod, L), -f(-x_mod, L))
    return f_base

def periodic_extension(f, L, x):
    x_mod = x % L
    return f(x_mod, L)

# ----------- Parameters -----------
L1 = L2 = 1.0
x1 = np.linspace(-2*L1, 2*L1, 2000)
x2 = np.linspace(-2*L2, 2*L2, 2000)

# Compute extensions
f1_even = even_extension(f1, L1, x1)
```

```
f1_odd  = odd_extension(f1, L1, x1)
f1_per  = periodic_extension(f1, L1, x1)

f2_even = even_extension(f2, L2, x2)
f2_odd  = odd_extension(f2, L2, x2)
f2_per  = periodic_extension(f2, L2, x2)

# ----------- Plot -----------
fig, axes = plt.subplots(2, 3, figsize=(12, 6), sharex=True)

titles = ["Cosine (Even)", "Sine (Odd)", "Fourier (Periodic)"]

for i, (data, title) in enumerate(zip([f1_even, f1_odd, f1_per], titles)):
    axes[0, i].plot(x1, data, 'b')
    axes[0, i].set_ylim([-0.6, 0.6])
    axes[0, i].set_title(f"Case A: {title}")
    axes[0, i].grid(True)

for i, (data, title) in enumerate(zip([f2_even, f2_odd, f2_per], titles)):
    axes[1, i].plot(x2, data, 'r')
    axes[1, i].set_ylim([-0.6, 0.6])
    axes[1, i].set_title(f"Case B: {title}")
    axes[1, i].grid(True)

for ax in axes[:, 0]:
    ax.set_ylabel("f(x)")
for ax in axes[-1, :]:
    ax.set_xlabel("x")

plt.tight_layout()
plt.show()
```



11. Consider the differential equation

$$(1-x^2)y'' - 2xy' + \lambda y = 0, \qquad -1 < x < 1$$

ed to remain bounded at $x = \pm 1$

(a) Show that this equation can be written in the Sturm–Liouville form

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + [\lambda p(x) + q(x)]\, y = 0$$

and identify the coefficient functions $r(x)$, $p(x)$, and $q(x)$

(b) Find all eigenvalues λ_n and the corresponding eigenfunctions $y_n(x)$ satisfying the boundedness condition

(c) Show that the eigenfunctions are orthogonal with respect to the weight function $w(x)$

(d) Expand the function $f(x) = x^2$ in terms of the orthogonal eigenfunctions obtained above

Solution

(a) Put into Sturm–Liouville form

The given ODE is

$$(1-x^2)y'' - 2xy' + \lambda y = 0, \qquad -1 < x < 1$$

Note that

$$\frac{d}{dx}[(1-x^2)y'] = (1-x^2)y'' - 2xy'$$

Hence the equation can be written as

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + [\lambda p(x) + q(x)]y = 0$$

with

$$\boxed{r(x) = 1 - x^2, \qquad p(x) = 1, \qquad q(x) = 0, \qquad -1 < x < 1}$$

The natural weight for this Sturm–Liouville problem is $w(x) = p(x) = 1$

(b) Eigenvalues and eigenfunctions (bounded at $x = \pm 1$)

This is the Legendre equation. Requiring solutions bounded on $[-1, 1]$ forces the parameter λ to take the values

$$\boxed{\lambda_n = n(n+1), \quad n = 0, 1, 2, ...}$$

and the corresponding bounded eigenfunctions are (constant multiples of) the Legendre polynomials $P_n(x)$. A standard closed form is Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$$

so

$$\boxed{y_n(x) = P_n(x), \quad n = 0, 1, 2, ...}$$

(Any nonzero constant multiple of P_n is also an eigenfunction)

Remark (why these λ_n): solving by power series around $x = 0$ (Frobenius-type) one finds a polynomial solution only when $\lambda = n(n + 1)$; otherwise the series does not terminate and the solution is singular at the endpoints. The polynomial solutions are exactly the Legendre polynomials which are bounded on $[-1, 1]$

(c) Orthogonality with respect to the weight $w(x) = 1$

Sturm–Liouville theory gives orthogonality, but a short direct derivation is instructive. Let y_n and y_m satisfy

$$\frac{d}{dx}[(1 - x^2)y_n'] + \lambda_n y_n = 0, \qquad \frac{d}{dx}[(1 - x^2)y_m'] + \lambda_m y_m = 0$$

Multiply the first by y_m, the second by y_n, subtract and integrate from -1 to 1:

$$(\lambda_n - \lambda_m) \int_{-1}^{1} y_n y_m \, dx = \Big[(1 - x^2)(y_n' y_m - y_m' y_n)\Big]_{x=-1}^{x=1}$$

But $1 - x^2$ vanishes at $x = \pm 1$. If y_n and y_m are bounded on $[-1, 1]$ then the boundary term is zero. Thus, for $\lambda_n \neq \lambda_m$

$$\int_{-1}^{1} y_n(x)\, y_m(x)\, dx = 0$$

Hence the eigenfunctions are orthogonal with respect to the weight $w(x) = 1$:

$$\boxed{\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0 \quad (m \neq n)}$$

(One also has the normalization formula)

$$\int_{-1}^{1} P_n(x)^2 \, dx = \frac{2}{2n + 1}$$

(d) Expand $f(x) = x^2$ in the eigenfunctions (Legendre series)

Any reasonable function on $[-1, 1]$ can be expanded in Legendre polynomials:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \qquad a_n = \frac{2n + 1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx$$

Since x^2 is a polynomial of degree 2, only P_0, P_2 contribute (orthogonality kills all other coefficients). Use

$$P_0(x) = 1, \qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$

Express x^2 in the P_n-basis by solving

$$x^2 = \alpha P_0(x) + \beta P_2(x)$$

From $P_2 = \frac{1}{2}(3x^2 - 1)$ we get

$$\boxed{x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)}$$

One can check the coefficients by the integral formula:

$$a_0 = \tfrac{1}{2}\int_{-1}^{1} x^2 dx = 1/3,\ a_2 = \tfrac{5}{2}\int_{-1}^{1} x^2 P_2(x)\,dx = 2/3,\ \text{all other } a_n = 0$$

12. Consider the differential equation

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (\lambda x^2 - \nu^2)\,y = 0, \qquad 0 < x < a$$

where $y(x)$ remains finite at $x = 0$ and satisfies $y(a) = 0$

(a) Show that this equation can be written in the Sturm–Liouville form

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + [\lambda p(x) + q(x)]\,y = 0$$

and identify the coefficient functions $r(x)$, $p(x)$, and $q(x)$

(b) Find all eigenvalues λ_n and the corresponding eigenfunctions $y_n(x)$ satisfying the given boundary conditions (**Hint** The solution should be expressed in terms of the Bessel function $J_\nu(x)$)

(c) Show that the eigenfunctions are orthogonal with respect to the weight function $w(x)$

(d) Expand the function $f(x) = x$ in terms of the orthogonal eigenfunctions obtained above

Solution

(a) Sturm–Liouville form

Write the equation as

$$x^2y'' + xy' + (\lambda x^2 - \nu^2)y = 0$$

A convenient SL form is

$$\frac{d}{dx}[r(x)y'(x)] + [\lambda p(x) + q(x)]y(x) = 0$$

with

$$r(x) = x, \qquad p(x) = x, \qquad q(x) = -\frac{\nu^2}{x}, \qquad 0 < x < a$$

Indeed,

$$\frac{d}{dx}(xy') + \Big(\lambda x - \frac{\nu^2}{x}\Big)y = 0$$

is equivalent to the given equation after multiplying by x. The natural weight function for this SL problem is

$$w(x) = p(x) = x$$

(b) Eigenvalues and eigenfunctions

The differential equation is the Bessel equation of order ν with parameter $k^2 = \lambda$:

$$x^2y'' + xy' + (k^2x^2 - \nu^2)y = 0, \qquad k = \sqrt{\lambda}$$

The two linearly independent solutions are $J_\nu(kx)$ and $Y_\nu(kx)$ (Bessel functions of first and second kind). Finiteness at $x = 0$ requires discarding Y_ν (which is singular at 0 for nonnegative ν), so the regular solution is

$$y(x) = A\, J_\nu(kx)$$

The boundary condition $y(a) = 0$ forces

$$J_\nu(ka) = 0$$

Let $j_{\nu,n}$ denote the n-th positive zero of J_ν (counting $n = 1, 2, ...$). Then

$$k_n = \frac{j_{\nu,n}}{a}, \qquad \lambda_n = k_n^2 = \left(\frac{j_{\nu,n}}{a}\right)^2, \qquad n = 1, 2, ...$$

and the corresponding eigenfunctions (choose the gauge $A = 1$ or any normalization constant) are

$$\boxed{y_n(x) = J_\nu\Big(\frac{j_{\nu,n}}{a}\,x\Big), \qquad \lambda_n = \Big(\frac{j_{\nu,n}}{a}\Big)^2}$$

(One may include $n = 0$ only if $J_\nu(0) = 0$ which does not happen for general ν; typically eigenindexing starts at $n = 1$)

(c) Orthogonality (weight $w(x) = x$)

From general Sturm–Liouville theory, eigenfunctions for distinct eigenvalues are orthogonal with respect to the weight $w(x) = p(x) = x$. Concretely, if y_n and y_m correspond to $\lambda_n \neq \lambda_m$, then

$$\int_0^a x\, y_n(x)\, y_m(x)\, dx = 0, \qquad m \neq n$$

You can verify this directly: multiply the n-equation by y_m, the m-equation by y_n, subtract, integrate 0 to a, and use the boundary conditions and the fact that $r(0) = 0$ to kill the boundary term

A standard normalization formula for the squared norm is

$$\int_0^a x\Big[J_\nu\Big(\frac{j_{\nu,n}}{a}x\Big)\Big]^2 dx = \frac{a^2}{2}\left[J_{\nu+1}(j_{\nu,n})\right]^2$$

You can derive this by the substitution $u = j_{\nu,n}x/a$ and the known integral

$$\int_0^{j_{\nu,n}} uJ_\nu^2(u)\,du = \tfrac{j_{\nu,n}^2}{2}[J_{\nu+1}(j_{\nu,n})]^2$$

Thus orthogonality and normalization are completely standard

(d) Expansion of $f(x) = x$ in the eigenfunctions

Because y_n form a complete orthogonal set in the weighted space $L^2([0,a], w(x)\,dx)$ with $w(x) = x$, we may expand

$$x = \sum_{n=1}^{\infty} c_n\, J_\nu\Big(\frac{j_{\nu,n}}{a}x\Big)$$

where the expansion coefficients are obtained by the weighted inner product:

$$c_n = \frac{\displaystyle\int_0^a x\cdot f(x)\,y_n(x)\,dx}{\displaystyle\int_0^a x\,y_n(x)^2\,dx} = \frac{\displaystyle\int_0^a x^2 J_\nu\Big(\frac{j_{\nu,n}}{a}x\Big)\,dx}{\displaystyle\frac{a^2}{2}\left[J_{\nu+1}(j_{\nu,n})\right]^2}$$

It is convenient to change variable $u = \dfrac{j_{\nu,n}}{a}x$. Then

$$\int_0^a x^2 J_\nu\Big(\frac{j_{\nu,n}}{a}x\Big)\,dx = \frac{a^3}{j_{\nu,n}^3}\int_0^{j_{\nu,n}} u^2 J_\nu(u)\,du$$

so

$$\boxed{c_n = \frac{2a}{j_{\nu,n}^3\left[J_{\nu+1}(j_{\nu,n})\right]^2}\int_0^{j_{\nu,n}} u^2 J_\nu(u)\,du}$$

Answers

Answer 1 (Section 11.1)

$(\phi_m, \phi_n) = \delta_{mn}$, so the set is orthonormal on $[-\pi, \pi]$

Answer 2 (Section 11.4)

$$c_n = \frac{(-1)^n \sinh\pi}{\pi\,(1+in)}, \qquad e^{-x} = \frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{1+in}\,e^{inx}$$

Spectrum: $\omega = 1$, points $\left(n,\ \dfrac{\sinh\pi}{\pi\sqrt{1+n^2}}\right)$

Answer 3 (Section 11.5)

$\lambda_0 = 0$: $y_0 = 1$; $\lambda_n = \left(\dfrac{n\pi}{L}\right)^2$: $y_n = A_n\cos\dfrac{n\pi x}{L} + B_n\sin\dfrac{n\pi x}{L}$ $(n = 1, 2, \dots)$

12 Parabolic Partial Differential Equations

Learning objectives After this chapter you should be able to

- classify second-order linear PDEs, and set up diffusion problems with Dirichlet, Neumann and Robin boundary conditions
- solve the heat equation on a finite interval by separation of variables and eigenfunction expansions
- remove nonhomogeneous boundary conditions with a steady-state solution, and simplify equations by substitutions
- solve problems with the finite sine and cosine transforms, the Fourier, Fourier sine and cosine transforms, and the Laplace transform
- treat convection terms, superposition of sources and time-dependent boundary conditions (Duhamel's principle)

12.1 Introduction to Partial Differential Equations

- Most physical phenomena, whether in the domain of fluid dynamics, electricity, magnetism, mechanics, optics, or heat flow, can be described in general by partial differential equations(PDEs): in fact, most of mathematical physics are PDEs
- **What Are PDEs?**

  A partial differential equation is an equation that contains partial derivatives. In contrast to ordinary differential equations (ODEs), where the unknown function depends on *only one variable*, **in PDEs, the unknown function depends on several variables**: $u(x,t)$ depends both on location x and time t
- A few well-known PDEs

$$
\begin{aligned}
u_t &= u_{xx}, && \text{heat equation in one dimension}\\
u_t &= u_{xx} + u_{yy}, && \text{heat equation in two dimensions}\\
u_{tt} &= u_{xx} + u_{yy} + u_{zz}, && \text{wave equation in three dimensions}\\
u_{rr} &+ \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, && \text{Laplace's equation in polar coordinates}
\end{aligned}
$$

- The unknown function u always depends on more than one variable. The variable u (which we differentiate) is called the **dependent** variable, whereas the ones we differentiate *with respect to* are called the **independent** variables
- **Classification** is an important concept because the general theory and methods of solution apply only to a given class of equations. Six basic classifications are:

– **Order of the PDE**

$$\begin{aligned} u_t &= u_{xx}, && \text{second order} \\ u_t &= u_x, && \text{first order} \\ u_t &= uu_{xxx} + \sin x, && \text{third order} \end{aligned}$$

– **Number of Independent Variables**

$$\begin{aligned} u_t &= u_{xx}, && \text{two variables: } x \text{ and } t \\ u_t &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}, && \text{three variables: } t,\ r \text{ and } \theta \end{aligned}$$

– **Linearity**

$$\begin{aligned} u_{tt} &= e^{-t}u_{xx} + \sin t, && \text{linear} \\ uu_{xx} + u_t &= 0, && \text{nonlinear} \\ u_{xx} + yu_{yy} &= 0, && \text{linear} \\ xu_x + yu_y + u^2 &= 0, && \text{nonlinear} \end{aligned}$$

- A second-order linear equation in two variables is an equation of the form:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{SL}$$

where A, B, C, D, E, F, and G can be constants or given functions of x and y

– **Homogeneity**

* (SL) is called **homogeneous** if the right hand side $G(x, y)$ is identically zero for all x and y.
* If $G(x, y)$ is not identically zero, then the equation is **nonhomogeneous**

– **Kinds of Coefficients**

The coefficients in (SL) are constants, then (SL) is said to have constant coefficients (otherwise, variable coefficients)

– **Three Basic Types of Linear Equations**

* **Parabolic**

  $B^2 - 4AC = 0$, heat flow and diffusion processes

  $u_t = u_{xx}$
* **Hyperbolic**

  $B^2 - 4AC > 0$, vibrating systems and wave motion

  $u_{tt} = u_{xx},\ u_{\xi\eta} = 0$
* **Elliptic**

  $B^2 - 4AC < 0$, steady-state phenomena

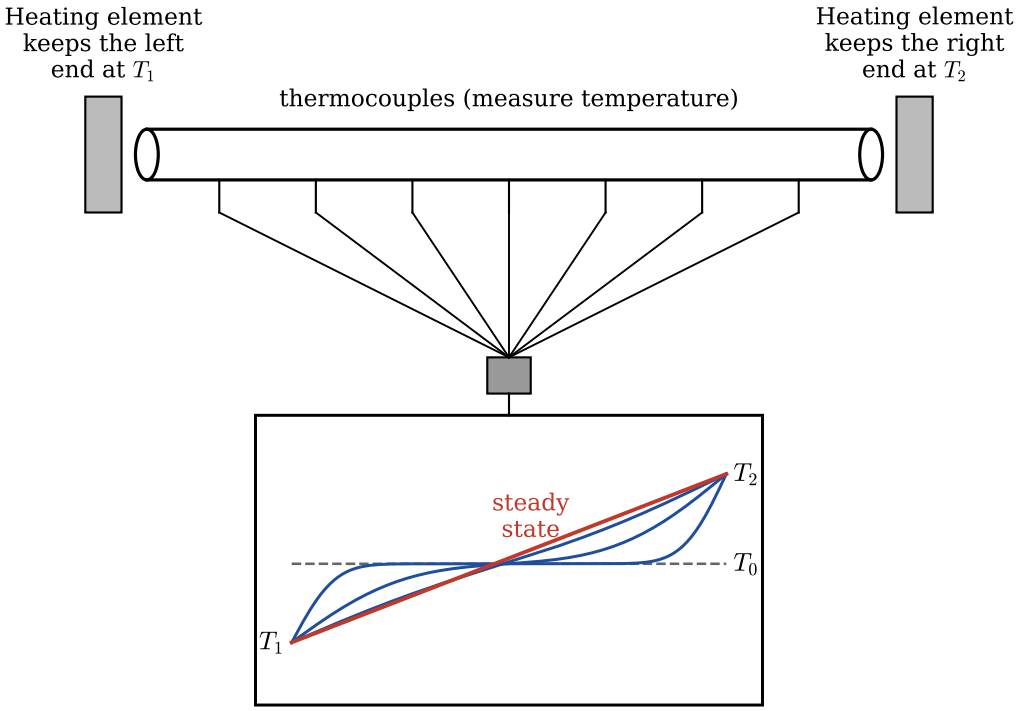
  $u_{xx} + u_{yy} = 0$

– In the case of variable coefficients, the situation can change from point to point

$yu_{xx} + u_{yy} = 0$

Linearity	Linear					Nonlinear	
Order	1	2	3	4	5	···	m
Kinds of coefficients (linear equations)	Constant					Variable	
Homogeneity (linear equations)	Homogeneous					Nonhomogeneous	
Number of variables	1	2	3	4	5	···	n
Basic type (linear equations)	Hyperbolic				Parabolic		Elliptic

12.2 Diffusion-Type Problems (Parabolic Equations)



screen showing the temperature profile at different times

- **The Mathematical Model of the Heat-Flow**

  The description of our physical problem requires three types of equations

  – The **PDE** describing the physical phenomenon of heat flow

  – The **boundary conditions** describing the physical nature of our problem on the boundaries

  – The **initial condition** describing the physical phenomenon at the start of the experiment

- **The Heat Equation**

  The basic equation of *one-dimensional* heat flow is

$$u_t = \alpha u_{xx},\ 0 < x < L,\ 0 < t < \infty \tag{HE}$$

  This equation is derived from the basic conservation of energy

- **Boundary Conditions**

  All physical problems have boundaries of some kind, so we must describe mathematically what goes on there in order to adequately describe the problem. Since the temperature u was fixed for all time $t > 0$ at T_1 and T_2 at the two ends $x = 0$ and $x = L$

$$\begin{matrix} u(0,t) = T_1 \\ u(L,t) = T_2 \end{matrix}\ ,\quad 0 < t < \infty \tag{BCs}$$

- **Initial Condition**

  All physical problems must start from some value of time (generally called $t = 0$), so we must specify the physical apparatus at this time. Since we started monitoring the rod temperature from time the rod has achieved a constant temperature T_0, we have

$$u(x,0) = T_0,\ \ 0 \le x \le L \tag{IC}$$

  By writing (HE), (BCs), and (IC) together, we have what is called an *initial-boundary-value problem*. The interesting thing here, which is not at all obvious, is that there is only one function $u(x,t)$ that satisfies the problem

- **More Diffusion-Type Equations**

  - **Lateral Heat Loss Proportional to the Temperature Difference**

    The equation

$$u_t = \alpha u_{xx} - \beta(u - u_0),\ \beta > 0$$

    describes heat flow in the rod with both diffusion αu_{xx} along the rod and heat loss (or gain) across the lateral sides of the rod. Heat loss ($u > u_0$) or gain ($u < u_0$) is proportional to the difference between the temperature $u(x,t)$ and the surrounding medium u_0

  - **Internal Heat Source**

    The nonhomogeneous equation

$$u_t = \alpha u_{xx} + f(x,t)$$

    corresponds to the situation where the rod is being supplied with an internal heat source

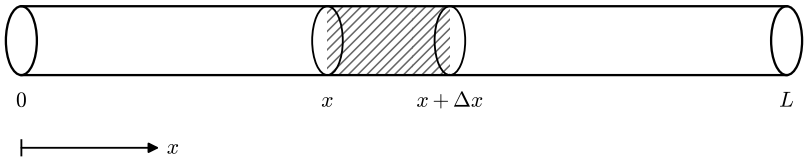
– **Diffusion-Convection Equation**

Suppose a pollutant is being carried along in a stream moving with velocity v. The rate of change u_t is measured by the diffusion-convection equation

$$u_t = \alpha u_{xx} - v u_x$$

The term αu_{xx} is the diffusion contribution and $-v u_x$ is the convection component

12.3 Derivation of the Heat Equation



- Suppose we have a one-dimension rod of length L for which we make the following assumptions:
  - The rod is made of a single homogeneous conducting material
  - The rod is laterally insulated (heat flows only in the x-direction)
  - The rod is thin (the temperature at all points of a cross section is constant)
- If we apply the principle of conservation of energy to the segment $[x, x + \Delta x]$, we can claim

$$\begin{aligned}\text{Net Change of Heat inside } [x, x+\Delta x] \; = \\ \text{Net Flux of Heat across the Boundaries} + \\ \text{Total Heat Generated inside } [x, x+\Delta x]\end{aligned}$$

  - The total amount of heat inside $[x, x + \Delta x]$ at any time t is measured by

$$\text{Total Heat inside } [x, x+\Delta x] = \int_x^{x+\Delta x} \rho c_p A u(s,t)\, ds$$

  - We can write the conservation of energy via calculus as

$$\frac{d}{dt}\int_x^{x+\Delta x} \rho c_p A u(s,t)\,ds = \rho c_p A \int_x^{x+\Delta x} u_t(s,t)\,ds$$
$$= kA\left[u_x(x+\Delta x,t) - u_x(x,t)\right] + A\int_x^{x+\Delta x} F(s,t)\,ds$$

$$\downarrow \text{Mean Value Theorem } \int_a^b f(x)\,dx = f(\xi)(b-a),\ a<\xi<b$$

$$\rho c_p A u_t(\xi_1,t)\Delta x = kA\left[u_x(x+\Delta x,t) - u_x(x,t)\right] + AF(\xi_2,t)\Delta x,\quad x<\xi_1,\xi_2<x+\Delta x$$

$$\downarrow$$

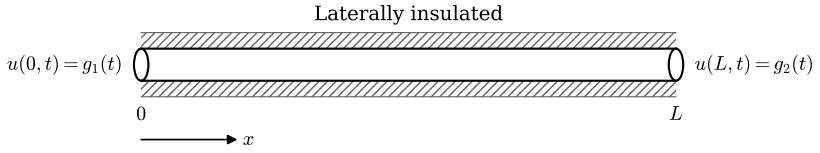
$$u_t(\xi_1,t) = \frac{k}{\rho c_p}\left\{\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}\right\} + \frac{1}{\rho c_p}F(\xi_2,t)$$

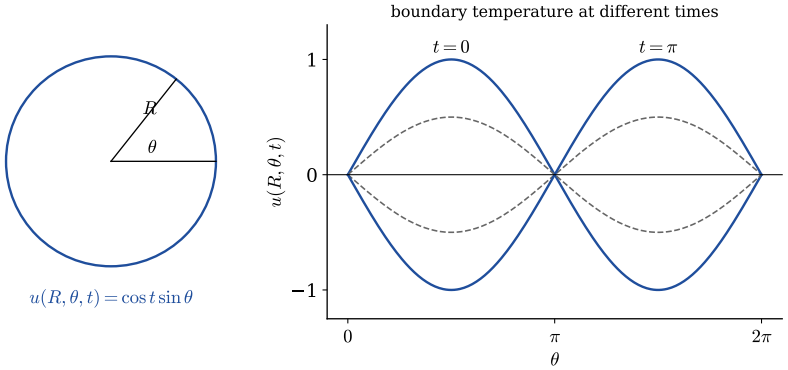
$$\downarrow \Delta x \to 0$$

$$u_t(x,t) = \alpha u_{xx}(x,t) + f(x,t)$$

12.4 Boundary Conditions for Diffusion-Type Problems

- When describing the various types of boundary conditions that can occur for heat-flow problems, three basic types generally come to mind
- **Type 1 BC (Temperature specified on the boundary)**





  - Of course, we'd have an initial temperature to get this experiment started, but in this case, the effects of our IC would vanish after a short period of time, and the resulting temperature inside the circle would depend on the boundary temperature

- **Type 2 BC (Flux specified - including the special case of insulated boundaries)**
  - **Insulated boundaries** are those that do not allow any heat flow to pass, and, hence, the normal derivative (inward or outward) must be zero on the boundary (since the normal derivative is proportional to the flux)
  - In the case of the one-dimensional rod with insulated ends at $x = 0$ and $x = L$, the BCs are

$$\begin{matrix} u_x(0,t) = 0 \\ u_x(L,t) = 0 \end{matrix} , \quad 0 < t < \infty$$

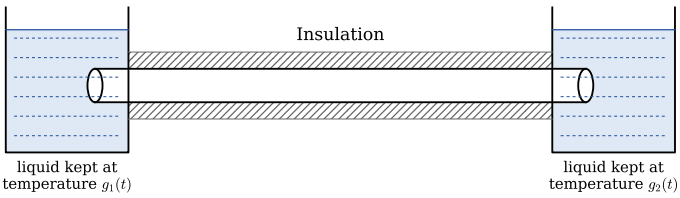
  - In two-dimensional domains, an insulated boundary would mean that the normal derivative of the temperature across the boundary is zero. For example, if the circular disc insulated on the boundary, then the BC would be

$$u_r(R,\theta,t) = 0 \text{ for } 0 \leq \theta < 2\pi \text{ and } 0 < t < \infty$$

    On the other hand, if we specify the amount of heat entering across the boundary of our disc, the BC is

$$u_r(R,\theta,t) = f(\theta,t)$$

- **Type 3 BC (Temperature of the surrounding medium specified)**



  - By specifying these types of BCs, we cannot say the boundary temperatures of the rod will be the same as the liquid temperature $g_1(t)$ and $g_2(t)$,
  - but we do know (Newton's law of cooling) that whenever the rod temperature at one of the boundaries is less than the respective liquid temperatures, then heat will flow into the rod at a rate proportional to this difference
  - In other words, for the one-dimensional rod with boundaries at $x = 0$ and L, Newton's law of cooling states:

$$\text{Outward Flux of Heat (at } x = 0) = h[u(0,t) - g_1(t)]$$

$$\text{Outward Flux of Heat (at } x = L) = h[u(L,t) - g_2(t)]$$

    where h is a heat transfer coefficient

– Fourier's law gives us another representation for the outward flux of heat. In our one-dimensional problem, Fourier's law takes the form:

$$\text{Outward Flux of Heat (at } x = 0) \ = k\frac{\partial u(0,t)}{\partial x}$$

$$\text{Outward Flux of Heat (at } x = L) \ = -k\frac{\partial u(L,t)}{\partial x}$$

where k is the thermal conductivity of the material

– Finally if we use the two expressions for heat flux, we have our desired BCs in purely mathematical terms;

$$\begin{aligned} \frac{\partial u(0,t)}{\partial x} &= \frac{h}{k}\left[u(0,t) - g_1(t)\right] \\ \frac{\partial u(L,t)}{\partial x} &= -\frac{h}{k}\left[u(L,t) - g_2(t)\right] \end{aligned}, \quad 0 < t < \infty$$

– In higher dimensions, we have similar BCs; for example, if the boundary of a circular disc is interfaced with a moving liquid that has a temperature $g(\theta, t)$, our BC would be

$$\frac{\partial u}{\partial r}(R,\theta,t) = -\frac{h}{k}\left[u(R,\theta,t) - g(\theta,t)\right]$$

12.5 Separation of Variables

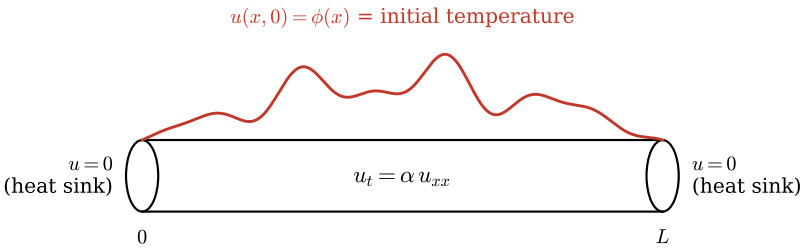
- Separation of variables is one of the oldest techniques for solving initial-boundary-value problems and applies to problems where
  - The PDE is linear and homogeneous (not necessarily constant coefficients)
  - The boundary conditions are of the form

$$\begin{aligned} \alpha u_x(0,t) + \beta u(0,t) &= 0 \\ \gamma u_x(L,t) + \delta u(L,t) &= 0 \end{aligned}$$

where α, β, γ, and δ are constants (boundary conditions of this form are called linear homogeneous BCs)

- We wish to find the function $u(x,t)$ that satisfies the following four conditions:

$$\begin{aligned} u_t = \alpha u_{xx} \quad & 0 < x < L, \ 0 < t < \infty \\ u(0,t) = 0 \quad & 0 < t < \infty \\ u(L,t) = 0 & \\ u(x,0) = \phi(x) \quad & 0 \le x \le L \end{aligned}$$



STEP 1 To begin, we look for solutions of the form $u(x,t)=X(x)T(t)$ by substituting $X(x)T(t)$ into the PDE

$$
\begin{aligned}
X(x)T'(t) &= \alpha X''(x)T(t) \\
&\Downarrow \\
\frac{T'(t)}{\alpha T(t)} &= \frac{X''(x)}{X(x)} = -\lambda < 0 \\
&\Downarrow \text{ We essentially change a second-order PDE to two ODEs} \\
T' &+ \alpha\lambda T = 0 \\
X'' &+ \lambda X = 0 \\
&\Downarrow \\
T(t) &= a_1 e^{-\alpha\lambda t} \\
X(x) &= a_2 \sin\sqrt{\lambda}x + a_3 \cos\sqrt{\lambda}x \\
&\Downarrow \\
u(x,t) &= e^{-\alpha\lambda t}\left[A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x\right]
\end{aligned}
$$

At this point, we have an infinite number of functions that satisfy the PDE

STEP 2 The next step is to choose a certain subset of our current crop of solutions

$$
e^{-\alpha\lambda t}\left[A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x\right] \tag{12.1}
$$

that satisfy the boundary conditions

$$
\begin{aligned}
u(0,t) &= 0 \\
u(L,t) &= 0
\end{aligned}
$$

To do this, we substitute Equation 12.1 into these BCs, getting

$$
\begin{aligned}
u(0,t) &= Be^{-\alpha\lambda t} = 0 \;\Rightarrow B = 0 \\
u(L,t) &= Ae^{-\alpha\lambda t}\sin\sqrt{\lambda}L = 0 \;\Rightarrow \sin\sqrt{\lambda}L = 0,\; A \neq 0 \\
&\Downarrow \\
\lambda &= \left(\frac{\pi}{L}\right)^2, \left(\frac{2\pi}{L}\right)^2, \left(\frac{3\pi}{L}\right)^2, \cdots \\
&\Downarrow \\
\lambda_n &= \left(\frac{n\pi}{L}\right)^2,\; n = 1,2,3,\cdots
\end{aligned}
$$

We have now finished the second step; we have an infinite number of functions

$$u_n(x,t) = c_n e^{-\alpha(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi}{L}x\right),\ n = 1, 2, 3, \cdots$$

each one satisfying the PDE and the BCs

STEP 3 The last step is to add the fundamental solutions

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi}{L}x\right) \tag{12.2}$$

in such a way (pick the coefficients c_n) that the initial condition

$$u(x,0) = \phi(x)$$

is satisfied. Substituting the sum into the IC gives

$$\phi(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)$$

Now the problem becomes how to find the coefficients c_n. This is actually very easy

- One uses properties of the functions known as orthogonality:

$$\int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right)\, dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases}$$

  We multiply each side of Equation 12.2 by $\sin(m\pi x/L)$ and integrate from zero to L; we get

$$\int_0^L \phi(x) \sin\left(\frac{m\pi}{L}x\right)\, dx = c_m \int_0^L \sin^2\left(\frac{m\pi}{L}x\right)\, dx = \frac{L}{2} c_m$$

  We're done; the solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi}{L}x\right) \quad \text{where} \quad c_n = \frac{2}{L}\int_0^L \phi(x) \sin\left(\frac{n\pi}{L}x\right)\, dx$$

NOTES

- Observe that the only difference between the Fourier sine expansion of $\phi(x)$ and the solution is the insertion of the time factor

$$e^{-\alpha(\frac{n\pi}{L})^2 t}$$

  in each term

- The terms in the series get small very fast due to the factor $e^{-\alpha(\frac{n\pi}{L})^2 t}$. Hence, for long time periods, the solution is approximately equal to the first term

$$u(x,t) \approx c_1 e^{-\alpha(\frac{\pi}{L})^2 t} \sin\left(\frac{\pi}{L}x\right)$$

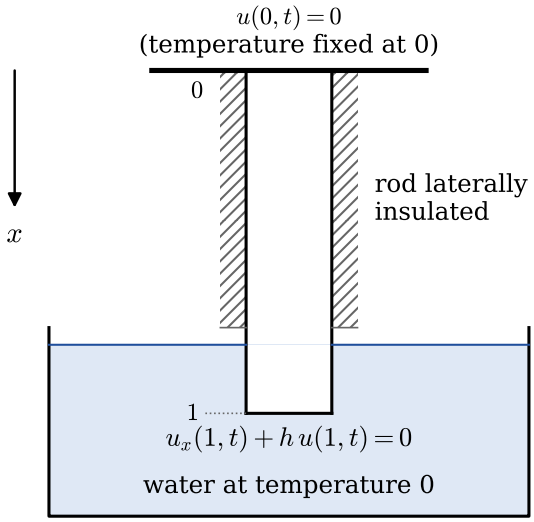
Example Solve the diffusion problem with insulated boundaries; that is

$$\begin{aligned} u_t &= u_{xx} && 0 < x < 1,\ 0 < t < \infty \\ u_x(0,t) &= 0 && 0 < t < \infty \\ u_x(1,t) &= 0 \\ u(x,0) &= 1 + \cos \pi x + 0.5 \cos 3\pi x && 0 \le x \le 1 \end{aligned}$$

Answer 1 (end of chapter) · Solution

12.6 Solving More Complicated Problems by Separation of Variables

- We start with a one-dimensional heat-flow problem where one of the BCs contains derivatives
- **Heat-Flow Problem with 3rd kind BC**



- We fix the temperature at the top of the rod at $u(0,t) = 0$ and immerse the bottom of the rod in a solution of water fixed at the same temperature of zero. The natural flow of heat says that the BC at $x = 1$ is

$$u_x(1,t) = -hu(1,t)$$

- Suppose now the initial temperature of the rod is $u(x,0) = x,$ but instantaneously thereafter $(t > 0),$ we apply our BCs. To find the ensuing temperature, we must solve the IBVP

$$\begin{aligned} u_t &= \alpha u_{xx} && 0 < x < 1,\ 0 < t < \infty \\ u(0,t) &= 0 && 0 < t < \infty \\ u_x(1,t) + hu(1,t) &= 0 \\ u(x,0) &= x && 0 \le x \le 1 \end{aligned}$$

STEP 1

- Substituting $u(x,t) = X(x)T(t)$ into the PDE gives

$$u(x,t) = e^{-\alpha\lambda t}\left[A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x\right]$$

  for any $\lambda > 0$ and any A and B

STEP 2

- Substituting the solution into the BCs gives us conditions on λ, A and B that must be satisfied;

$$Be^{-\alpha\lambda t} = 0 \;\Rightarrow\; B = 0$$

$$Ae^{-\alpha\lambda t}(\sqrt{\lambda}\cos\sqrt{\lambda} + h\sin\sqrt{\lambda}) = 0 \;\Rightarrow\; \tan\sqrt{\lambda} = -\frac{\sqrt{\lambda}}{h},\; A \neq 0$$

- To find λ, we must find the intersections of the curves $\tan\sqrt{\lambda}$ and $-\frac{\sqrt{\lambda}}{h}$. These values λ_1, λ_2, $\cdots$ can be computed numerically for a given h and are called the **eigenvalues** of the boundary-value problem

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) &= 0 \\ X'(1) + hX(1) &= 0 \end{aligned} \tag{12.3}$$

  The eigenvalue problem Equation 12.3 is a special case of the general **Sturm-Liouville problem**

```
import numpy as np
from scipy import optimize
import matplotlib.pyplot as plt

plt.rcParams['font.size'] = 12
plt.rcParams['xtick.direction'] = 'in'
plt.rcParams['ytick.direction'] = 'in'
plt.rcParams['text.usetex'] = True
plt.rcParams['text.latex.preamble'] = r'\usepackage{amsmath}'

def example_plot(h, n_eig, sqrt_eig):

    x = np.linspace(0, n_eig*np.pi, 1000)
    y = np.tan(x)

    threshold = 100
    y[y > threshold] = np.inf
    y[y <-threshold] = np.inf

    plt.figure(figsize=(7, 4))

    plt.plot(x, y, linewidth=1.2, color="blue")
```

```
    plt.plot(x,-x/h, linewidth=1.2, color="green")
    plt.scatter(sqrt_eig,-sqrt_eig/h, color='red')

    plt.xticks([0, np.pi, 2*np.pi, 3*np.pi, 4*np.pi, 5*np.pi],
            ['0',r'$\pi$',r'$2\pi$',r'$3\pi$',r'$4\pi$',r'$5\pi$'])
    plt.xlim(0, 5.0*np.pi)
    plt.ylim(-20, 20)

    plt.xlabel(r'$\sqrt{\lambda}$')
    plt.ylabel(r'$\tan \sqrt{\lambda}$, $-\sqrt{\lambda}/h$')
    plt.title(r'$h=%3.1f$' % h)
    plt.grid()

    plt.show()
```

```
def calc_eig(h, n_eig):
    return np.array([optimize.brentq(lambda x: np.tan(x) +x/h,
      (i +0.5001)*np.pi, (i +1)*np.pi) for i in range(n_eig)])

h = 1
n_eig = 5

sqrt_eig = calc_eig(h, n_eig)
example_plot(h, n_eig, sqrt_eig)
```

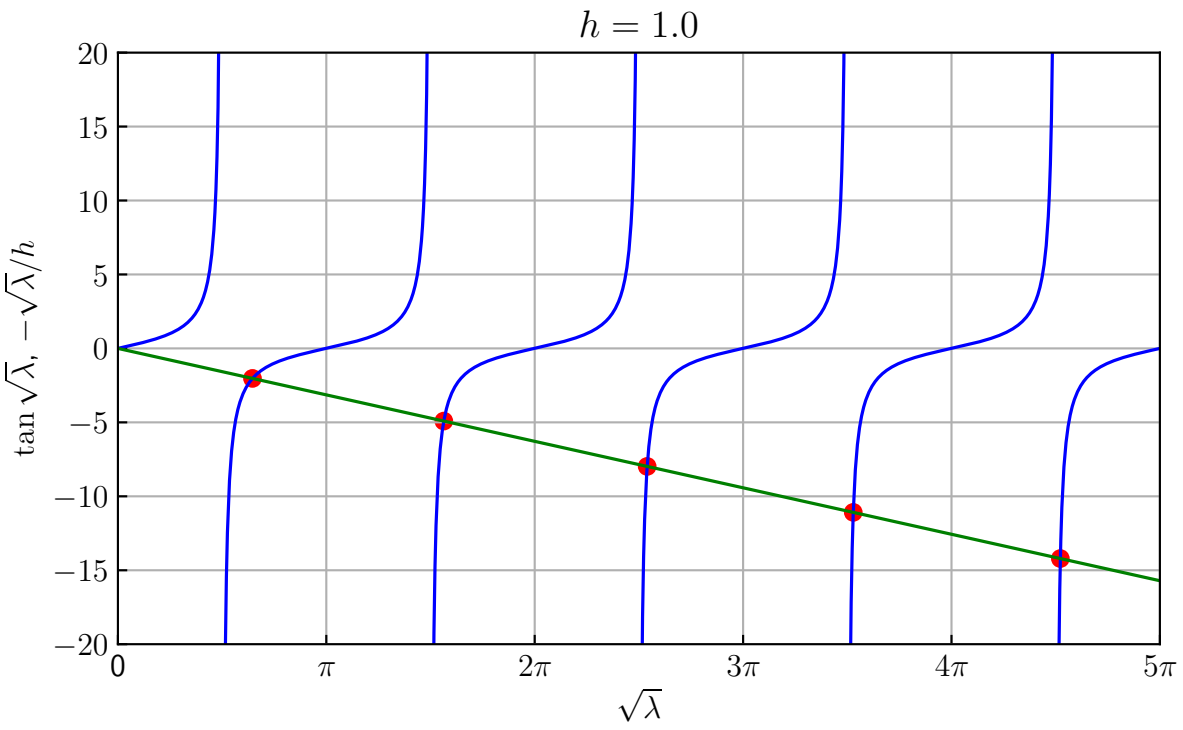


Figure 12.1: Eigenvalues

```
for i in range(n_eig):
  print(rf'$$ \lambda_{i +1} = {np.round(sqrt_eig[i]**2, 2)} $$')
```

$$\lambda_1 = 4.12$$
$$\lambda_2 = 24.14$$
$$\lambda_3 = 63.66$$
$$\lambda_4 = 122.89$$
$$\lambda_5 = 201.85$$

- The solutions of Equation 12.3 corresponding to the eigenvalues λ_n are called **eigenfunctions** $X_n(x)$

$$X_n(x) = \sin \sqrt{\lambda_n} x$$

```
def example_plot2(x, n_eig, sqrt_eig, eigfunc):

    plt.figure(figsize=(7, 4))
    for n in range(n_eig):
        plt.plot(x, eigfunc(x, sqrt_eig[n]), label=rf'$X_{n +1}(x)$')

    plt.xlim(0, 1)
    plt.ylim(-1.2, 1.2)
    plt.grid(ls=':')
    plt.legend(loc='upper left', bbox_to_anchor=(0, -0.1))

    plt.xlabel('$x$')
    plt.ylabel('$X_n(x)$')
    plt.title(r'$X_n(x)=\sin\sqrt{\lambda_n} x$')

    plt.show()
```

```
def eigfunc(x, sqrt_eig):
    return np.sin(sqrt_eig *x)

x = np.linspace(0, 1, 150)
example_plot2(x, n_eig, sqrt_eig, eigfunc)
```

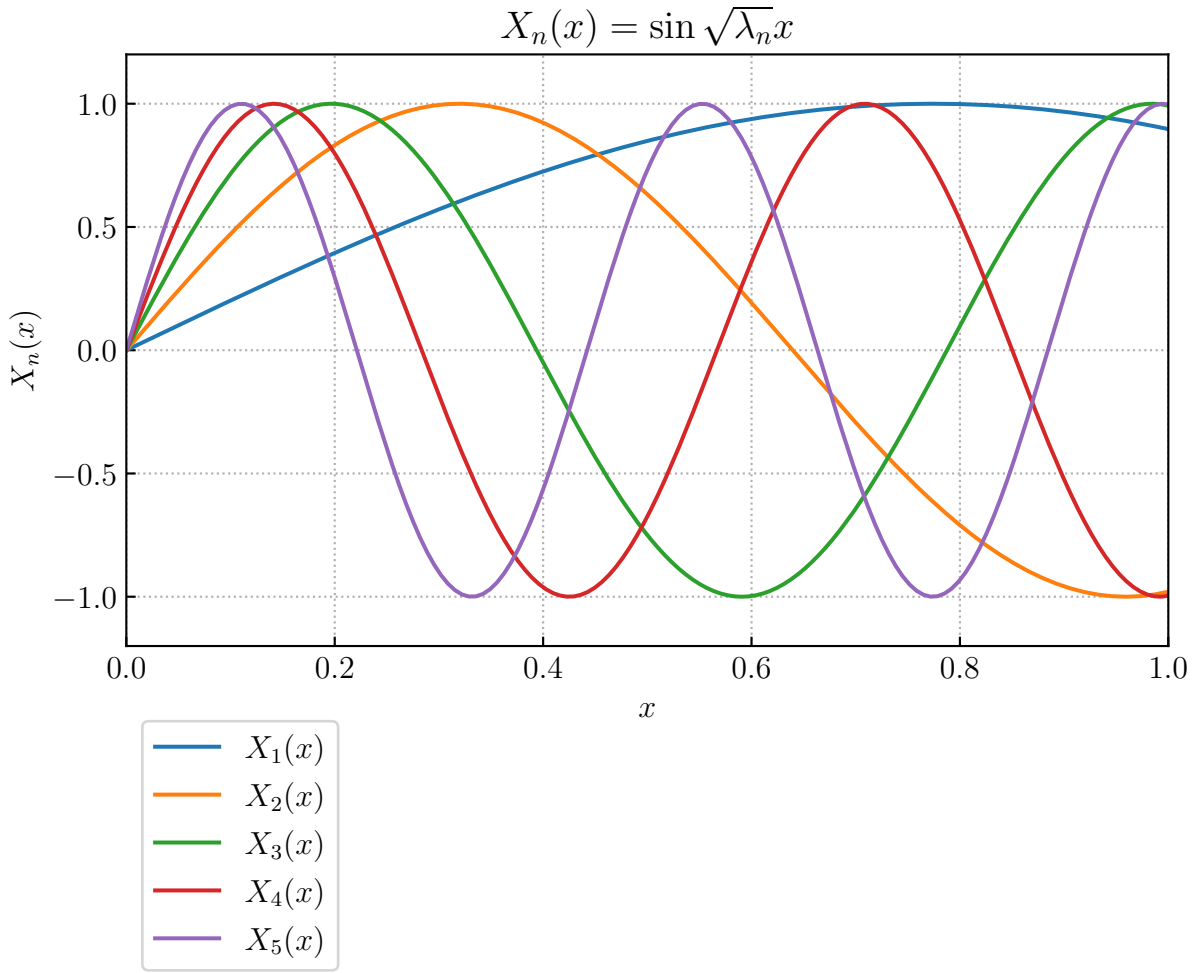


Figure 12.2: Eigenfunctions

STEP 3

- We now have an infinite number of solutions

$$u_n(x,t) = e^{-\alpha\lambda_n t} \sin \sqrt{\lambda_n} x$$

each one satisfying the PDE and the BCs. The final step is to add these functions together (the sum will still satisfy the PDE and BCs, since both the PDE and BCs are linear and homogeneous)

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha\lambda_n t} \sin \sqrt{\lambda_n} x$$

in such a way that they agree with the IC

$$u(x,0) = x = \sum_{n=1}^{\infty} c_n \sin \sqrt{\lambda_n} x$$

- To find the coefficients c_n, we must multiply each side of the equation by $\sin\sqrt{\lambda_m}x$ and integrate x from 0 to 1

$$\int_0^1 x\sin\sqrt{\lambda_m}x\,dx = \sum_{n=1}^{\infty} c_n \int_0^1 \sin\sqrt{\lambda_n}x\,\sin\sqrt{\lambda_m}x\,dx$$

$$\Downarrow$$

$$\frac{\sin\sqrt{\lambda_m} - \sqrt{\lambda_m}\cos\sqrt{\lambda_m}}{\lambda_m} = c_m \int_0^1 \sin^2\sqrt{\lambda_m}x\,dx = c_m\left[\frac{\sqrt{\lambda_m} - \sin\sqrt{\lambda_m}\cos\sqrt{\lambda_m}}{2\sqrt{\lambda_m}}\right]$$

$$\Downarrow$$

$$c_n = \frac{2}{\sqrt{\lambda_n}}\left[\frac{\sin\sqrt{\lambda_n} - \sqrt{\lambda_n}\cos\sqrt{\lambda_n}}{\sqrt{\lambda_n} - \sin\sqrt{\lambda_n}\cos\sqrt{\lambda_n}}\right]$$

In this problem, the first five constants c_n have been computed:

```
def cal_c_n(sqrt_eig):

    sin_sqrt_eig = np.sin(sqrt_eig)
    cos_sqrt_eig = np.cos(sqrt_eig)

    return (2.0 /sqrt_eig
           *(sin_sqrt_eig -sqrt_eig *cos_sqrt_eig)
           /(sqrt_eig -sin_sqrt_eig *cos_sqrt_eig))

c_n = cal_c_n(sqrt_eig)
print(r'$,\;\;$'.join([rf'$c_{i} = {c_i: 6.4f}$'
              for i, c_i in enumerate(c_n, 1)]))
```

$c_1 = 0.7292,\ c_2 = -0.1562,\ c_3 = 0.0614,\ c_4 = -0.0322,\ c_5 = 0.0197$

```
def example_plot3(alpha, t, x, n_eig, sqrt_eig, eigfunc, u_solution):
    plt.figure(figsize=(7, 4))
    plt.plot(x, x, 'r:', label='$t=0.0$')
    for tt in t:
        plt.plot(x, u_solution(alpha, tt, x, n_eig, sqrt_eig, eigfunc),
                 label='$t=%0.1f$' % tt)

    plt.xlim(0, 1)
    plt.ylim(0, 1)
    plt.legend(loc='upper left')
    plt.xlabel('$x$')
    plt.ylabel('$u(x,t)$')

    plt.show()
```

```
def u_solution(alpha, t, x, n_eig, sqrt_eig, eigfunc):
    u = 0
    for i in range(n_eig):
        u += c_n[i] *np.exp(-alpha *sqrt_eig[i]**2 *t) \
```

```
                    *eigfunc(x, sqrt_eig[i])
    return u

alpha = 1
t = np.linspace(0.1, 0.5, 4)
example_plot3(alpha, t, x, n_eig, sqrt_eig, eigfunc, u_solution)
```

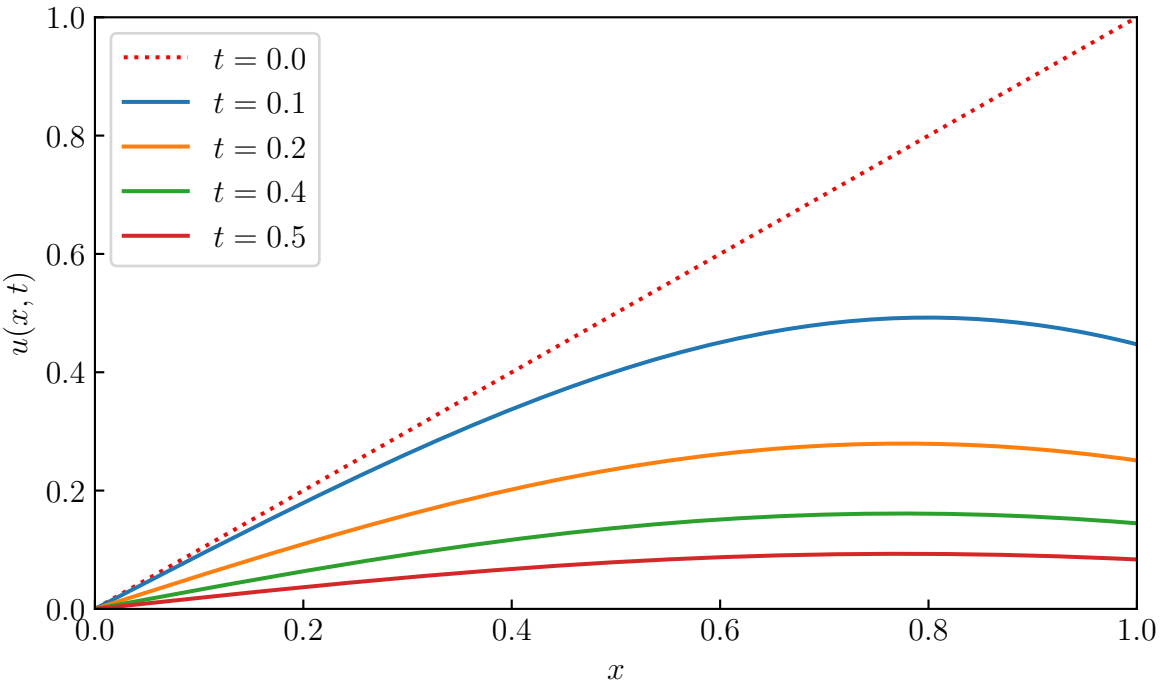


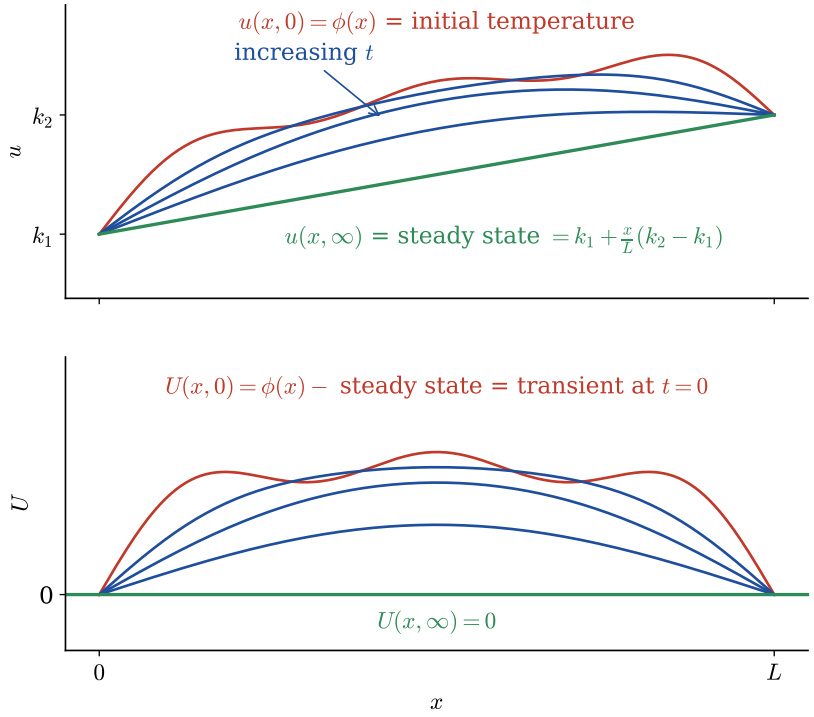
Figure 12.3: Solutions

12.7 Transforming Nonhomogeneous BCs into Homogeneous Ones

- Consider heat flow in an insulated rod where the two ends are kept at constant temperatures k_1 and k_2:

$$\begin{matrix} u(0,t) = k_1 \\ u(L,t) = k_2 \end{matrix}, \qquad 0 < t < \infty \tag{12.4}$$

- The difficulty here is that since the BCs are not homogeneous, we cannot solve this problem by separation of variables
- However, it is obvious that the solution will have a steady-state solution (when $t = \infty$) that varies linearly between the boundary temperatures k_1 and k_2



- In other words, it seems reasonable to think of our temperature $u(x,t)$ as the sum of two parts

$$u(x,t) = \underbrace{\text{steady state}}_{\text{Eventual Solution for Large Time}} + \underbrace{\text{transient}}_{\substack{\text{Part of the Solution that depends on the IC} \\ \text{(and will go to zero)}}}$$

$$\Downarrow$$

$$= \left[k_1 + \frac{x}{L}(k_2 - k_1)\right] + U(x,t)$$

- By substituting the above relation in the original problem Equation 12.4, we arrive at a new problem in $U(x,t)$:

$$U_t = \alpha U_{xx}, \quad 0 < x < L,\ 0 < t < \infty$$

$$\begin{matrix} U(0,t) = 0 \\ U(L,t) = 0 \end{matrix}, \quad 0 < t < \infty, \quad U(x,0) = \phi(x) - \left[k_1 + \frac{x}{L}(k_2 - k_1)\right] = \bar{\phi}(x), \quad 0 \le x \le L$$

- **Transforming Time Varying BCs to Zero BCs**
  - What about more realistic-type derivative BCs with time-varying right-hand sides? Consider the typical problem

$$\begin{aligned} u_t &= \alpha u_{xx} && 0 < x < L,\ 0 < t < \infty \\ u(0,t) &= g_1(t) && 0 < t < \infty \\ u_x(L,t) + hu(L,t) &= g_2(t) && \\ u(x,0) &= \phi(x) && 0 \le x \le L \end{aligned} \tag{12.5}$$

– To change these nonzero BCs to homogeneous ones, we (after some trial and error) seek a solution of the form:

$$u(x,t) = \underbrace{A(t)\left[1-\frac{x}{L}\right] + B(t)\frac{x}{L}}_{S(x,t)} + U(x,t)$$

where $A(t)$ and $B(t)$ are chosen so that $S(x,t)$ satisfies the BCs and thus

$$\begin{aligned} U(0,t) &= 0 \\ U_x(L,t) + hU(L,t) &= 0 \end{aligned}, \quad 0 < t < \infty$$

– Substituting $S(x,t)$ into the BCs gives

$$\begin{aligned} S(0,t) &= g_1(t) \\ S_x(L,t) + hS(L,t) &= g_2(t) \end{aligned}, \quad 0 < t < \infty$$

in which we get

$$\begin{aligned} A(t) &= g_1(t) \\ B(t) &= \frac{g_1(t) + Lg_2(t)}{1+Lh}, \end{aligned} \quad 0 < t < \infty$$

– So if we substitute this into the original problem Equation 12.5, we get the transformed problem in $U(x,t)$:

$$\begin{aligned} U_t &= \alpha U_{xx} - S_t && 0 < x < L,\ 0 < t < \infty \\ U(0,t) &= 0 && 0 < t < \infty \\ U_x(L,t) + hU(L,t) &= 0 \\ U(x,0) &= \phi(x) - S(x,0) && 0 \le x \le L \end{aligned}$$

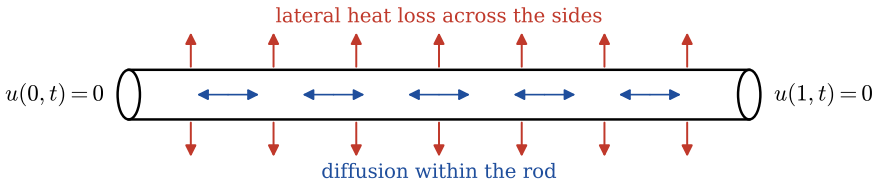
* We now have our new problem with zero BCs but unfortunately the PDE is nonhomogeneous
* We cannot solve this problem by separation of variables, but we will solve it by integral transforms and eigenfunction expansions

Example Solve the initial-boundary-value problem

$$\begin{aligned} u_t &= \alpha u_{xx} && 0 < x < 1,\ 0 < t < \infty \\ u(0,t) &= 1 && 0 < t < \infty \\ u_x(1,t) + hu(1,t) &= 1 \\ u(x,0) &= \sin \pi x + x && 0 \le x \le 1 \end{aligned}$$

Answer 2 (end of chapter) · Solution

12.8 Transforming Hard Equations into Easier Ones



- Consider the following problem:

$$\begin{aligned} u_t &= \alpha u_{xx} - \beta u \quad && 0 < x < 1,\ 0 < t < \infty \\ u(0,t) &= 0 && 0 < t < \infty \\ u(1,t) &= 0 \\ u(x,0) &= \phi(x) && 0 \le x \le 1 \end{aligned} \tag{12.6}$$

  where the term $-\beta u$ represents heat flow across the lateral boundary

- We introduce a new temperature $w(x,t)$ in place of $u(x,t)$, so that the PDE in w is simpler than the original one
- The transformation is generally based on an intuitive feeling of how the solution of the original PDE behaves
- In our problem Equation 12.6, the temperature $u(x,t)$ at any point x_0 is changing as a result of two phenomena

  1. **diffusion** of heat within the rod due to αu_{xx}
  2. **heat flow** across the lateral boundary due to $-\beta u$

- The important point is that if there were no diffusion within the rod ($\alpha = 0$) then the temperature at each point x_0 would decay exponentially to zero according to

$$u(x_0,t) = u(x_0,0)e^{-\beta t}$$

- By means of this observation, we can essentially decompose the temperature $u(x,t)$ into two factors

$$u(x,t) = e^{-\beta t} w(x,t)$$

  where $w(x,t)$ would represent the temperature due to diffusion only

- When we substitute this expression into Equation 12.6, we arrive at

$$\begin{aligned} w_t &= \alpha w_{xx} \quad && 0 < x < 1,\ 0 < t < \infty \\ w(0,t) &= 0 && 0 < t < \infty \\ w(1,t) &= 0 \\ w(x,0) &= \phi(x) && 0 \le x \le 1 \end{aligned}$$

- The diffusion-convection equation

$$u_t = \alpha u_{xx} - v u_x$$

(v is a constant) can also be transformed to

$$w_t = \alpha w_{xx}$$

In this case, the transformation is

$$u(x,t) = \exp\left[\frac{v}{2\alpha}\left(x - \frac{v}{2}t\right)\right] w(x,t)$$

Example Solve the diffusion-convection problem

$$\begin{aligned} u_t &= u_{xx} - u_x \quad && 0 < x < 1,\ 0 < t < \infty \\ u(0,t) &= 0 && 0 < t < \infty \\ u(1,t) &= 0 \\ u(x,0) &= e^{x/2} && 0 \le x \le 1 \end{aligned}$$

by transforming it into an easier one

$$\begin{aligned} w_t &= w_{xx} \quad && 0 < x < 1,\ 0 < t < \infty \\ w(0,t) &= 0 && 0 < t < \infty \\ w(1,t) &= 0 \\ w(x,0) &= 1 && 0 \le x \le 1 \end{aligned}$$

Solution

$$\begin{aligned} u(x,t) &= \exp\left[\frac{1}{2}\left(x - \frac{1}{2}t\right)\right] w(x,t) \\ &\Downarrow \\ w_t = w_{xx},\ w(0,t) &= 0,\ w(1,t) = 0,\ w(x,0) = 1 \\ &\Downarrow \\ w(x,t) &= 2\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} e^{-n^2\pi^2 t} \sin n\pi x \\ &\Downarrow \\ u(x,t) &= 2\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} e^{-n^2\pi^2 t + \frac{1}{2}(x-\frac{1}{2}t)} \sin n\pi x \\ &= 4\sum_{m=1}^{\infty} \frac{1}{(2m-1)\pi} e^{-(2m-1)^2\pi^2 t + \frac{1}{2}(x-\frac{1}{2}t)} \sin(2m-1)\pi x \end{aligned}$$

```
def example_plot4(t, x, n_terms, u_solution):
    plt.figure(figsize=(6, 4))
    plt.plot(x, np.exp(x /2.0), 'r:', label='$t=0$')

    for tt in t:
        plt.plot(x, u_solution(tt, x, n_terms), label='$t=%0.3f$' % tt)

    plt.xlim(0, 1)
    plt.ylim(0, 2)
    plt.legend(loc='upper left', bbox_to_anchor=(1.0, 1.0))
    plt.xlabel('$x$')
    plt.ylabel('$u(x,t)$')
    plt.title(r'$\alpha=1,\;v=1$')

    plt.show()
```

```
def u_solution(t, x, n_terms):
    u = 0
    for m in range(1, n_terms +1):
        a = (2.0 *m -1.0) *np.pi
        u += 1 /a *np.exp(-a *a *t +(x -t /2.0) /2.0) *np.sin(a *x)
    return 4.0*u

x = np.linspace(0, 1, 150)
t = np.array([0.001, 0.002, 0.004, 0.008, 0.05, 0.10, 0.15, 0.20])

n_terms = 25

example_plot4(t, x, n_terms, u_solution)
```

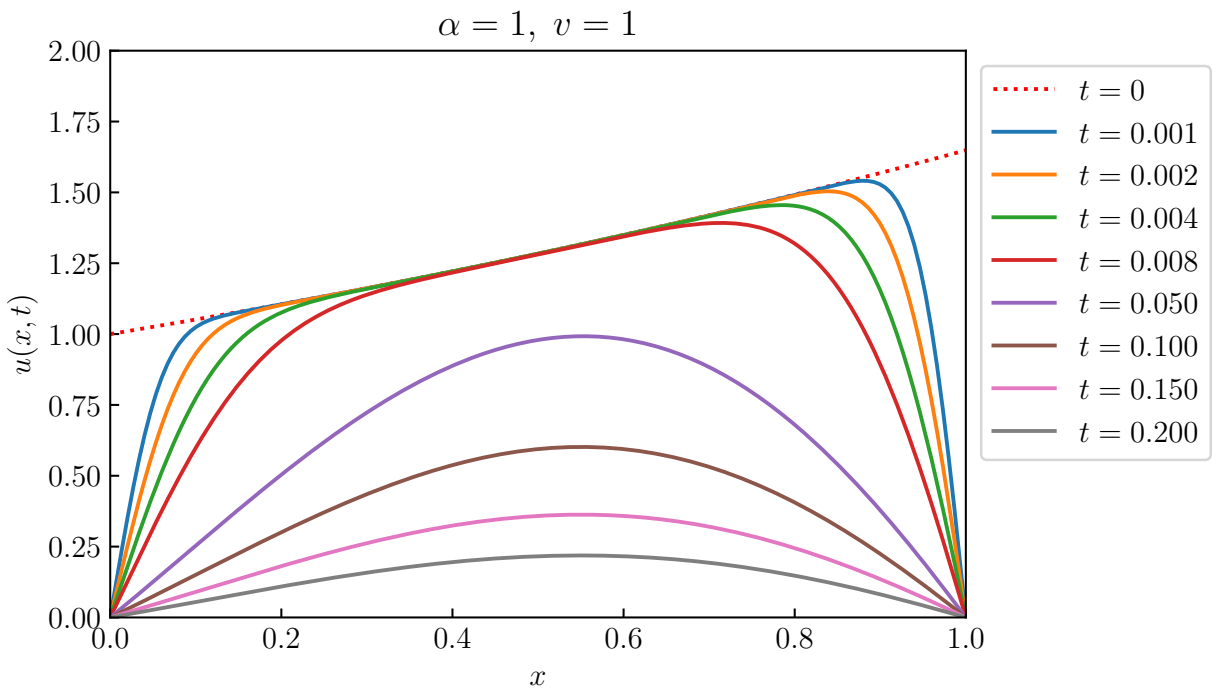


Figure 12.4: Solutions

12.9 Solving Nonhomogeneous PDEs (Eigenfunction Expansion)

- Consider the nonhomogeneous problem

$$\begin{aligned} u_t &= \alpha u_{xx} + f(x,t) && 0 < x < 1,\ 0 < t < \infty \\ u(0,t) &= 0 && 0 < t < \infty \\ u(1,t) &= 0 \\ u(x,0) &= \phi(x) && 0 \le x \le 1 \end{aligned} \tag{12.7}$$

  The purpose of this section is to solve this problem by a method that is analogous to the method of variation of parameters in ODEs and is known as the eigenfunction expansion

STEP 1

- The basic idea in this method is to decompose the heat source $f(x,t)$ into simple components

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) X_n(x)$$

  It turns out that the $X_n(x)$ are the **eigenfunctions of the Sturm-Liouville system** we get when solving the associated homogeneous system by separation of variables

- In this case, the Sturm-Liouville problem we find when separating variables is

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) &= 0 \\ X(1) &= 0 \end{aligned}$$

  and, hence, the $X_n(x)$ are

$$X_n(x) = \sin n\pi x,\ n = 1, 2, 3, \cdots$$

- Finally, to find the functions $f_n(t)$, we merely multiply each side of this equation by $\sin m\pi x$ and integrate from zero to one:

$$\int_0^1 f(x,t) \sin m\pi x\, dx = \sum_{n=1}^{\infty} f_n(t) \int_0^1 \sin m\pi x \, \sin n\pi x\, dx = \frac{1}{2} f_m(t)$$

$$\Downarrow \text{ changing } m \text{ to } n$$

$$f_n(t) = 2 \int_0^1 f(x,t) \sin n\pi x\, dx$$

STEP 2

- The responses $u_n(x,t) = T_n(t)\sin n\pi x$ to each of these individual components $f_n(t)\sin n\pi x$ are added to construct the solution of our problem:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin n\pi x$$

Substituting $u(x,t)$ and $f(x,t)$ into Equation 12.7 gives us

$$\sum_{n=1}^{\infty} T_n'(t)\sin n\pi x = -\alpha\sum_{n=1}^{\infty}(n\pi)^2\,T_n(t)\sin n\pi x + \sum_{n=1}^{\infty} f_n(t)\sin n\pi x$$
$$\sum_{n=1}^{\infty} T_n(0)\sin n\pi x = \phi(x)$$
$$\Downarrow$$
$$\sum_{n=1}^{\infty} \underbrace{[T_n' + \alpha(n\pi)^2\,T_n - f_n(t)]}_{=0}\sin n\pi x = 0$$
$$T_n(0) = 2\int_0^1 \phi(x)\sin n\pi x\,dx = a_n$$
$$\Downarrow$$
$$T_n(t) = a_n e^{-\alpha(n\pi)^2 t} + \int_0^t e^{-\alpha(n\pi)^2(t-\tau)} f_n(\tau)\,d\tau$$

Hence, the solution of our problem Equation 12.7 is

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin n\pi x = \underbrace{\sum_{n=1}^{\infty} a_n e^{-\alpha(n\pi)^2 t}\sin n\pi x}_{\text{Transient Part due to the IC}} + \underbrace{\sum_{n=1}^{\infty}\int_0^t e^{-\alpha(n\pi)^2(t-\tau)} f_n(\tau)\,d\tau\cdot\sin n\pi x}_{\text{Forcing Part due to } f(x,t)}$$

- The eigenfunctions $X_n(x)$ in the expansion change from problem to problem and depend on the PDE and BCs

Example Find the solution

$$\begin{aligned} u_t &= u_{xx} + \sin\sqrt{\lambda_1}x \quad && 0 < x < 1,\ 0 < t < \infty \\ u(0,t) &= 0 && 0 < t < \infty \\ u_x(1,t) + hu(1,t) &= 0 \\ u(x,0) &= 0 && 0 \le x \le 1 \end{aligned}$$

where $h = 1$ and λ_1 is the first root of the equation $\tan\sqrt{\lambda} = -\sqrt{\lambda}$

Answer 3 (end of chapter) · Solution

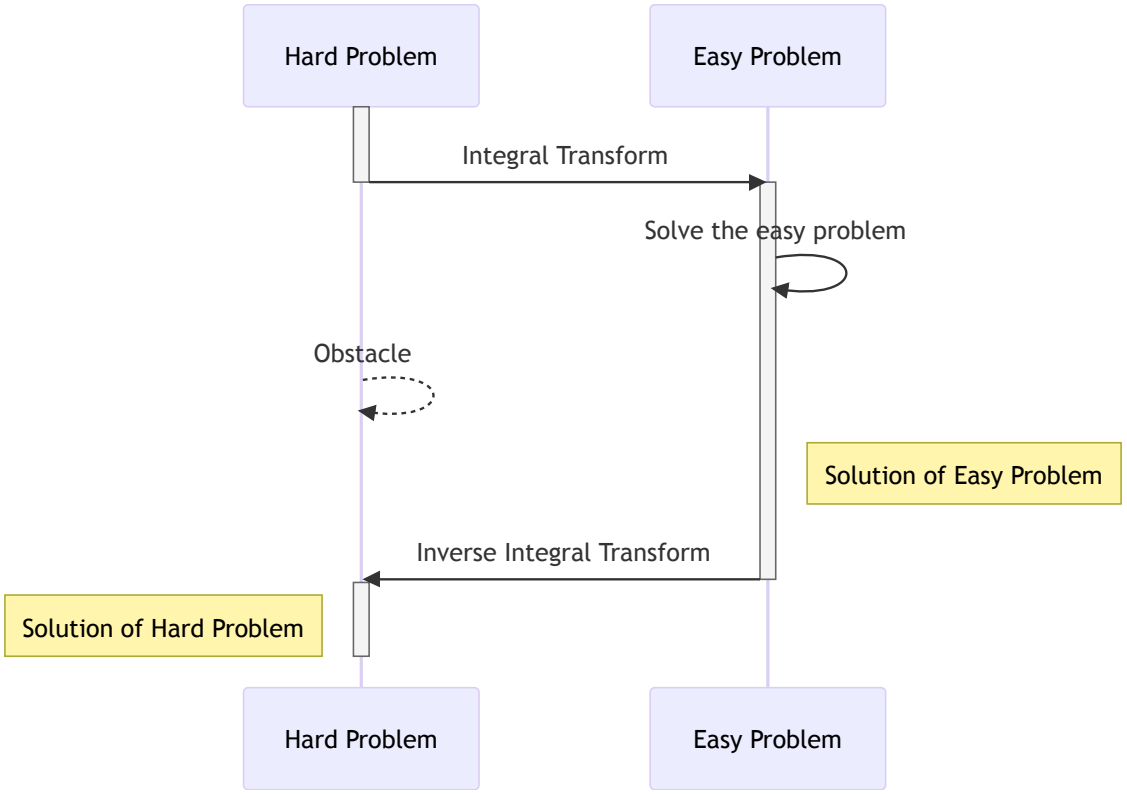
12.10 The Finite Sine and Cosine Transforms

- An integral transformation is merely a transformation that assigns to one function $f(t)$ a new function $F(s)$ by means of a formula like

$$F(s) = \int_a^b K(s,t)\, f(t)\, dt$$

  Note that we start with a function of t and end with a function of s

- The function $K(s,t)$ is called the **kernel** of the transformation and is the major ingredient that distinguishes one transform from another; it is chosen so that the transform has certain desirable properties. The limits a and b also change from transformation to transformation
- With every integral transform, there is an inverse transform that will reproduce that original function from its transform. The transform and its inverse together form what is called **a transform pair**
- The general philosophy behind integral transformation is that they eliminate *partial derivatives* with respect to one of the variables; hence, the new equation has one less variable
- In other words, integral transformations change problems into easier ones. The transformed problem is then solved, and its inverse is obtained to find the solution to the original problem



- We first start with a function $f(x)$ defined on an interval $[0, L]$. The finite sine and cosine transforms of this function are defined by

$$\mathcal{F}_s[f] = \frac{2}{L}\int_0^L f(x)\,\sin\frac{n\pi x}{L}\,dx = b_n \qquad \text{Finite Sine Transform}$$

$$\mathcal{F}_c[f] = \frac{2}{L}\int_0^L f(x)\,\cos\frac{n\pi x}{L}\,dx = a_n \qquad \text{Finite Cosine Transform}$$

- The student will note that these transforms do nothing more than transform a function into the Fourier sine and cosine coefficients. The inverse transform of these transforms are the Fourier sine and cosine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad \text{Inverse Sine Transform}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \qquad \text{Inverse Cosine Transform}$$

  Note that the inverse cosine transform includes the $n = 0$ term $a_0/2$, while the inverse sine starts at $n = 1$

Examples of the Sine Transform

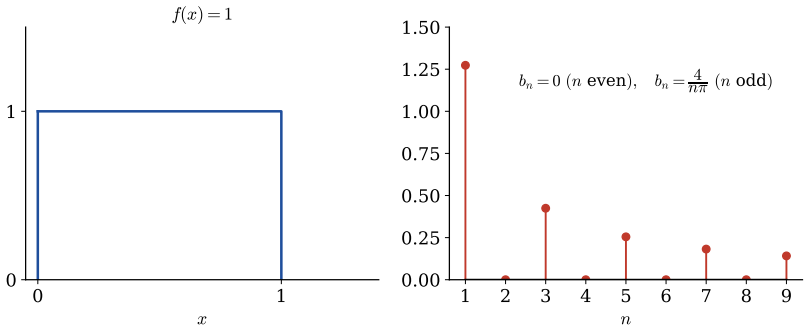
$$f(x) = 1, \quad 0 \le x \le 1$$

$$\Downarrow$$

$$b_n = 2\int_0^1 \sin n\pi x\,dx = \begin{cases} 0 & n \text{ even} \\ 4/n\pi & n \text{ odd} \end{cases}$$

$$\Downarrow$$

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{2n-1}\right]\sin(2n-1)\pi x$$



- **Properties of the Finite Sine and Cosine Transforms**

$$\mathcal{F}_s[u_t] = \frac{d\mathcal{F}_s[u]}{dt}$$
$$\mathcal{F}_s[u_{tt}] = \frac{d^2\mathcal{F}_s[u]}{dt^2}$$

$$\mathcal{F}_s[u_x] = -\frac{n\pi}{L}\mathcal{F}_c[u]$$
$$\mathcal{F}_s[u_{xx}] = -\left(\frac{n\pi}{L}\right)^2 \mathcal{F}_s[u] + \frac{2n\pi}{L^2}\left[u(0,t) - (-1)^n u(L,t)\right]$$
$$\mathcal{F}_c[u_x] = \frac{n\pi}{L}\mathcal{F}_s[u] + \frac{2}{L}\left[(-1)^n u(L,t) - u(0,t)\right]$$
$$\mathcal{F}_c[u_{xx}] = -\left(\frac{n\pi}{L}\right)^2 \mathcal{F}_c[u] + \frac{2}{L}\left[(-1)^n u_x(L,t) - u_x(0,t)\right]$$

Example Solve the general problem

$$\begin{aligned} u_t &= \alpha u_{xx} - \beta u + f(x,t) && 0 < x < 1,\ 0 < t < \infty \\ u(0,t) &= 0 && 0 < t < \infty \\ u(1,t) &= 0 && \\ u(x,0) &= 0 && 0 \le x \le 1 \end{aligned}$$

Answer 4 (end of chapter) · Solution

12.11 Superposition (The Backbone of Linear Systems)

- For an engineer who wishes to find the response u to a linear system from input f, a common approach is

  1. Break f into elementary parts, $f = \sum f_k$
  2. Find the system response u_k to f_k
  3. Add(superimpose) the simple responses u_k to get $u = \sum u_k$

- It turns out if the system is *linear*, then the sum u is the response we get if the function f were input directly; this is the **principle of superposition**

- **Superposition Used to Break an IBVP into Two Simpler Problems**

  Suppose we have the linear problem

$$\begin{aligned} u_t &= u_{xx} + \sin \pi x && 0 < x < 1,\ 0 < t < \infty \\ u(0,t) &= 0 && 0 < t < \infty \\ u(1,t) &= 0 && \\ u(x,0) &= \sin 2\pi x && 0 \le x \le 1 \end{aligned} \tag{P}$$

  Here, we have a *nonhomogeneous* heat equation, so separation of variables is not a viable method of attack

– We could, of course, use the finite sine transform on the variable x or the Laplace transform on t, but still another idea would be to consider two subproblems

$$\begin{aligned} u_t &= u_{xx} + \sin \pi x && 0 < x < 1,\ 0 < t < \infty \\ u(0,t) &= 0 && 0 < t < \infty \\ u(1,t) &= 0 && \\ u(x,0) &= 0 && 0 \le x \le 1 \end{aligned} \tag{A}$$

and

$$\begin{aligned} u_t &= u_{xx} && 0 < x < 1,\ 0 < t < \infty \\ u(0,t) &= 0 && 0 < t < \infty \\ u(1,t) &= 0 && \\ u(x,0) &= \sin 2\pi x && 0 \le x \le 1 \end{aligned} \tag{B}$$

– These two problems can be solved individually with a little effort, and it should be clear here that the *sum* of the solutions to (A) and (B) is the solution to the original problem (P); that is

$$u(x,t) = \underbrace{\frac{1}{\pi^2}(1 - e^{-\pi^2 t}) \sin \pi x}_{\text{Solution to (A)}} + \underbrace{e^{-(2\pi)^2 t} \sin 2\pi x}_{\text{Solution to (B)}}$$

- **Separation of Variables and Integral Transforms as Superpositions**

  – In separation of variables, we generally break down the initial conditions into an infinite number of simple parts and find the response to each part. We then sum these individual responses to find the solution to the problem

  – On the other hand, integral transforms also use superposition, for instance, let's show how the finite sine transform uses this principle. Consider the nonhomogeneous heat equation

$$\begin{aligned} u_t &= u_{xx} + f(x,t) && 0 < x < 1,\ 0 < t < \infty \\ u(0,t) &= 0 && 0 < t < \infty \\ u(1,t) &= 0 && \\ u(x,0) &= 0 && 0 \le x \le 1 \end{aligned}$$

  and its solution by use of the finite sine transform. What we're really doing is resolving the input $f(x,t)$ into components, finding the response U_n due to each component, and adding these responses

$$
\begin{aligned}
& u_t = u_{xx} + f(x,t) \\
& \quad \Downarrow \text{ finite sine transform, } u(0,t)=u(1,t)=0 \\
& \sum_{n=1}^{\infty} \left[U_n' + (n\pi)^2 U_n - F_n(t)\right] \sin n\pi x = 0 \\
& \quad \Downarrow \ u(x,0)=0 \\
& U_n'(t) + (n\pi)^2 U_n(t) = F_n(t), \quad U_n(0) = 0 \quad n = 1,2,3,\cdots \\
& \quad \Downarrow \\
& U_n(t) = e^{-(n\pi)^2 t} \int_0^t e^{(n\pi)^2 \tau} F_n(\tau)\, d\tau = \int_0^t e^{-(n\pi)^2 (t-\tau)} F_n(\tau)\, d\tau \\
& \quad \Downarrow \\
& u(x,t) = \sum_{n=1}^{\infty} U_n(t) \sin n\pi x
\end{aligned}
$$

NOTES

In the finite sine transform, the resolutions were infinite series, whereas in most other integral transforms, the resolutions are integrals (continuous resolutions)

Example Find four initial-boundary-value problems whose solutions sum to the solution of the following problem:

$$
\begin{aligned}
u_t &= u_{xx} + f(x,t) && 0 < x < 1,\ 0 < t < \infty \\
u(0,t) &= g_1(t) && 0 < t < \infty \\
u(1,t) &= g_2(t) && \\
u(x,0) &= \phi(x) && 0 \le x \le 1
\end{aligned}
$$

Answer 5 (end of chapter) · Solution

Example Solve the problem

$$
\begin{aligned}
u_t &= u_{xx} + \sin 3\pi x && 0 < x < 1,\ 0 < t < \infty \\
u(0,t) &= 0 && 0 < t < \infty \\
u(1,t) &= 1 && \\
u(x,0) &= \sin \pi x && 0 \le x \le 1
\end{aligned}
$$

Answer 6 (end of chapter) · Solution

12.12 The Fourier Integral and Transform

- An integral transformation can be thought of as a resolution of a function into a certain spectrum of components:
- Let's consider the resolution of a periodic function $f(x)$ into Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

- Here the coefficients a_n and b_n represent the amount of the function $f(x)$ made up of $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$, respectively
- While the square root

$$\sqrt{a_n^2 + b_n^2}$$

(called the spectrum of the function) measures the amount of $f(x)$ with frequency $\frac{n\pi}{L}$

- Functions that are *periodic* can be resolved into *infinite series* (they have discrete spectra)
- Of course, if a function is defined only on a *finite interval*, we could extend the function outside the interval in a periodic way, so that a Fourier series representation could be obtained for the function inside the interval
- Whereas functions that are *not periodic* must be resolved into a *continuous spectrum* of values
- Although a nonperiodic function $f(x)$ cannot be represented by an infinite series of sines and cosines, we might write it as *continuous analog* of the Fourier series:

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] \\
&= \frac{1}{2L} \int_{-L}^{L} f(\xi)\, d\xi + \frac{1}{L} \sum_{n=1}^{\infty} \left[\left(\int_{-L}^{L} f(\xi) \cos \frac{n\pi\xi}{L}\, d\xi \right) \cos \frac{n\pi x}{L} + \left(\int_{-L}^{L} f(\xi) \sin \frac{n\pi\xi}{L}\, d\xi \right) \sin \frac{n\pi x}{L} \right] \\
&\Downarrow\ \omega_n = \frac{n\pi}{L},\ \Delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{L} \\
&= \frac{1}{2\pi} \left(\int_{-L}^{L} f(\xi)\, d\xi \right) \Delta\omega + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\left(\int_{-L}^{L} f(\xi) \cos \omega_n \xi\, d\xi \right) \cos \omega_n x + \left(\int_{-L}^{L} f(\xi) \sin \omega_n \xi\, d\xi \right) \sin \omega_n x \right] \\
&\Downarrow\ L \to \infty,\ \Delta\omega \to 0,\ \lim_{\Delta\omega \to 0} \sum_{n=1}^{\infty} F(\omega_n)\, \Delta\omega = \int_0^{\infty} F(\omega)\, d\omega
\end{aligned}$$

$$f(x) = \int_0^{\infty} \left[a(\omega) \cos \omega x + b(\omega) \sin \omega x \right] d\omega$$

$$a(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \omega x\, dx$$

$$b(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x\, dx$$

where the functions $a(\omega)$ and $b(\omega)$ are known as the Fourier cosine and sine transforms and measure the cosine and sine component of $f(x)$ and

$$\sqrt{a^2(\omega) + b^2(\omega)}$$

measures the ω frequency component of $f(x)$ and is called the spectrum of $f(x)$. With this intuitive explanation of the spectrum of a function, we now get to the nuts and bolts of integral transforms

- **The Exponential Fourier Transform**

  We are now in a position to define what is generally known as the exponential Fourier transform:

$$\begin{aligned}
f(x) &= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(\xi)\left[\cos\omega\xi\cos\omega x + \sin\omega\xi\sin\omega x\right]\, d\xi\, d\omega \\
&= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(\xi) \underbrace{\cos\omega(\xi - x)}_{\cos\omega(x-\xi)\,-\,\text{even function w.r.t. }\omega} d\xi\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(\xi)\cos\omega(x-\xi)\, d\xi\, d\omega = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(\xi)\left[\cos\omega(x-\xi) + \underbrace{i\sin\omega(x-\xi)}_{\text{odd in }\omega\text{: integrates to 0}}\right] d\xi\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(\xi)e^{i\omega(x-\xi)}\, d\xi\, d\omega \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(\xi)e^{-i\omega\xi}\, d\xi\right] e^{i\omega x}\, d\omega
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
\mathcal{F}[f(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{-i\omega x}\, dx = F(\omega) \\
\mathcal{F}^{-1}[F(\omega)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(\omega)e^{i\omega x}\, d\omega = f(x)
\end{aligned}$$

Try it in Python: Fourier and Laplace transforms with `sympy`: Integral transforms

12.13 The Fourier Transform and its Application to PDEs

- The usefulness of the Fourier transform comes from the fact that it changes the operation of differentiation into multiplication; that is, partial differential equations are changed into ordinary differential equations

- **Useful Properties of the Fourier Transform**

  Property 1 Fourier Transform Pair

  The Fourier transform of $f(x), -\infty < x < \infty,$ produces a new function $F(\omega)$:

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx = F(\omega)$$

  and the inverse Fourier transform of $F(\omega), -\infty < \omega < \infty,$ will produce the original function $f(x)$:

$$\mathcal{F}^{-1}[F(\omega)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x}\, d\omega = f(x)$$

  For example,

$$e^{-|x|} \xrightarrow{\mathcal{F}} \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2} \xrightarrow{\mathcal{F}^{-1}} e^{-|x|}$$

  Property 2 Linear Transformation

$$\mathcal{F}[af(x) + bg(x)] = a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)]$$

  Property 3 Transformation of Partial Derivatives

$$\mathcal{F}[f_x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_x e^{-i\omega x}\, dx = i\omega \mathcal{F}[f]$$

$$\mathcal{F}[f_{xx}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_{xx} e^{-i\omega x}\, dx = -\omega^2 \mathcal{F}[f]$$

  Property 4 Convolution Property

  Every integral transform has what is called a convolution property. The general idea is that the transform of a product of two functions $f(x)g(x)$ is not the product of the individual transforms:

$$\mathcal{F}[f(x)g(x)] \neq \mathcal{F}[f(x)]\mathcal{F}[g(x)]$$

However, in transform theory, there is something called the **convolution** $f * g$ of two functions that more or less plays the role of the product:

$$\mathcal{F}[f * g] = \mathcal{F}[f(x)]\mathcal{F}[g(x)]$$

So what is this mysterious convolution $f * g$? It's given by formula

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\, d\xi$$

The importance of the convolution in applications is due to the fact that quite often, the final step in solving a PDE boils down to finding the inverse transform of some expression that we can interpret as the product of two transforms $\mathcal{F}[f]\mathcal{F}[g]$

- **Solution of an Initial-Value Problem**

  Consider the heat flow in an infinite rod where the initial temperature is $u(x,0) = \phi(x)$

$$\begin{aligned} u_t &= \alpha u_{xx} \quad && -\infty < x < \infty,\ 0 < t < \infty \\ u(x,0) &= \phi(x) && -\infty < x < \infty \end{aligned}$$

STEP 1 Transforming the Problem

$$\begin{aligned} \mathcal{F}[u_t] &= \alpha\mathcal{F}[u_{xx}] \\ \mathcal{F}[u(x,0)] &= \mathcal{F}[\phi(x)] \\ &\Downarrow \\ \frac{dU(t)}{dt} &= -\alpha\omega^2 U(t) \\ U(0) &= \Phi(\omega) \end{aligned}$$

STEP 2 Solving the Transformed Problem

$$U(t) = \Phi(\omega)e^{-\alpha\omega^2 t}$$

STEP 3 Finding the Inverse Transform

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left[\Phi(\omega)e^{-\alpha\omega^2 t}\right] \\ &= \mathcal{F}^{-1}\left[\Phi(\omega)\right] * \mathcal{F}^{-1}\left[e^{-\alpha\omega^2 t}\right] \\ &= \phi(x) * \frac{1}{\sqrt{2\alpha t}} e^{-x^2/4\alpha t} \\ &= \frac{1}{2\sqrt{\alpha\pi t}} \int_{-\infty}^{\infty} \phi(\xi) e^{-(x-\xi)^2/4\alpha t}\, d\xi \end{aligned}$$

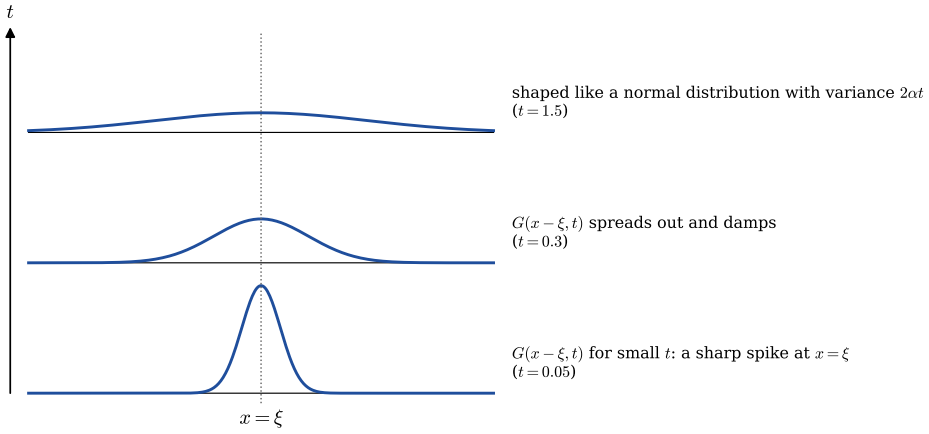
Note that the integrand is made up of two terms:

1. The initial temperature $\phi(x)$

2. The Green's function or impulse-response function

$$G(x-\xi,t)=\frac{1}{2\sqrt{\alpha\pi t}}e^{-(x-\xi)^2/4\alpha t},\quad \lim_{t\to 0^+}G(x-\xi,t)=\underbrace{\delta(x-\xi)}_{\text{Dirac delta function}}$$

It can be shown that this impulse-response function $G(x-\xi,t)$ is the temperature response to an initial temperature impulse at $x=\xi$



Hence, the interpretation of solution is that the initial temperature $u(x,0)=\phi(x)$ is decomposed into a continuum of impulses of $\phi(\xi)$ at each point $x=\xi$ and the resulting temperature $\phi(\xi)G(x-\xi,t)$ is found. These resulting temperatures are then added (integrated) to obtain the solution. This general idea is also superposition

```
def example_plot5(alpha, xi, t, x):

    plt.figure(figsize=(7, 5))

    for tt in t:
      plt.plot(x, 0.5 /np.sqrt(alpha *np.pi *tt)
            *np.exp(-(x -xi)**2 /(4.0 *alpha *tt)),
          label=f'$t={tt:0.2f}$')
    plt.legend(loc='upper right')

    plt.text(x[0] +1, 1.5,
      r'$$G(x-\xi,t)=\frac{1}{2\sqrt{\alpha\pi t}}'
      r'e^{-(x-\xi)^2/4\alpha t}$$')

    plt.xlim(x[0], x[-1])
    plt.ylim(0, 2)

    plt.xlabel('$x$')
    plt.ylabel(r'$G(x-\xi,t)$')

    plt.title(rf'$\alpha={alpha:2.1f}, \;\xi={xi:2.1f}$')

    plt.show()
```

```
alpha = 1
xi = 0

t = [0.05, 0.1, 0.2, 0.4, 0.8, 1.6, 3.2]
x = np.linspace(-6, 6, 1000)

example_plot5(alpha, xi, t, x)
```

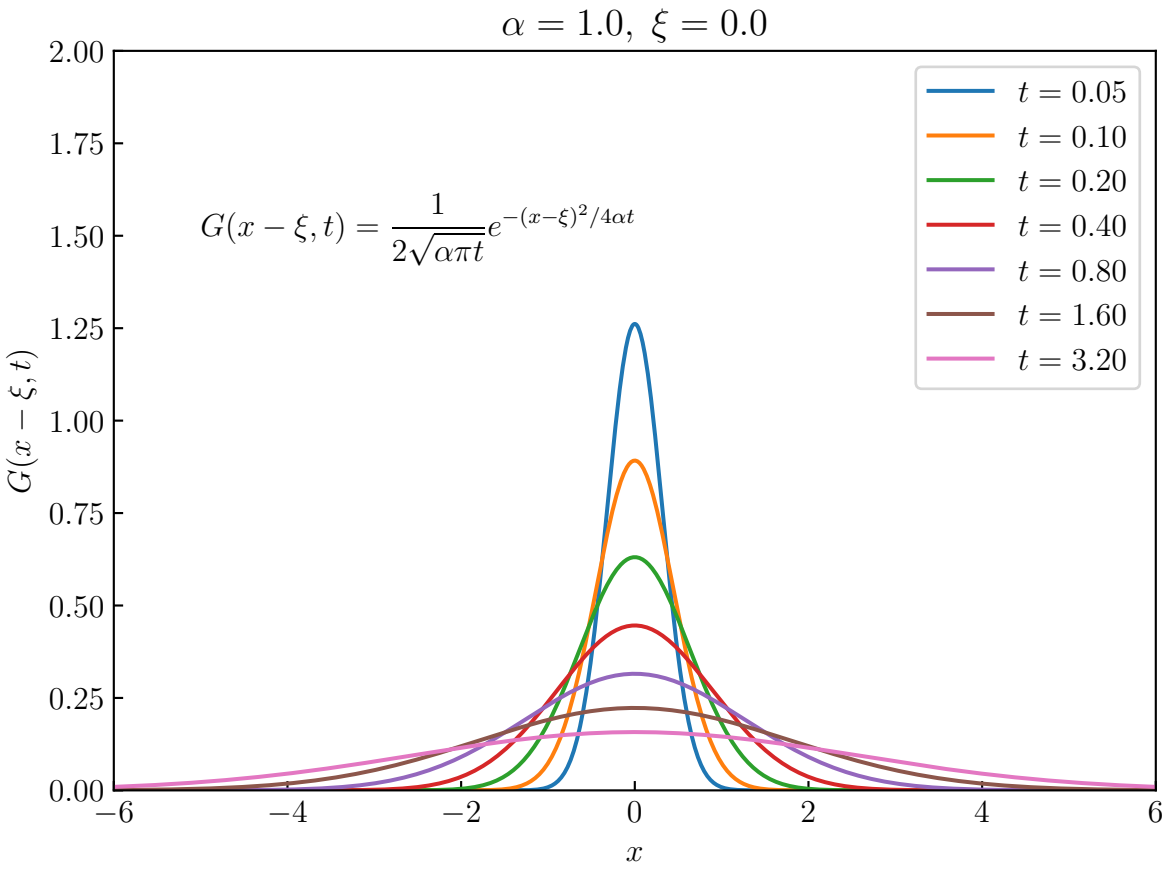


Figure 12.5: Impulse-response function

NOTES

The major drawback of the Fourier transform is that not all functions can be transformed. For example, even simple functions like

$$f(x) = \text{constant}$$
$$f(x) = e^x$$
$$f(x) = \sin x$$

cannot be transformed, since the integral

$$\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$$

does not exist. Only functions that damp to zero sufficiently fast as $|x| \to \infty$ have transforms

12.14 Sine and Cosine Transforms

- **Fourier Sine Transform Pair**

$$\mathcal{F}_s[f(x)] = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx = F(\omega)$$
$$\mathcal{F}_s^{-1}[F(\omega)] = \int_0^\infty F(\omega)\sin\omega x\,d\omega = f(x)$$

- **Fourier Cosine Transform Pair**

$$\mathcal{F}_c[f(x)] = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx = F(\omega)$$
$$\mathcal{F}_c^{-1}[F(\omega)] = \int_0^\infty F(\omega)\cos\omega x\,d\omega = f(x)$$

- The next step is to list a few properties of these transforms that make them work

$$\begin{aligned}\mathcal{F}_s[f'(x)] &= \frac{2}{\pi}\int_0^\infty f'(x)\sin\omega x\,dx \\ &= \frac{2}{\pi}f(x)\sin\omega x\Big|_0^\infty - \omega\frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx = -\omega\mathcal{F}_c[f(x)]\end{aligned}$$

$$\begin{aligned}\mathcal{F}_c[f'(x)] &= \frac{2}{\pi}\int_0^\infty f'(x)\cos\omega x\,dx \\ &= \frac{2}{\pi}f(x)\cos\omega x\Big|_0^\infty + \omega\frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx = -\frac{2}{\pi}f(0) + \omega\mathcal{F}_s[f(x)]\end{aligned}$$

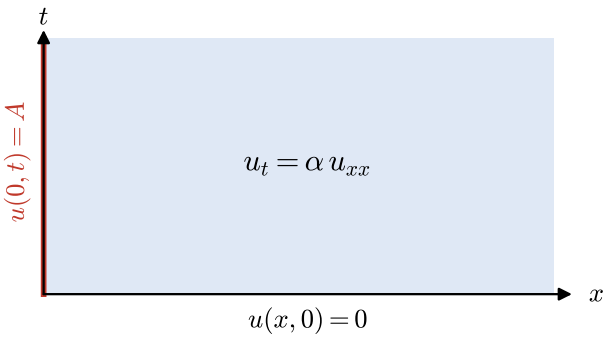
$$\begin{aligned}\mathcal{F}_s[f''(x)] &= \frac{2}{\pi}\int_0^\infty f''(x)\sin\omega x\,dx = \frac{2}{\pi}f'(x)\sin\omega x\Big|_0^\infty - \omega\frac{2}{\pi}\int_0^\infty f'(x)\cos\omega x\,dx \\ &= \frac{2}{\pi}\omega f(0) - \omega^2\mathcal{F}_s[f(x)]\end{aligned}$$

$$\begin{aligned}\mathcal{F}_c[f''(x)] &= \frac{2}{\pi}\int_0^\infty f''(x)\cos\omega x\,dx = \frac{2}{\pi}f'(x)\cos\omega x\Big|_0^\infty + \omega\frac{2}{\pi}\int_0^\infty f'(x)\sin\omega x\,dx \\ &= -\frac{2}{\pi}f'(0) - \omega^2\mathcal{F}_c[f(x)]\end{aligned}$$

- **Solution of a Semi-infinite Diffusion Problem via the Sine Transform**

  The problem we are interested in is the *semi-infinite diffusion problem*

$$\begin{aligned} u_t &= \alpha u_{xx} && 0 < x < \infty,\ 0 < t < \infty \\ u(0,t) &= A && 0 < t < \infty \\ u(x,0) &= 0 && 0 \le x < \infty \end{aligned}$$



To solve this, we break into three simple steps. First our strategy is to transform the x-variable via Fourier sine transform so that we get an ODE in time

STEP 1

We start by transforming each side of the PDE;

$$\begin{aligned}
\mathcal{F}_s[u_t] &= \alpha \mathcal{F}_s[u_{xx}] \\
&\Downarrow \ \mathcal{F}_s[u(x,t)] = U(t) \\
\frac{d}{dt}U(t) &= \alpha\left[\frac{2}{\pi}\omega u(0,t) - \omega^2 \mathcal{F}_s[u]\right] = \alpha\left[A\frac{2}{\pi}\omega - \omega^2 U(t)\right] \\
&\Downarrow \ \mathcal{F}_s[u(x,0)] = U(0) \\
\frac{dU}{dt} &+ \alpha\omega^2 U = A\frac{2}{\pi}\alpha\omega, \ U(0) = 0
\end{aligned}$$

We arrive at the ODE

STEP 2

To solve this IVP, we could use a variety of elementary techniques from ordinary differential equations. The solution is

$$U(t) = A\frac{2}{\pi}\left[\frac{1-e^{-\alpha\omega^2 t}}{\omega}\right]$$

We have now found the sine transformation for the answer $u(x,t)$

STEP 3

To find the solution, we can either evaluate the inverse transform directly from the integral or else resort to the tables. Using the tables, we get

$$u(x,t) = A\,\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

where $\text{erfc}(\eta),\ 0<\eta<\infty,$ is called the complementary-error function and is given by

$$\text{erfc}(\eta) = \frac{2}{\sqrt{\pi}}\int_{\eta}^{\infty} e^{-\beta^2}\, d\beta$$

```
from scipy.special import erf, erfc

def example_plot6(A, alpha, t, x):
    plt.figure(figsize=(7, 5))
    for tt in t:
        plt.plot(x, A *erfc(x /(2 *np.sqrt(alpha *tt))),
          label=rf'$t={tt:0.1f}$')
    plt.xlim(0, x[-1])
    plt.ylim(0, 1.2)
    plt.legend(loc='upper right')
    plt.xlabel('$x$')
    plt.ylabel('$u(x,t)$')
    plt.text(2, 0.8,
      r'$$u(x,t)=A\, \mathrm{erfc}'
      r'\left( \dfrac{x}{2\sqrt{\alpha t}} \right)$$')
    plt.title(rf'$A={A:3.1f}, \; \alpha={alpha:3.1f}$')
    plt.show()

A = 1
alpha = 1

t = [0.1, 0.2, 0.4, 0.8, 1.6, 3.2]
x = np.linspace(0, 8)

example_plot6(A, alpha, t, x)
```

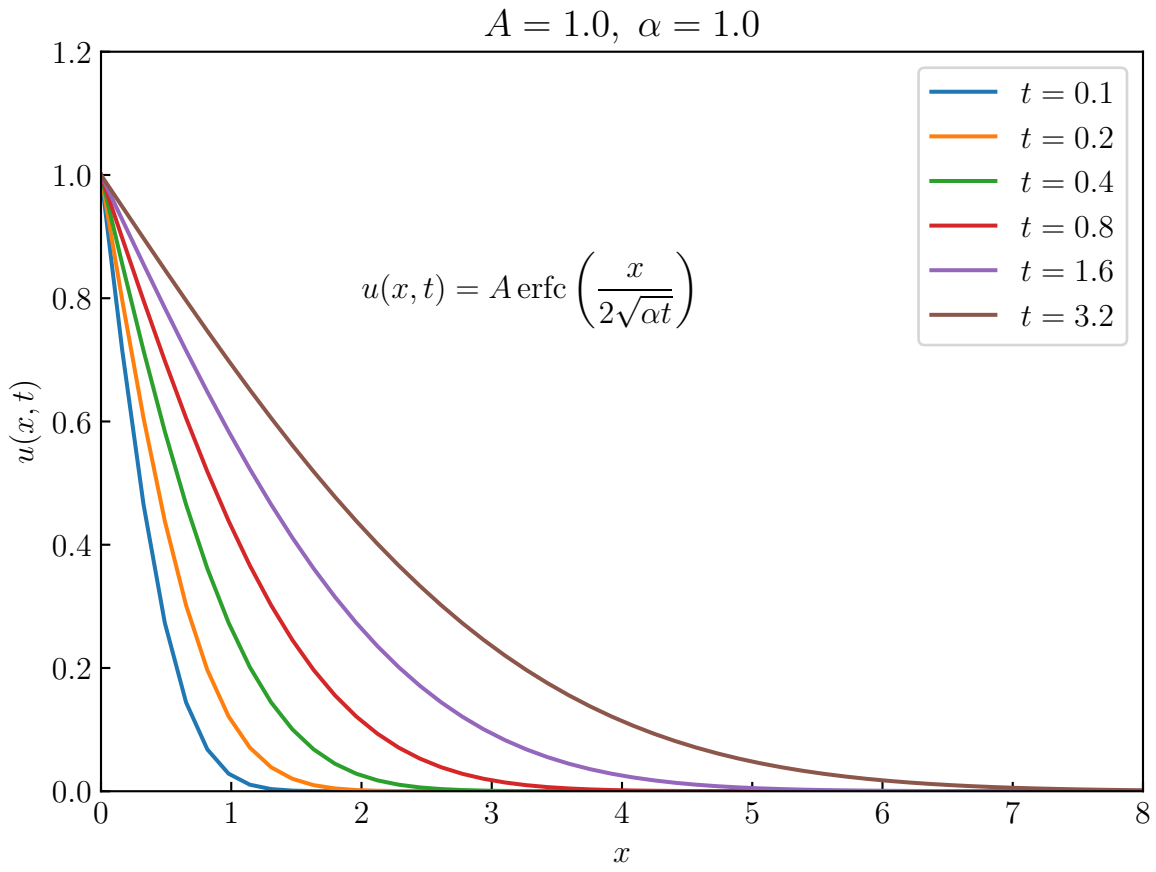


Figure 12.6: Heat conduction in semi-infinite domain with fixed temperature at the boundary

Example Solve by means of the cosine transform

$$\begin{aligned} u_t &= \alpha u_{xx} && 0 < x < \infty,\ 0 < t < \infty \\ u_x(0,t) &= 0 && 0 < t < \infty \\ u(x,0) &= H(1-x) && 0 \le x < \infty \end{aligned}$$

where $H(x)$ is the Heaviside function

Answer 7 (end of chapter) · Solution

```
x0 = [0, 1, 1, 8]
u0 = [1, 1, 0, 0]

def example_plot6b(alpha, t, x, u_solution):
    plt.figure(figsize=(7, 5))
    plt.plot(x0, u0, 'r:', label='$t=0.0$')
    for tt in t:
        u = u_solution(alpha, tt, x)
        plt.plot(x, u, label='$t=%0.1f$' % tt)
    plt.xlim(0, x[-1])
    plt.ylim(0, 1.1)
```

```
    plt.legend(loc='upper right')
    plt.xlabel('$x$')
    plt.ylabel('$u(x,t)$')
    plt.text(2, 0.35,
        (r'$$u(x,t)=\dfrac{2}{\pi}\,'
         r'\int_0^\infty \dfrac{\sin\omega}{\omega}\,'
         r'\exp(-\alpha\omega^2 t)\, \cos\omega x \,d\omega$$'))
    plt.title(r'$\alpha=%3.1f$' % alpha)
    plt.show()

alpha = 1

t = [0.1, 0.2, 0.4, 0.8, 1.6, 3.2, 6.4]
x = np.linspace(0, 8)

def u_solution(alpha, t, x):
    # closed form of the cosine-transform integral
    s = 2.0 *np.sqrt(alpha *t)
    return 0.5 *(erf((1.0 +x) /s) +erf((1.0 -x) /s))

example_plot6b(alpha, t, x, u_solution)
```

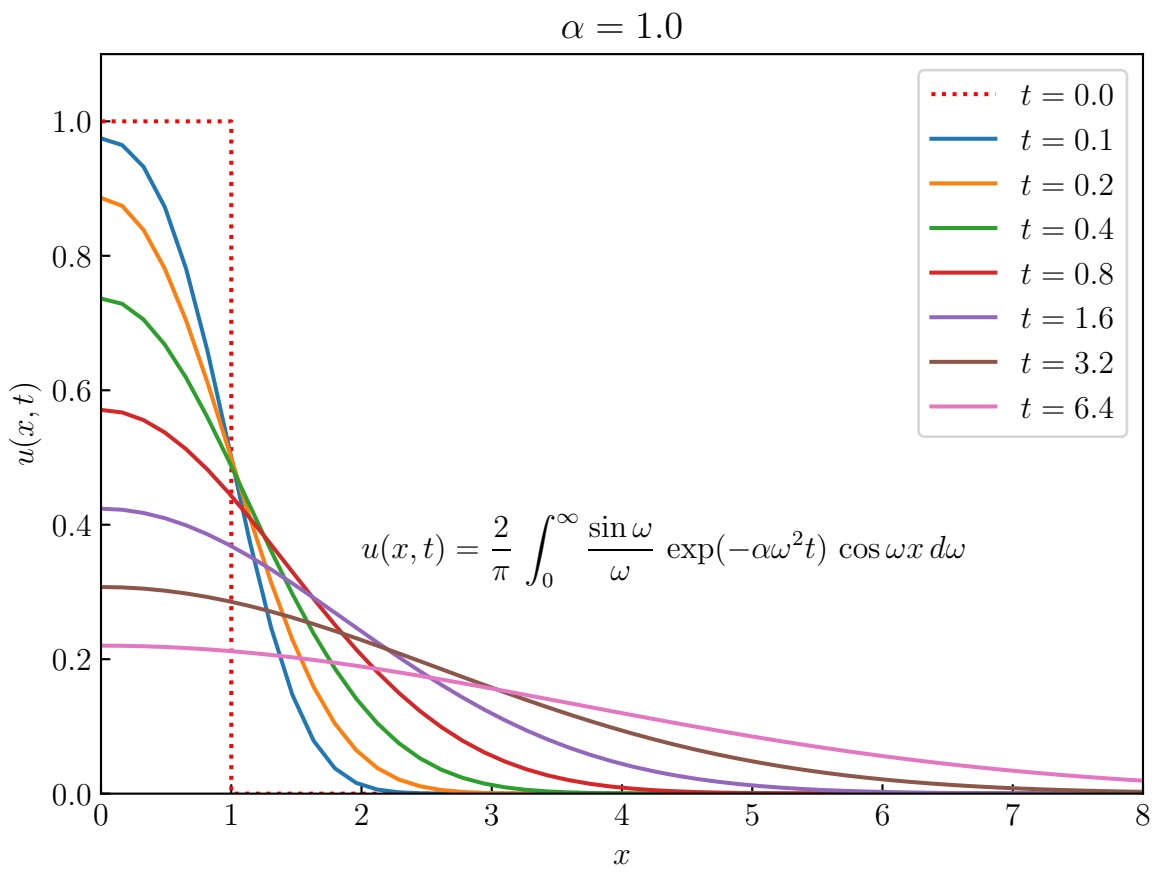


Figure 12.7: Heat conduction in semi-infinite domain insulated at the boundary

12.15 The Laplace Transform

- The Laplace transform

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

  is probably the only integral transform the student has seen before, since it is a very powerful tool for transforming initial-value problems in ODEs into algebraic equations

- Not only is the Laplace transform useful in transforming ODEs into algebraic equations, but now we will use the Laplace transform to transform PDEs into ODEs

- The major difference in applying the Laplace transform to PDEs in contrast to ODEs is that now when the original PDE is transformed, the new resulting equation will be either a new PDE with one less independent variable or else an ODE in one variable

- We must then decide how to solve this new problem (maybe by another transform, by separation of variables, and so on)

Property 1 Transform Pair

- The Laplace transform and its inverse are given by

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$
$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}\,ds$$

- The Laplace transform has one major advantage over the Fourier transform in that the damping factor e^{-st} in the integrand allows us to transform a wider class of functions (the factor $e^{-i\omega x}$ in the Fourier transform doesn't do any damping since its absolute value is one)

- In fact, the exact conditions that ensure that a function $f(t)$ has a Laplace transform are given by the following theorem:

  If we can find constants M and a such that

$$|f(t)| \le Me^{at} \quad \text{for} \quad t > T,$$

  then $\displaystyle\int_0^\infty f(t)\,e^{-st}\,dt$ exists for $s > a$

- We now list a few functions that have Laplace transforms

$$\begin{aligned}
&f(t) = 1 && (\text{pick } M = 1, a = 0) && \longrightarrow && F(s) = \frac{1}{s} \\
&f(t) = e^{2t} && (\text{pick } M = 1, a = 2) && \longrightarrow && F(s) = \frac{1}{s-2}, \quad s > 2 \\
&f(t) = \sin\omega t && (\text{pick } M = 1, a = 0) && \longrightarrow && F(s) = \frac{\omega}{s^2+\omega^2} \\
&f(t) = e^{t^2} && (\text{doesn't have a Laplace transform}) &&&&
\end{aligned}$$

- In the definition of the Laplace transform, the variable s is taken to be a real variable $0 < s < \infty$. It is possible to extend this definition to complex values of s and, in fact, to evaluate the inverse Laplace transform

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}\,ds$$

We must often resort to contour integration in the complex plane and the theory of residues. We won't bother ourselves with this topic here but will use the tables for finding inverse transforms

Property 2 Transforms of Partial Derivatives

- Suppose we have a function $u(x,t)$ of two variables. Since the Laplace transform transforms the t-variable, the rules of transformation for partial derivatives are

$$\mathcal{L}[u_t] = \int_0^\infty u_t(x,t)e^{-st}\,dt = sU(x,s) - u(x,0)$$

$$\mathcal{L}[u_{tt}] = \int_0^\infty u_{tt}(x,t)e^{-st}\,dt = s^2U(x,s) - su(x,0) - u_t(x,0)$$

$$\mathcal{L}[u_x] = \int_0^\infty u_x(x,t)e^{-st}\,dt = \frac{\partial U}{\partial x}(x,s)$$

$$\mathcal{L}[u_{xx}] = \int_0^\infty u_{xx}(x,t)e^{-st}\,dt = \frac{\partial^2 U}{\partial x^2}(x,s)$$

Property 3 Convolution Theory

- Convolution plays the same role here as it did in the Fourier transform, but now the convolution is defined slightly differently. The finite convolution of two functions f and g is defined by

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

As in the case of the infinite convolution, the important property of this new convolution is that

$$\mathcal{L}[f*g] = \mathcal{L}[f]\,\mathcal{L}[g]$$

or the equivalent formula

$$\mathcal{L}^{-1}\left\{\mathcal{L}[f]\,\mathcal{L}[g]\right\} = f*g$$

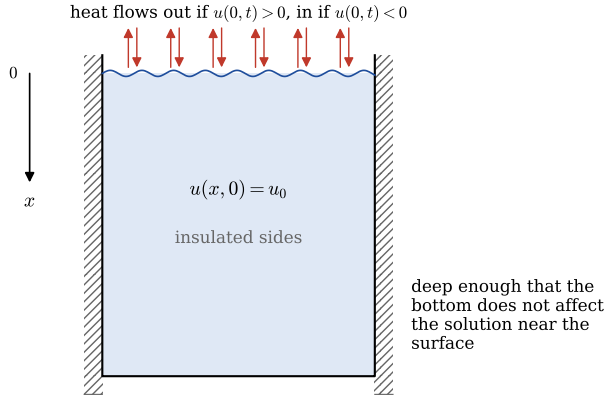
For example

$$\mathcal{L}^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+1}\right] = \int_0^t \sin\tau\,d\tau = 1-\cos t$$

NOTES

The Laplace transform can also be applied to problems where the PDE is nonhomogeneous (in separation of variables, the equation had to be homogeneous), but the Laplace transform will generally work only if the equation has constant coefficients (in separation of variables, we could have variable coefficients)

- **Heat Conduction in a Semi-infinite Medium**



  - Consider a large (deep) container of liquid that is insulated on the sides. Suppose the liquid has an initial temperature of u_0 and that the temperature of the air above liquid is zero. Our goal is to find the temperature of the liquid at various depths of the container at different t values:

$$\begin{aligned} u_t &= u_{xx} \quad && 0<x<\infty,\ 0<t<\infty \\ u_x(0,t)-u(0,t) &= 0 && 0<t<\infty \\ u(x,0) &= u_0 && 0<x<\infty \end{aligned}$$

  - Transforming our problem, we arrive at an ODE in x

$$sU(x)-u_0 = \frac{d^2U}{dx^2} \quad 0<x<\infty$$
$$\frac{dU}{dx}(0) = U(0)$$

    We first find the general solution of the ODE, which is

$$U(x) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{u_0}{s}$$

  - Substituting the BCs into this expression allows us to find the constants c_1 and c_2 (first note that $c_1 = 0$ or else the temperature will go to infinity as x gets large). Finding c_2 from the BC at $x = 0$ gives us the answer for $U(x)$

$$U(x) = -u_0\left[\frac{e^{-\sqrt{s}x}}{s(\sqrt{s}+1)}\right] + \frac{u_0}{s} = \frac{u_0}{s} - u_0\left[\frac{e^{-\sqrt{s}x}}{s} - \frac{e^{-\sqrt{s}x}}{\sqrt{s}(\sqrt{s}+1)}\right]$$

    Now for the last step, to find $u(x,t)$, we compute

$$u(x,t) = \mathcal{L}^{-1}\left[U(x,s)\right] = u_0 - u_0\left[\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) - \operatorname{erfc}\left(\sqrt{t} + \frac{x}{2\sqrt{t}}\right)e^{x+t}\right]$$

See Inverse Laplace Transform Table

```
def example_plot7(t, x, u0, u_solution):

    plt.figure(figsize=(7, 5))
    plt.plot(x, np.full_like(x, u0), label='$t=0$')
    for tt in t:
        plt.plot(x, u_solution(tt, x, u0), label='$t=%1.3f$' % tt)
    plt.legend(loc='lower right')

    plt.xlim(0, 5)
    plt.ylim(0.4, 1.2)

    plt.xlabel('$x$')
    plt.ylabel('$u(x,t)$')
    plt.title(r'$u_0=%3.1f$' % u0)

    plt.show()

def u_solution(t, x, u0):
    sqrt_t = np.sqrt(t)
    x_2sqrt_t = x /(2 *sqrt_t)
    return u0 -u0*(erfc(x_2sqrt_t)
              -erfc(sqrt_t +x_2sqrt_t) *np.exp(x +t))

u0 = 1
t = np.logspace(-3, 0, 6)
x = np.linspace(0, 5, 300)

example_plot7(t, x, u0, u_solution)
```

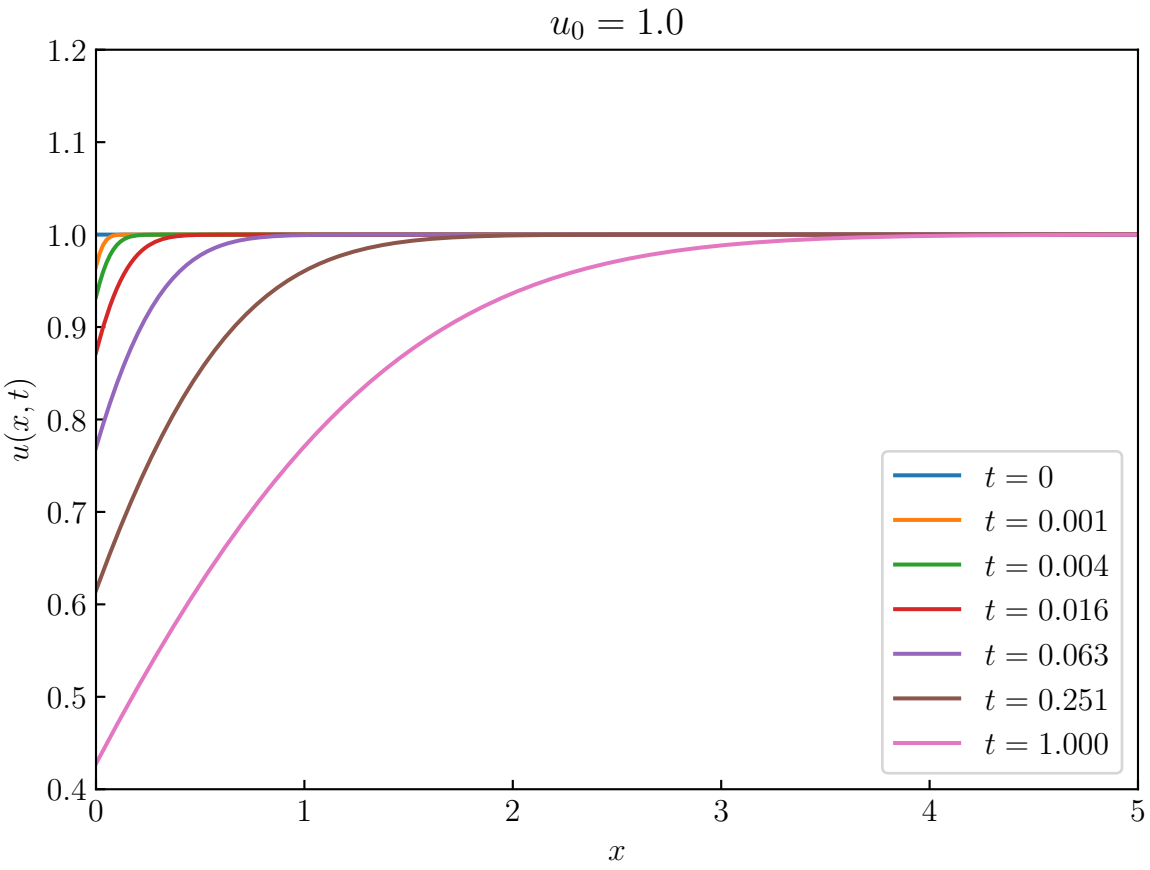


Figure 12.8: Heat conduction in semi-infinite domain with 3rd kind BC

Try it in Python: Laplace transforms with `sympy`: Sympy: Laplace Transform

12.16 The Convection Term u_x in the Diffusion Problems

- Suppose now we consider the problem of finding the concentration of a substance upwards from the surface of the earth where the substance both diffuses through the air and is carried upwards (convected) by moving currents (moving with velocity V)
- In any case, it is our purpose here to solve the diffusion-convection equation

$$u_t = Du_{xx} - Vu_x$$

- To get an idea of what solutions look like or how they behave with the convection term included, let's first work a problem that is pure convection (the diffusion term is zero)
- A typical problem would be dumping a substance into clean air (moving with upward velocity V) and observing the concentration of the substance at the upperside. Then the concentration of substance $u(x,t)$ can be found by solving the following mathematical model:

$$
\begin{aligned}
u_t &= -Vu_x \qquad 0 < x < \infty,\ 0 < t < \infty \\
u(0,t) &= P \qquad \longleftarrow \text{ Constant Input of the Substance} \\
u(x,0) &= 0 \qquad \longleftarrow \text{ Initially Clean Air}
\end{aligned}
\tag{CV}
$$

- Since the x-variable is unbounded, we use the Laplace transform on t

$$
\begin{aligned}
sU(x) &= -V\frac{dU}{dx}, \quad 0 < x < \infty \\
U(0) &= \frac{P}{s} \\
&\Downarrow \\
U(x) &= \frac{P}{s}e^{-\frac{x}{V}s} \\
&\Downarrow \\
u(x,t) &= P\,H(t - x/V) \\
&\Downarrow \\
u(x,t) &= \begin{cases} P & x \le Vt \\ 0 & x > Vt \end{cases}
\end{aligned}
$$

- To see what happens when a moving wave diffuses, we solve the following problem ($P = 1$)

$$
\begin{aligned}
u_t &= Du_{xx} - Vu_x \qquad -\infty < x < \infty,\ 0 < t < \infty \\
u(x,0) &= P\,(1 - H(x)) \qquad -\infty < x < \infty
\end{aligned}
\tag{DC}
$$

  Note that in the new problem (DC), we have moved the boundary to $-\infty$ (we now have an initial-value problem)

- In this case, we introduce a new coordinate ξ, which moves along the x-axis with velocity V. In other words, we now place our coordinate system so that it moves with the wave front (of course, now when diffusion in addition to convection, we won't have a sharp wave front)
- Mathematically this says that we change our space coordinate x to $\xi = x - Vt$. It's now clear that when $\xi = 0$, we are on the wave front
- To begin, we introduce new coordinates (ξ, τ) and use the chain rule to get the new IVP in terms of ξ and τ:

$$
\begin{aligned}
u_t &= Du_{xx} - Vu_x \\
&\Downarrow\ \xi = x - Vt,\ \tau = t \\
u_t &= u_\xi \xi_t + u_\tau \tau_t = -Vu_\xi + u_\tau \\
u_x &= u_\xi \xi_x + u_\tau \tau_x = u_\xi \\
u_{xx} &= (u_\xi)_x = u_{\xi\xi}\xi_x = u_{\xi\xi} \\
&\Downarrow \\
-Vu_\xi + u_\tau &= Du_{\xi\xi} - Vu_\xi \\
&\Downarrow\!\Downarrow\ \xi = x \text{ at } t = 0 \\
u_\tau &= Du_{\xi\xi}, \quad -\infty < \xi < \infty \\
u(\xi,0) &= P(1 - H(\xi)) = \phi(\xi)
\end{aligned}
$$

- This problem has already been solved in Section 12.13 by the Fourier transform and has the solution

$$u(\xi,\tau)=\frac{1}{2\sqrt{D\pi\tau}}\int_{-\infty}^{\infty}\phi(\omega)e^{-(\xi-\omega)^2/4D\tau}\,d\omega=\frac{P}{2\sqrt{D\pi\tau}}\int_{-\infty}^{0}e^{-(\xi-\omega)^2/4D\tau}\,d\omega$$

  where $\phi(\omega)$ is the initial condition

- By letting

$$\beta=\frac{\xi-\omega}{2\sqrt{D\tau}},\quad d\beta=-\frac{1}{2\sqrt{D\tau}}d\omega$$

  we get the interesting result

$$u(\xi,\tau)=\frac{P}{2}\left[\frac{2}{\sqrt{\pi}}\int_{\frac{\xi}{2\sqrt{D\tau}}}^{\infty}e^{-\beta^2}\,d\beta\right]=\begin{cases}\dfrac{P}{2}\left[1+\operatorname{erf}\left(-\dfrac{\xi}{2\sqrt{D\tau}}\right)\right], & \xi<0\\ \dfrac{P}{2}\operatorname{erfc}\left(\dfrac{\xi}{2\sqrt{D\tau}}\right), & \xi\geq 0\end{cases}$$

- Finally, the solution of our problem in terms of the coordinates x and t is

$$u(x,t)=\begin{cases}\dfrac{P}{2}\left[1+\operatorname{erf}\left(\dfrac{Vt-x}{2\sqrt{Dt}}\right)\right], & Vt>x\\ \dfrac{P}{2}\operatorname{erfc}\left(\dfrac{x-Vt}{2\sqrt{Dt}}\right), & Vt\leq x\end{cases}$$

```
def example_plot8(D, V, t, x, u_solution):
    plt.figure(figsize=(8, 6))
    plt.plot([-1, 0, 0, x[-1]], [1, 1, 0, 0], 'r:', label='$t=0.0$')
    for tt in t:
        x_tt = V*tt
        plt.plot(x, u_solution(D, V, tt, x), label=rf'$t={tt:3.1f}$')
        plt.plot([-1, x_tt, x_tt, x[-1]], [1, 1, 0, 0], 'k:')
    plt.xlim(-1, x[-1])
    plt.ylim(0, 1.2)
    plt.legend(loc='upper left', bbox_to_anchor=(1.0, 1.0))
    plt.xlabel('$x$')
    plt.ylabel('$u(x,t)$')
    plt.title(rf'$D={D:4.1f},\;V={V:3.1f},\;P=1.0$')
    plt.show()

def u_solution(D, V, t, x):
    x_t = (x -V*t) /(2.0 *np.sqrt(D*t))
    return [0.5*(1.0 +erf(-q)) if q <= 0.0 else 0.5*erfc(q) for q in x_t]

D = 0.1
V = 1.0

t = [2.0, 4.0, 6.0, 8.0]
x = np.linspace(0, 16, 400)
```

```
example_plot8(D, V, t, x, u_solution)
```

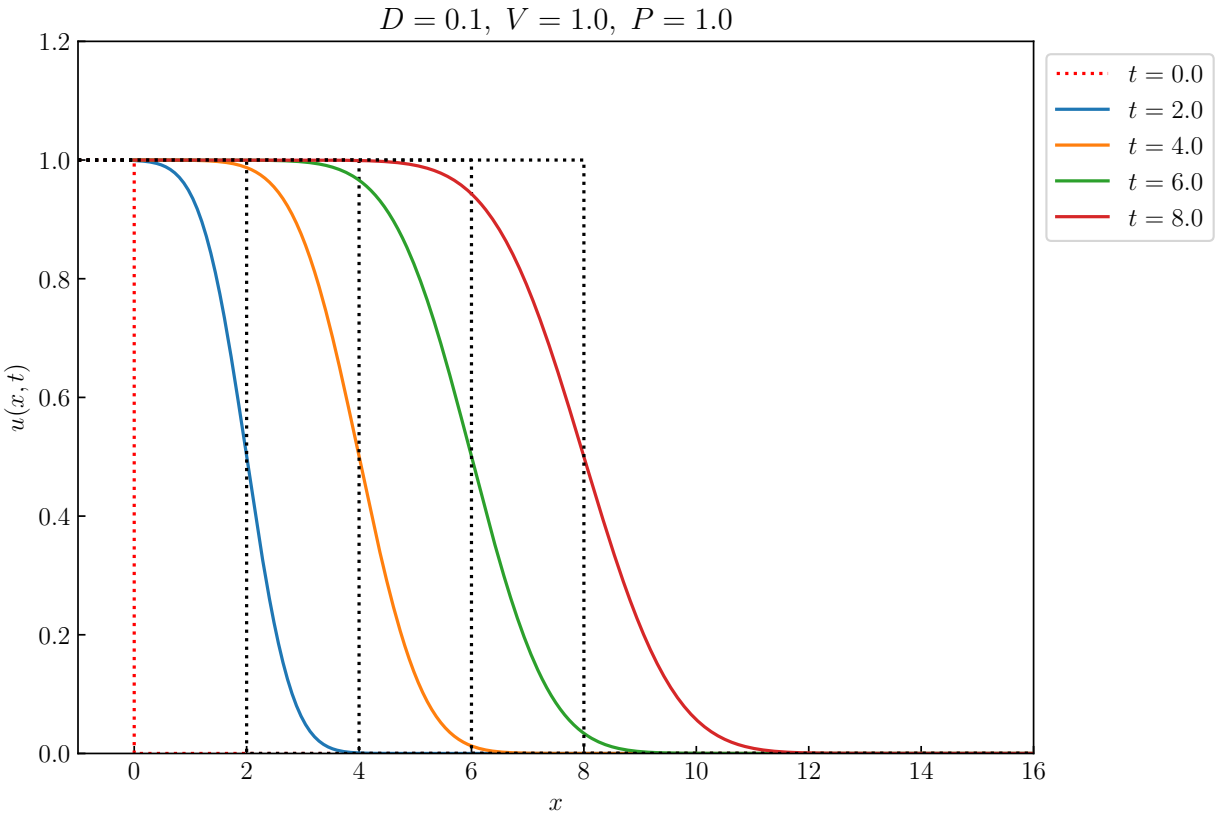


Figure 12.9: Diffusion-convection on the infinite domain

12.17 Duhamel's Principle

- Quite often, it is important to find the temperature inside a medium due to time-varying boundary conditions. For example, consider an insulated rod with temperature specified as $f(t)$ on the right end

$$\begin{aligned} u_t &= u_{xx} \quad && 0 < x < 1,\ 0 < t < \infty \\ u(0,t) &= 0 && 0 < t < \infty \\ u(1,t) &= f(t) \\ u(x,0) &= 0 && 0 \le x \le 1 \end{aligned} \tag{D1}$$

- We may think that the solution to (D1) can be easily found once we know the solution to the simpler version (constant temperature on the boundaries)

$$\begin{aligned} w_t &= w_{xx} \quad && 0 < x < 1,\ 0 < t < \infty \\ w(0,t) &= 0 && 0 < t < \infty \\ w(1,t) &= 1 \\ w(x,0) &= 0 && 0 \le x \le 1 \end{aligned} \tag{D2}$$

In fact, if we solve (D1) and (D2) side by side by the Laplace transform, we will see a striking result (Duhamel's principle) that will give us the solution to (D1) in terms of the solution of (D2)

- Solving (D2) by Laplace transform, we have

$$\begin{aligned}
&\frac{d^2W}{dx^2} - sW = 0 \\
&W(0) = 0 \\
&W(1) = 1/s \\
&\qquad \Downarrow \text{ Solve the ODE} \\
&W(x,s) = \frac{1}{s}\left[\frac{\sinh x\sqrt{s}}{\sinh\sqrt{s}}\right] \\
&\qquad \Downarrow \text{ Find the inverse transform} \\
&w(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-(n\pi)^2 t}\sin n\pi x
\end{aligned}$$

- And we also obtain the following by transforming (D1):

$$\begin{aligned}
&\frac{d^2U}{dx^2} - sU = 0 \\
&U(0) = 0 \\
&U(1) = F(s) \\
&\qquad \Downarrow \text{ Solve the ODE} \\
&U(x,s) = F(s)\left[\frac{\sinh x\sqrt{s}}{\sinh\sqrt{s}}\right] = F(s)s\frac{1}{s}\left[\frac{\sinh x\sqrt{s}}{\sinh\sqrt{s}}\right] = F(s)sW(x,s) \\
&\qquad \Downarrow \mathcal{L}[w_t]=sW-w(x,0)=sW \\
&U(x,s) = F(s)\mathcal{L}[w_t] \\
&\qquad \Downarrow \text{ Find the inverse transform} \\
&u(x,t) = \mathcal{L}^{-1}\left[F(s)\mathcal{L}[w_t]\right] = f(t) * w_t(x,t) \\
&\qquad = \int_0^t w_{t-\tau}(x,t-\tau)\, f(\tau)\, d\tau \\
&\qquad = -w(x,t-\tau)\, f(\tau)\Big|_0^t + \int_0^t w(x,t-\tau) f'(\tau)\, d\tau \\
&\qquad = \int_0^t w(x,t-\tau) f'(\tau)\, d\tau + w(x,t)\, f(0)
\end{aligned}$$

We have found the solution $u(x,t)$ to the time-varying problem in terms of the solution to the easy problem (constant BCs). The above result is known as Duhamel's principle

NOTES

- There is another interesting version of Duhamel's principle that gives the answer to (D1) in terms of the solution $v(x,t)$ of the alternative simple problem

$$
\begin{aligned}
v_t &= v_{xx} \qquad 0 < x < 1,\ 0 < t < \infty \\
v(0,t) &= 0 \qquad 0 < t < \infty \\
v(1,t) &= \delta(t) \\
v(x,0) &= 0 \qquad 0 \le x \le 1
\end{aligned}
$$

- Knowing this formula

$$u(x,t) = \int_0^t v(x, t-\tau)\, f(\tau)\, d\tau$$

allows us to find $u(x,t)$ to an arbitrary boundary temperature $f(t)$ once we have carried out an experiment to determine the temperature response $v(x,t)$ from an impulse temperature

Summary

Topic	Key result
Classification	$Au_{xx} + Bu_{xy} + Cu_{yy} + \cdots = 0$: $B^2 - 4AC < 0$ elliptic, $= 0$ parabolic, > 0 hyperbolic
Heat equation	$u_t = \alpha u_{xx}$
Separation of variables	$u(0,t) = u(L,t) = 0$: $u = \sum_{n=1}^{\infty} b_n e^{-\alpha(n\pi/L)^2 t} \sin \frac{n\pi x}{L}$, $b_n = \frac{2}{L}\int_0^L \phi(x) \sin \frac{n\pi x}{L}\, dx$
Nonhomogeneous BCs	u = steady state + transient, the transient satisfying homogeneous BCs
Substitutions	$u_t = \alpha u_{xx} - \beta u$: $u = e^{-\beta t} w$; $u_t = \alpha u_{xx} - v u_x$: $u = \exp\left[\frac{v}{2\alpha}\left(x - \frac{v}{2}t\right)\right] w$
Eigenfunction expansion	$u_t = \alpha u_{xx} + f(x,t)$: $u = \sum T_n(t) X_n(x)$, one ODE for each T_n
Finite sine transform	$\mathcal{F}_s[u_{xx}] = -\left(\frac{n\pi}{L}\right)^2 \mathcal{F}_s[u] + \frac{2n\pi}{L^2}\left[u(0,t) - (-1)^n u(L,t)\right]$
Fourier transform	$\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$, $\mathcal{F}[f_x] = i\omega \mathcal{F}[f]$, $\mathcal{F}[f_{xx}] = -\omega^2 \mathcal{F}[f]$
Infinite rod	$u(x,t) = \frac{1}{2\sqrt{\alpha\pi t}} \int_{-\infty}^{\infty} \phi(\xi)\, e^{-(x-\xi)^2/4\alpha t}\, d\xi$
Duhamel's principle	$u(x,t) = \int_0^t \frac{\partial w}{\partial t}(x, t-\tau)\, f(\tau)\, d\tau$, where w solves the problem with the constant boundary value 1

Worked Exercises

1. Solve the problem

$$\begin{aligned} u_t &= \alpha u_{xx}, \;\; a < x < b, \;\; t > 0 \\ u(a,t) &= 0, \; u(b,t) = 0, \; u(x,0) = (x-a)(b-x) \end{aligned}$$

Solution

Step 1: Separation of Variables

We look for solutions of the form:

$$u(x,t) = X(x)T(t)$$

Substitute into the PDE:

$$X(x)T'(t) = \alpha X''(x)T(t) \;\Rightarrow\; \frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

We get two ODEs:

$$\begin{aligned} T'(t) + \alpha\lambda T(t) &= 0 \\ X''(x) + \lambda X(x) &= 0 \end{aligned}$$

with boundary conditions:

$$X(a) = 0, \quad X(b) = 0$$

Step 2: Solve the Spatial Part

We solve:

$$X'' + \lambda X = 0, \quad X(a) = 0, \quad X(b) = 0$$

This is a standard Sturm–Liouville problem

The eigenfunctions are:

$$X_n(x) = \sin\left(\frac{n\pi(x-a)}{b-a}\right), \quad n = 1, 2, 3, ...$$

The corresponding eigenvalues are:

$$\lambda_n = \left(\frac{n\pi}{b-a}\right)^2$$

Step 3: Solve the Temporal Part

$$T_n(t) = e^{-\alpha\lambda_n t} = \exp\left(-\alpha\left(\frac{n\pi}{b-a}\right)^2 t\right)$$

Step 4: General Solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi(x-a)}{b-a}\right) e^{-\alpha(\frac{n\pi}{b-a})^2 t}$$

Step 5: Apply Initial Condition

$$u(x,0) = (x-a)(b-x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi(x-a)}{b-a}\right)$$

To find coefficients B_n, use Fourier sine series:

$$B_n = \frac{2}{b-a}\int_a^b (x-a)(b-x)\sin\left(\frac{n\pi(x-a)}{b-a}\right) dx$$

Let's use substitution: Let $\xi = \frac{x-a}{b-a}$, so $x = (b-a)\xi + a$, $dx = (b-a)d\xi$, and when $x = a \Rightarrow \xi = 0, x = b \Rightarrow \xi = 1$

Then:

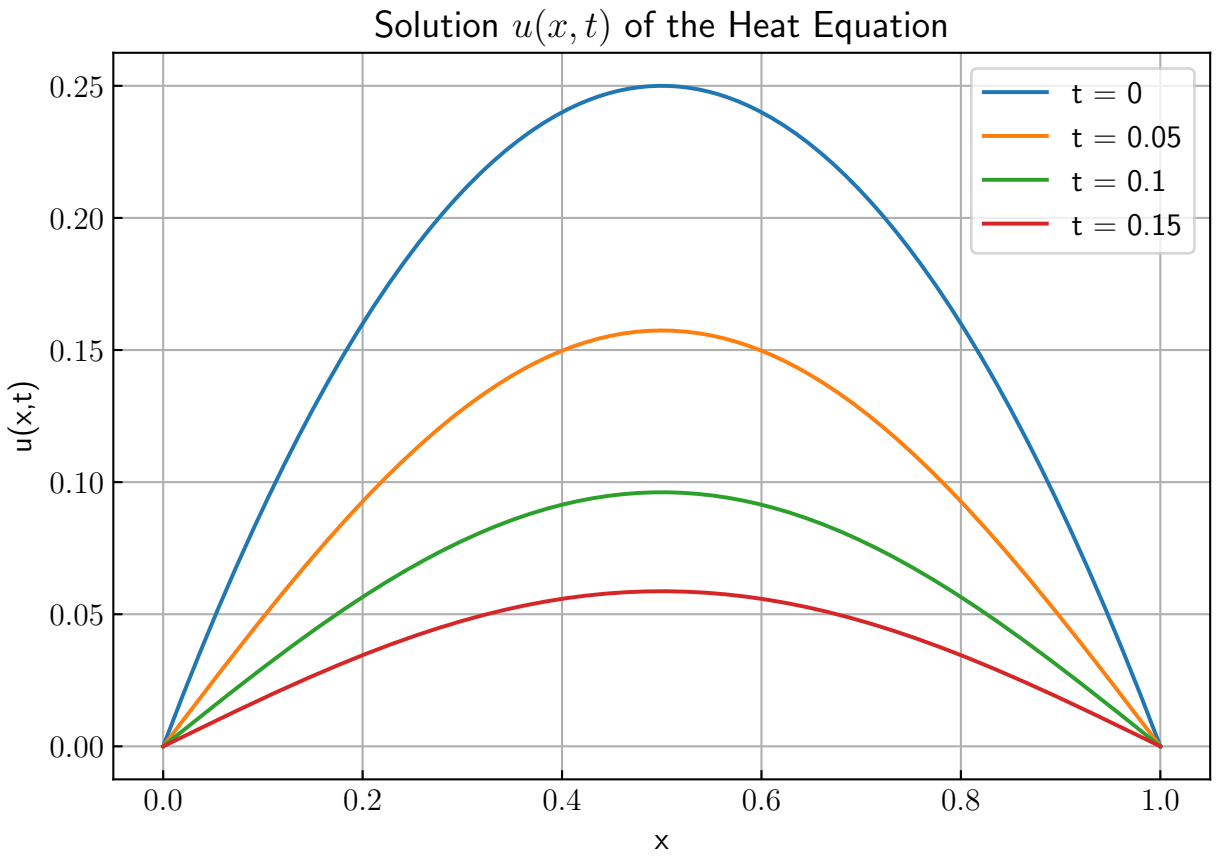
$$(x-a)(b-x) = (b-a)^2\xi(1-\xi)$$

and:

$$\frac{B_n}{(b-a)} = 2\int_0^1 \xi(1-\xi)\sin(n\pi\xi)d\xi = \begin{cases} \frac{8}{n^3\pi^3}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
a = 0
b = 1
alpha = 1
t_values = [0, 0.05, 0.10, 0.15]  # Times to plot
x_vals = np.linspace(a, b, 400)
N_terms = 50  # Number of terms in Fourier series

# Solution series function
def u_series(x, t, N=N_terms):
    u = np.zeros_like(x)
    for n in range(1, N + 1, 2):  # Only odd n
        coeff = 8 /(n**3 * np.pi**3)
        term = coeff *(np.sin(n *np.pi *(x - a) /(b - a))
          *np.exp(-alpha *(n *np.pi /(b - a))**2 *t))
        u += term
    return u

# Plotting
plt.figure(figsize=(7.5, 5))
```

```
for t in t_values:
    u_vals = u_series(x_vals, t)
    plt.plot(x_vals, u_vals, label=f"t = {t}")

plt.title("Solution $u(x,t)$ of the Heat Equation")
plt.xlabel("x")
plt.ylabel("u(x,t)")
plt.legend()
plt.grid(True)
plt.show()
```



2. Solve the problem

$$u_t = \alpha u_{xx}, \quad 0 < x < \pi, \quad t > 0$$
$$u_x(0,t) = 0, \; u_x(\pi,t) = 0, \; u(x,0) = 1 + \frac{1}{2}\cos 3x$$

Solution

Step 1: Separation of Variables

Let's look for a solution of the form:

$$u(x,t) = X(x)T(t)$$

Plug into the PDE:

$$X(x)T'(t) = \alpha X''(x)T(t) \;\Rightarrow\; \frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

Which gives us two ODEs:

$$T'(t) + \alpha\lambda T(t) = 0$$
$$X''(x) + \lambda X(x) = 0$$

With Neumann boundary conditions:

$$X'(0) = 0, \quad X'(\pi) = 0$$

Step 2: Solve the Spatial ODE

$$X'' + \lambda X = 0$$

Solutions depend on the value of λ:

- The eigenfunctions satisfying Neumann BCs are:

$$X_n(x) = \begin{cases} 1, & n = 0 \\ \cos(nx), & n = 1, 2, 3, ... \end{cases}$$

- Corresponding eigenvalues:

$$\lambda_n = n^2, \quad n = 0, 1, 2, ...$$

Step 3: Solve the Time Part

$$T_n(t) = e^{-\alpha n^2 t}$$

Step 4: General Solution

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) e^{-\alpha n^2 t}$$

Step 5: The initial condition is:

$$u(x,0) = 1 + \cos 3x$$

This already matches our eigenfunctions: * 1 corresponds to $\frac{A_0}{2}$ * $\frac{1}{2}\cos 3x$ corresponds to $A_3 \cos(3x)$

So the expansion is:

$$u(x,t) = \frac{A_0}{2} + A_3 \cos(3x) e^{-9\alpha t}$$

Comparing with $u(x,0) = 1 + \frac{1}{2}\cos 3x$, we get:

$$A_0 = 2, \quad A_3 = \frac{1}{2}$$

All other $A_n = 0$

Final Answer:

$$u(x,t) = 1 + \frac{1}{2}\cos(3x)e^{-9\alpha t}$$

3. Solve the problem

$$u_t = u_{xx} + F(x,t), \ \ -\infty < x < \infty, \ \ t > 0$$
$$u(x,0) = 0, \ u(x,t) \sim \text{bounded}$$

Solution

For the heat equation on $\mathbb{R}$, the Green's function (or heat kernel) is:

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}$$

The solution to the nonhomogeneous heat equation with zero initial condition is given by the Duhamel's principle:

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-\xi, t-\tau) F(\xi,\tau)\, d\xi\, d\tau$$

So,

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-\tau)}} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right) F(\xi,\tau)\, d\xi\, d\tau$$

4. Solve the problem

$$u_t = u_{xx}, \ \ 0 < x < \infty, \ \ t > 0$$
$$u(0,t) = g(t), \ u(x,0) = 0, \ u(x,t) \sim \text{bounded}$$

Solution

This is a classical problem with the Dirichlet boundary condition on a semi-infinite domain.

Since the initial condition is zero and the boundary condition is nonzero at $x = 0$, we apply a Green's function method for the semi-infinite domain

The appropriate solution is:

$$u(x,t) = \int_0^t \frac{x}{\sqrt{4\pi(t-\tau)^3}} \exp\left(-\frac{x^2}{4(t-\tau)}\right) g(\tau)\, d\tau$$

This is the Poisson kernel representation for the half-line heat problem with a source at the boundary

5. Solve the problem

$$u_t = \alpha u_{xx} - \nu u_x - \beta u, \;\; 0 < x < 1, \quad t > 0$$
$$u(0,t) = 0, \; u(1,t) = 0, \; u(x,0) = \exp\left(\frac{\nu}{2\alpha} x\right)$$

Solution

Step 1: Change of variables

We use the substitution:

$$u(x,t) = e^{rx-\lambda t} w(x,t)$$

with the goal of choosing r and so that the PDE for $w(x,t)$ becomes the standard heat equation

Compute derivatives:

$$u_x = re^{rx-\lambda t} w + e^{rx-\lambda t} w_x = e^{rx-\lambda t}(rw + w_x)$$
$$u_{xx} = e^{rx-\lambda t}(r^2 w + 2rw_x + w_{xx})$$
$$u_t = e^{rx-\lambda t}(-\lambda w + w_t)$$

Now plug into the PDE:

$$u_t = \alpha u_{xx} - \nu u_x - \beta u$$

Substituting everything:

$$e^{rx-\lambda t}(-\lambda w + w_t) = \alpha e^{rx-\lambda t}(r^2 w + 2rw_x + w_{xx}) - \nu e^{rx-\lambda t}(rw + w_x) - \beta e^{rx-\lambda t} w$$

Cancel the common exponential $e^{rx-\lambda t}$:

$$-\lambda w + w_t = \alpha(r^2 w + 2rw_x + w_{xx}) - \nu(rw + w_x) - \beta w$$

Group terms:

$$w_t = \alpha w_{xx} + (2\alpha r - \nu) w_x + (\alpha r^2 - \nu r - \beta + \lambda) w$$

We want to choose r and λ so that:

- Coefficient of w_x is 0: $2\alpha r - \nu = 0$
- Coefficient of w is 0: $\alpha r^2 - \nu r - \beta + \lambda = 0$

Solve for r:

$$2\alpha r = \nu \;\Rightarrow\; r = \frac{\nu}{2\alpha}$$

Plug $r = \frac{\nu}{2\alpha}$ into the second equation to find λ:

$$\lambda = \alpha r^2 - \nu r + \beta = \alpha\left(\frac{\nu}{2\alpha}\right)^2 - \nu \cdot \frac{\nu}{2\alpha} + \beta = -\frac{\nu^2}{4\alpha} + \beta$$

Step 2: Resulting equation for w

We now have:

$$u(x,t) = e^{\frac{\nu}{2\alpha}x - \lambda t} w(x,t), \quad \text{with } \lambda = -\frac{\nu^2}{4\alpha} + \beta$$

Then $w(x,t)$ satisfies the heat equation:

$$w_t = \alpha w_{xx}, \quad 0 < x < 1,\ t > 0$$

with boundary conditions:

$$w(0,t) = 0, \quad w(1,t) = 0$$

and initial condition:

$$w(x,0) = u(x,0) \cdot e^{-\frac{\nu}{2\alpha}x} = \exp\left(\frac{\nu}{2\alpha}x\right) \cdot \exp\left(-\frac{\nu}{2\alpha}x\right) = 1$$

Step 3: Solve the heat equation with constant initial condition

We solve:

$$\begin{aligned} w_t &= \alpha w_{xx}, \quad 0 < x < 1, \\ w(0,t) &= w(1,t) = 0, \\ w(x,0) &= 1. \end{aligned}$$

Use Fourier sine series:

$$w(x,t) = \sum_{n=1}^{\infty} b_n e^{-\alpha(n\pi)^2 t} \sin(n\pi x)$$

where the coefficients b_n come from:

$$b_n = 2\int_0^1 1 \cdot \sin(n\pi x)\, dx = \frac{2}{n\pi}(1 - \cos(n\pi)) = \begin{cases} \frac{4}{n\pi}, & \text{if } n \text{ odd}, \\ 0, & \text{if } n \text{ even}. \end{cases}$$

So,

$$w(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} e^{-\alpha(n\pi)^2 t} \sin(n\pi x)$$

Step 4: Final answer for $u(x,t)$

$$u(x,t) = e^{\frac{\nu}{2\alpha}x + \frac{\nu^2}{4\alpha}t - \beta t} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} e^{-\alpha(n\pi)^2 t} \sin(n\pi x)$$

6. Solve the problem

$$\begin{aligned} &u_t = u_{xx}, \;\; 0 < x < 1, \;\; 0 < t < \infty \\ &\quad u(0,t) = 0 \\ &\quad u(1,t) = \cos t \\ &\quad u(x,0) = x \end{aligned}$$

by

(a) transforming it to one with zero BCs

(b) solving the resulting problem

Solution

(a)

Let's define:

$$u(x,t) = v(x,t) + w(x,t)$$

where $w(x,t)$ is chosen so that it satisfies the boundary conditions, and thus the new function $v(x,t)$ will satisfy homogeneous (zero) boundary conditions.

We want:

$$\begin{cases} w(0,t) = 0 \\ w(1,t) = \cos t \end{cases}$$

We can try a linear function in x:

$$w(x,t) = x \cos t$$

Then we compute the equation for v:

- $u_t = v_t + \partial_t(x \cos t) = v_t - x \sin t$
- $u_{xx} = v_{xx} + \partial_{xx}(x \cos t) = v_{xx} + 0$ (since $x \cos t$ is linear in x)

So the equation becomes:

$$v_t - x\sin t = v_{xx} \;\Rightarrow\; v_t = v_{xx} + x\sin t$$

And the transformed problem is:

$$\begin{aligned} v_t &= v_{xx} + x\sin t, && 0 < x < 1,\ t > 0 \\ v(0,t) &= 0, \\ v(1,t) &= 0, \\ v(x,0) &= u(x,0) - x\cos(0) = x - x = 0 \end{aligned}$$

(b)

Step 1: Apply the Fourier sine transform

Let's define the sine transform:

$$\mathcal{F}[v(x,t)] = \hat{v}_n(t) = \int_0^1 v(x,t)\sin(n\pi x)\,dx$$

Apply this transform to both sides of the PDE:

Left-hand side (time derivative):

$$\mathcal{F}[v_t(x,t)] = \frac{d}{dt}\hat{v}_n(t)$$

Right-hand side:

$$\mathcal{F}[v_{xx}(x,t)] = -n^2\pi^2\hat{v}_n(t)$$

$$\mathcal{F}[x\sin t] = \sin t \cdot \int_0^1 x\sin(n\pi x)\,dx = \sin t \cdot \frac{(-1)^{n+1}}{n\pi}$$

Step 2: Solve the ODE for $\hat{v}_n(t)$

Let's write the equation again:

$$\hat{v}_n{}'(t) + n^2\pi^2\hat{v}_n(t) = \frac{(-1)^{n+1}}{n\pi}\sin t$$

Use integrating factor:

$$\mu(t) = e^{n^2\pi^2 t} \Rightarrow \frac{d}{dt}\left[e^{n^2\pi^2 t}\hat{v}_n(t)\right] = \frac{(-1)^{n+1}}{n\pi}\sin t \cdot e^{n^2\pi^2 t}$$

Now integrate both sides from 0 to t:

$$e^{n^2\pi^2 t}\hat{v}_n(t) = \frac{(-1)^{n+1}}{n\pi}\int_0^t \sin s \cdot e^{n^2\pi^2 s}\,ds$$

So the solution in the transform domain is:

$$\hat{v}_n(t) = \frac{(-1)^{n+1}}{n\pi} \int_0^t \sin s \cdot e^{n^2\pi^2(s-t)}\, ds$$

Step 3: Invert the Fourier sine transform

Recall:

$$v(x,t) = \sum_{n=1}^{\infty} \hat{v}_n(t) \sin(n\pi x)$$

Substitute the expression for $\hat{v}_n(t)$:

$$v(x,t) = \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n\pi} \int_0^t \sin s \cdot e^{n^2\pi^2(s-t)}\, ds \right] \sin(n\pi x)$$

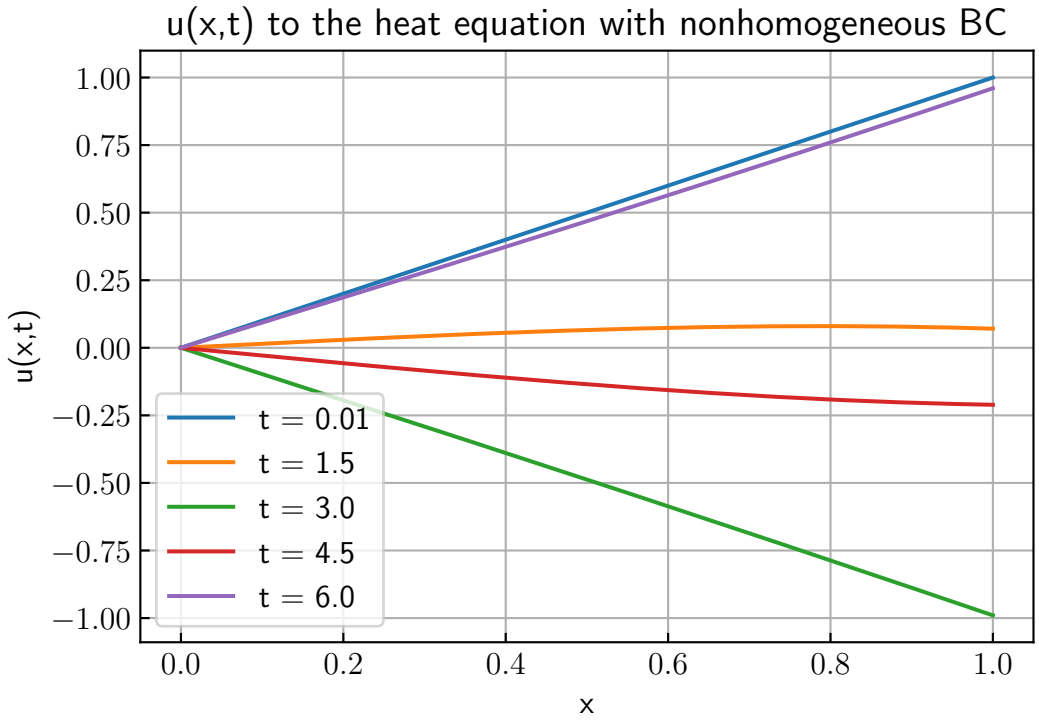
This gives us the complete solution for $v(x,t)$ using the Fourier sine transform method

So the full solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n\pi} \int_0^t \sin s \cdot e^{n^2\pi^2(s-t)}\, ds \right] \sin(n\pi x) + x \cos t$$

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import quad

# number of Fourier-terms
N_terms = 20

# Define parameters
L = 1
x_vals = np.linspace(0, L, 200)

# times at which to plot u(x, t)
t_vals = [1e-2, 1.5, 3.0, 4.5, 6.0]

# Compute u(x, t)
def compute_u(x, t, N=N_terms):
    u = x *np.cos(t)  # add x cos t term
    for n in range(1, N +1):
        n_pi = n *np.pi
        prefactor = (-1)**(n +1) /n_pi

        # Integrate I_n(t)
        integrand = lambda s: np.sin(s) *np.exp(n_pi**2 *(s -t))
        I_n, _ = quad(integrand, 0, t)

        A_n = prefactor *I_n
        u += A_n *np.sin(n_pi *x)
    return u
```

```
# Plot
plt.figure(figsize=(6, 4))
for t in t_vals:
    u_vals = [compute_u(x, t) for x in x_vals]
    plt.plot(x_vals, u_vals, label=f"t = {t}")

plt.title("u(x,t) to the heat equation with nonhomogeneous BC")
plt.xlabel("x")
plt.ylabel("u(x,t)")
plt.legend()
plt.grid(True)
plt.show()
```



7. Solve the problem

$$
\begin{aligned}
u_t &= u_{xx} - u + x, \quad 0 < x < 1, \quad 0 < t < \infty \\
u(0,t) &= 0 \\
u(1,t) &= 1 \\
u(x,0) &= 0
\end{aligned}
$$

by

(a) changing the nonhomogeneous BCs to homogeneous ones

(b) transforming into a new equation without the term $-u$

(c) solving the resulting problem

Solution

(*a*) Make the boundary conditions homogeneous

Let's define a new function:

$$u(x,t) = v(x,t) + w(x)$$

Choose $w(x)$ to satisfy the boundary conditions:

$$\begin{cases} w(0) = 0 \\ w(1) = 1 \end{cases}$$

Let's take the linear function:

$$w(x) = x$$

Then:

$$u(x,t) = v(x,t) + x \;\Rightarrow\; v(x,t) = u(x,t) - x$$

Now we compute: * $u_t = v_t$ * $u_{xx} = v_{xx} + w''(x) = v_{xx} + 0 = v_{xx}$ (since $x'' = 0$) * $u = v + x$

Substitute into the original PDE:

$$u_t = u_{xx} - u + x \Rightarrow v_t = v_{xx} - (v + x) + x = v_{xx} - v$$

So the PDE for v is:

$$v_t = v_{xx} - v$$

And the initial and boundary conditions become:

- $v(0,t) = u(0,t) - 0 = 0$
- $v(1,t) = u(1,t) - 1 = 0$
- $v(x,0) = u(x,0) - x = -x$

Transformed problem for $v(x,t)$:

$$\begin{aligned} v_t = v_{xx} - v, \quad 0 < x < 1,\ t > 0 \\ v(0,t) = v(1,t) = 0 \\ v(x,0) = -x \end{aligned}$$

(*b*) Eliminate the reaction term $-v$ by a second transformation

Let:

$$v(x,t) = e^{-t} z(x,t)$$

Then:

- $v_t = -e^{-t} z + e^{-t} z_t = e^{-t}(z_t - z)$
- $v_{xx} = e^{-t} z_{xx}$

Substitute into $v_t = v_{xx} - v$:

$$e^{-t}(z_t - z) = e^{-t}z_{xx} - e^{-t}z \Rightarrow z_t - z = z_{xx} - z \Rightarrow z_t = z_{xx}$$

So now we have the heat equation!

From $v = e^{-t}z$ at $t = 0$:

$$z(x, 0) = v(x, 0) = -x$$

And since $v(0, t) = v(1, t) = 0$, we have:

$$z(0, t) = z(1, t) = 0$$

Final simplified problem:

$$\begin{aligned} z_t = z_{xx}, \quad 0 < x < 1,\ t > 0 \\ z(0, t) = z(1, t) = 0 \\ z(x, 0) = -x \end{aligned}$$

(c) Solve the simplified heat equation

We solve this using separation of variables / Fourier series

Step 1: Expand initial condition in sine series

$$z(x, 0) = -x = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

Compute coefficients:

$$A_n = 2\int_0^1 (-x)\sin(n\pi x)dx = -2\int_0^1 x\sin(n\pi x)dx$$

Use integration by parts:

$$\int_0^1 x\sin(n\pi x)dx = \frac{(-1)^{n+1}}{n\pi} \Rightarrow A_n = \frac{2(-1)^n}{n\pi}$$

Step 2: General solution for $z(x, t)$

$$z(x, t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t}\sin(n\pi x) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} e^{-n^2\pi^2 t}\sin(n\pi x)$$

Step 3: Back-substitute to get $u(x, t)$

So finally:

$$u(x, t) = x + e^{-t}\sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} e^{-n^2\pi^2 t}\sin(n\pi x)$$

Final Answer:

$$\boxed{u(x,t) = x + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} e^{-(n^2\pi^2+1)t} \sin(n\pi x)}$$

```
# Re-import necessary libraries after kernel reset
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.animation as animation
from IPython.display import HTML

# Spatial domain
x = np.linspace(0, 1, 200)

# Number of Fourier terms
N = 50

# Time parameters
t_vals = np.linspace(0, 1, 100)  # from t=0 to t=1.5

# Precompute sin(nπx) terms
sin_terms = np.array([np.sin(n *np.pi *x) for n in range(1, N +1)])

# Compute the solution u(x, t) at each time t
def compute_u_xt(t):
    coeffs = np.array([
        (2 *(-1)**n) /(n *np.pi)
          *np.exp(-(n**2 *np.pi**2 +1) *t)
        for n in range(1, N +1)
    ])
    v_xt = np.dot(coeffs, sin_terms)
    return x +v_xt  # u = x +e^{-t} *z(x,t)

# Set up the plot
fig, ax = plt.subplots(figsize=(6, 4))

line, = ax.plot([], [], lw=2)
ax.set_xlim(0, 1)
ax.set_ylim(-0.1, 1.1)
ax.set_xlabel('x')
ax.set_ylabel('u(x,t)')
title = ax.set_title("")

# Initialization function
def init():
    line.set_data([], [])
    title.set_text("")
    return line, title

# Animation update function
def update(frame):
    t = t_vals[frame]
```

```
    u = compute_u_xt(t)
    line.set_data(x, u)
    title.set_text(f"u(x, t) at t = {t:.2f}")
    return line, title

# Create animation
ani = animation.FuncAnimation(
    fig, update, frames=len(t_vals),
    init_func=init, blit=True, interval=80
)

plt.close()
HTML('<center>' + ani.to_html5_video() + '</center>')
```

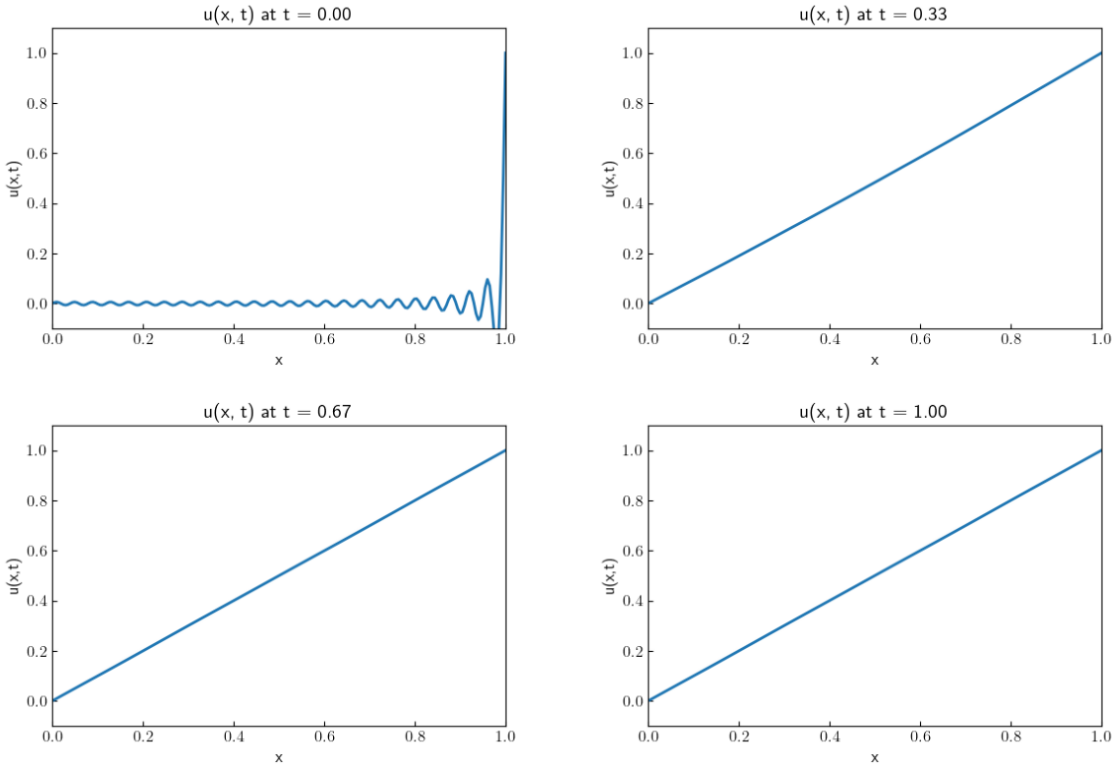


Figure 12.10: Four frames of the animation; play it on the web page

8. Consider the heat equation on the interval $0 < x < 1$:

$$u_t = u_{xx}$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0$$

and the initial condition

$$u(x,0) = f(x) = x(1-x)$$

(a) Find the solution $u(x,t)$ using separation of variables and Fourier sine series

(b) Show explicitly that $u(x,t) \to 0$ uniformly as $t \to \infty$

Solution

(a) Using separation of variables

The eigenfunctions for the Dirichlet problem on $(0,1)$ are

$$X_n(x) = \sin(n\pi x), \quad n = 1, 2, 3, ...$$

with eigenvalues

$$\lambda_n = (n\pi)^2$$

Thus the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

We compute coefficients:

$$b_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx$$

A standard integral computation gives

$$b_n = \frac{4}{(n\pi)^3}\left[1-(-1)^n\right]$$

Thus

$$u(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{(n\pi)^3} e^{-(n\pi)^2 t} \sin(n\pi x)$$

(b) Show $u(x,t) \to 0$ **as** $t \to \infty$

Each term satisfies

$$\left|\frac{8}{(n\pi)^3} e^{-(n\pi)^2 t} \sin(n\pi x)\right| \leq \frac{8}{(n\pi)^3} e^{-(n\pi)^2 t}$$

and the series

$$\sum_{n=1}^{\infty} \frac{1}{(n\pi)^3}$$

converges

By the Weierstrass M-test, the solution converges uniformly for each t, and the factor $e^{-(n\pi)^2 t} \to 0$ exponentially. Therefore

$$u(x,t) \to 0 \quad \text{uniformly}$$

9. Consider the inhomogeneous heat equation:

$$u_t = u_{xx} + e^{-t}x^2$$

on the whole line $-\infty < x < \infty$, with initial condition

$$u(x,0) = 0$$

(a) Use Duhamel's principle to express $u(x,t)$ in integral form involving the heat kernel

$$G(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

(b) Evaluate the integral explicitly and simplify the final closed-form expression

Solution

(a) Duhamel formula

For inhomogeneous forcing $F(x,t) = e^{-t}x^2$, Duhamel's principle gives

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-\xi, t-\tau) e^{-\tau} \xi^2 \, d\xi \, d\tau$$

Thus the integral representation is

$$u(x,t) = \int_0^t e^{-\tau} \left[\frac{1}{2\sqrt{\pi(t-\tau)}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right) \xi^2 \, d\xi \right] d\tau$$

(b) Evaluate the Gaussian integral

We use the standard identity

$$\int_{-\infty}^{\infty} \xi^2 \exp\left(-\frac{(x-\xi)^2}{4a}\right) d\xi = 2\sqrt{\pi a}\,(x^2 + 2a), \qquad a = t - \tau$$

Thus the inner integral is

$$\int_{-\infty}^{\infty} G(x-\xi, t-\tau)\, \xi^2 \, d\xi = x^2 + 2(t-\tau)$$

Therefore

$$u(x,t) = \int_0^t e^{-\tau} \left[x^2 + 2(t-\tau)\right] d\tau$$

Hence the final closed form is

$$\boxed{u(x,t) = 2t + (x^2 - 2)(1 - e^{-t})}$$

10. Consider the heat equation on the half-line $x > 0$ with Dirichlet boundary at $x = 0$:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & x > 0,\ t > 0, \\ u(0,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \geq 0 \end{cases}$$

(a) Using the method of images, derive the integral representation of the solution in terms of the heat kernel $G(x,t) = \dfrac{1}{2\sqrt{\pi t}} e^{-x^2/(4t)}$

(b) Let $u_0(x) = \mathbf{1}_{[0,a]}(x)$ for some fixed $a > 0$. Compute $u(x,t)$ explicitly (expressions with erf are acceptable). Then determine the pointwise limit $\lim_{t\to\infty} u(x,t)$ for each fixed $x \geq 0$ and briefly justify the result

Solution

(a) Method of images and integral representation

Let the heat kernel on the whole line be

$$G(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/(4t)}, \qquad t > 0$$

To enforce the Dirichlet condition $u(0,t) = 0$, extend the initial data u_0 to an odd function $\tilde{u}_0$ on $\mathbb{R}$:

$$\tilde{u}_0(\xi) = \begin{cases} u_0(\xi), & \xi \geq 0, \\ -u_0(-\xi), & \xi < 0 \end{cases}$$

The solution on the whole line with initial data $\tilde{u}_0$ is

$$\tilde{u}(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t)\, \tilde{u}_0(\xi)\, d\xi$$

Restricting to $x \geq 0$ and using oddness of $\tilde{u}_0$ gives the half-line solution

$$\boxed{u(x,t) = \int_0^{\infty} \left(G(x-\xi,t) - G(x+\xi,t)\right) u_0(\xi)\, d\xi, \quad x \geq 0,\ t > 0}$$

One readily checks $u(0,t) = 0$ and that u satisfies the heat equation for $x > 0$

(b) Indicator initial data and long-time limit

Take $u_0(\xi) = \mathbf{1}_{[0,a]}(\xi)$ with fixed $a > 0$. Then

$$u(x,t) = \int_0^a G(x-\xi,t) - G(x+\xi,t)\, d\xi$$

Compute each convolution term by the primitive of the Gaussian. Using the error function

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\xi^2}\, d\xi \quad \text{and} \quad \int_{-\infty}^{z} \frac{1}{2\sqrt{\pi t}} e^{-\xi^2/(4t)}\, d\xi = \frac{1}{2} \operatorname{erf}\left(\frac{z}{2\sqrt{t}}\right) + \frac{1}{2}$$

one convenient form for $u(x,t)$ is

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x-a}{2\sqrt{t}}\right)\right] - \frac{1}{2}\left[\operatorname{erf}\left(\frac{x+a}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right] \\ &= \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) - \frac{1}{2}\left[\operatorname{erf}\left(\frac{x-a}{2\sqrt{t}}\right) + \operatorname{erf}\left(\frac{x+a}{2\sqrt{t}}\right)\right] \end{aligned}$$

Fix $x \geq 0$. As $t \to \infty$, we have

$$\frac{x \pm a}{2\sqrt{t}} \to 0, \qquad \frac{x}{2\sqrt{t}} \to 0$$

and $\operatorname{erf}(0) = 0$. Therefore each error-function term tends to 0, so

$$\lim_{t\to\infty} u(x,t) = 0, \qquad \text{for every fixed } x \geq 0$$

Answers

Answer 1 (Section 12.5)

$u(x,t) = 1 + e^{-\pi^2 t}\cos \pi x + 0.5\, e^{-9\pi^2 t}\cos 3\pi x$

Answer 2 (Section 12.7)

$$u(x,t) = 1 + \frac{1-h}{1+h}\,x + \sum_{n=1}^{\infty} a_n e^{-\alpha\lambda_n t}\sin\sqrt{\lambda_n}\,x$$

λ_n: positive roots of $\tan\sqrt{\lambda} = -\sqrt{\lambda}/h$, $\left(n-\frac{1}{2}\right)^2\pi^2 < \lambda_n < n^2\pi^2$; $\quad a_n = \dfrac{2(\lambda_n + h^2)}{\lambda_n + h^2 + h}\left[\dfrac{\pi\sin\sqrt{\lambda_n}}{\pi^2 - \lambda_n} + \dfrac{h\sin\sqrt{\lambda_n}}{\lambda_n}\right.$

Answer 3 (Section 12.9)

$$u(x,t) = \frac{1 - e^{-\lambda_1 t}}{\lambda_1}\sin\sqrt{\lambda_1}\,x, \qquad \sqrt{\lambda_1} = 2.028758, \quad \lambda_1 = 4.115858$$

Answer 4 (Section 12.10)

$$u(x,t) = \sum_{n=1}^{\infty}\left[\int_0^t e^{-(\alpha n^2\pi^2+\beta)(t-\tau)}F_n(\tau)\,d\tau\right]\sin n\pi x, \qquad F_n(t) = 2\int_0^1 f(x,t)\sin n\pi x\,dx$$

Answer 5 (Section 12.11)

$u = u_1 + u_2 + u_3 + u_4$, each solving $u_t = u_{xx}$ with one source of nonhomogeneity and zero for the rest:

(P1) heat source $f(x,t)$ only (P2) $u(0,t) = g_1(t)$ only (P3) $u(1,t) = g_2(t)$ only (P4) $u(x,0) = \phi(x)$ only

Answer 6 (Section 12.11)

$$u(x,t) = x + \frac{\sin 3\pi x}{9\pi^2} + e^{-\pi^2 t}\sin\pi x - \frac{e^{-9\pi^2 t}}{9\pi^2}\sin 3\pi x + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}\,e^{-n^2\pi^2 t}\sin n\pi x$$

Answer 7 (Section 12.14)

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega}{\omega}e^{-\alpha\omega^2 t}\cos\omega x\,d\omega = \frac{1}{2}\left[\operatorname{erf}\left(\frac{1+x}{2\sqrt{\alpha t}}\right) + \operatorname{erf}\left(\frac{1-x}{2\sqrt{\alpha t}}\right)\right]$$

13 Hyperbolic Partial Differential Equations

Learning objectives After this chapter you should be able to

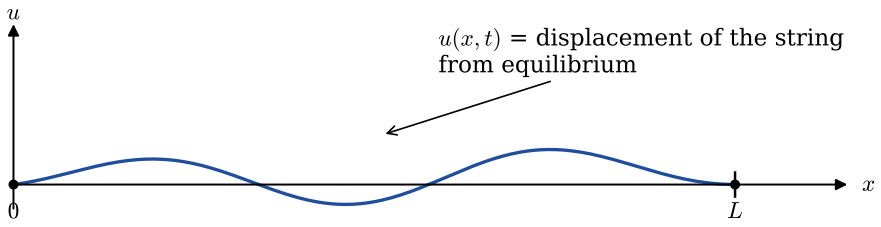
- solve the infinite string by d'Alembert's formula and interpret the solution as traveling waves, including reflection at a boundary
- solve the wave equation in two and three dimensions in free space, and explain Huygens' principle
- solve vibrating strings and drumheads by separation of variables, and nonhomogeneous problems by finite transforms
- nondimensionalize problems, classify second-order PDEs and reduce a hyperbolic equation to canonical form
- solve first-order linear and nonlinear (conservation) equations and hyperbolic systems by the method of characteristics, including shocks

13.1 The One-dimensional Wave Equation

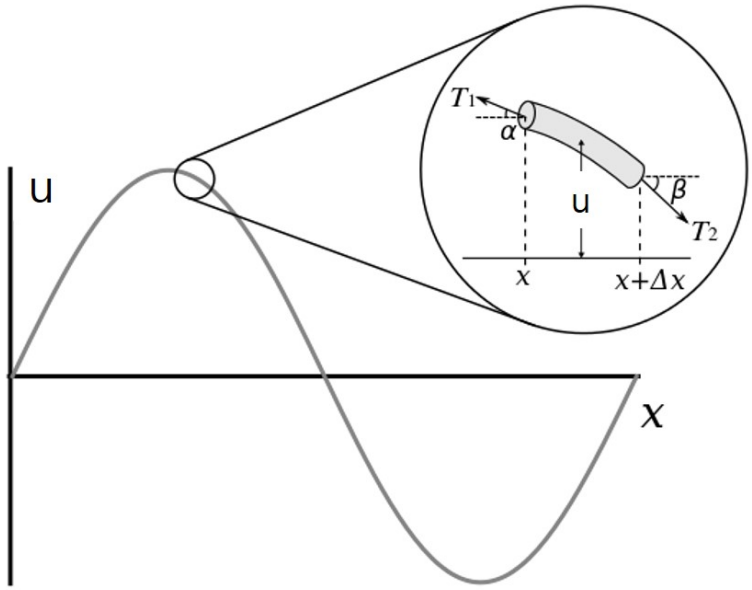
We will now begin to study the second major class of PDEs, **hyperbolic equations**

- **Vibrating-String Problem**

  We consider the small vibrations of a string that is fastened at each end



- To mathematically describe the vibrations of this string, we consider all the forces acting on a small section Δx of the string



If the horizontal component of tension is constant T, then the tension acting on each side of the string segment is given by

$$T_1 \cos\alpha \approx T$$
$$T_2 \cos\beta \approx T$$

- In the vertical component of Newton's second law, the mass of this piece $\rho\Delta x$ times its acceleration u_{tt} will be equal to the net force on the piece:

$$\begin{aligned}
\rho\Delta x u_{tt} &= T_2 \sin\beta + T_1 \sin\alpha \\
&\Downarrow \\
\frac{\rho\Delta x}{T} u_{tt} &= \frac{T_2 \sin\beta}{T_2 \cos\beta} + \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \tan\beta + \tan\alpha \\
&= u_x(x+\Delta x) - u_x(x) \\
&\Downarrow \\
u_{tt} &= \frac{T}{\rho} \frac{u_x(x+\Delta x) - u_x(x)}{\Delta x} \\
&\Downarrow \; c^2 = T/\rho, \; \Delta x \to 0 \\
u_{tt} &= c^2 u_{xx}
\end{aligned}$$

This is the wave equation for $u(x,t)$ and c is the speed of propagation of the wave in the string

13.2 D'Alembert Solution of the Wave Equation

- If the student recalls the parabolic case, we started solving diffusion problems when the space variable was bounded (by separation of variables), and then went on to solve the unbounded case (where $-\infty < x < \infty$) by the Fourier transform

- In the hyperbolic case (wave equation), we will do the opposite. We start by solving the one-dimensional wave equation in free space:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} && -\infty < x < \infty,\ 0 < t < \infty \\ u(x,0) &= f(x) && -\infty < x < \infty \\ u_t(x,0) &= g(x) \end{aligned} \tag{DA}$$

- We could solve this problem by using the Fourier transform (transforming x) or the Laplace transform (transforming t), but we will introduce yet a new technique (canonical coordinate), which will introduce the reader to several new and exciting ideas

STEP 1 Replacing (x,t) by new canonical coordinates (ξ,η)

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ &\Downarrow\ \xi = x + ct,\ \eta = x - ct \\ u_t &= u_\xi \xi_t + u_\eta \eta_t = c(u_\xi - u_\eta) \\ u_{tt} &= c(u_{\xi\xi} - u_{\eta\xi})\xi_t + c(u_{\xi\eta} - u_{\eta\eta})\eta_t \\ &= c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) \\ u_x &= u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta \\ u_{xx} &= u_{\xi\xi}\xi_x + u_{\eta\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\eta}\eta_x \\ &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \\ &\Downarrow \\ u_{\xi\eta} &= 0 \end{aligned}$$

STEP 2 Solving the Transformed Equations

$$\begin{gathered} u_{\xi\eta} = 0 \\ \Downarrow \\ \text{Integration with respect to } \xi \\ \Downarrow \\ u_\eta(\xi,\eta) = \varphi(\eta) \\ \Downarrow \\ \text{Integration with respect to } \eta \\ \Downarrow\ \ \phi = \int \varphi\, d\eta \\ u(\xi,\eta) = \phi(\eta) + \psi(\xi) \end{gathered}$$

STEP 3 Transforming back to the Original Coordinates x and t

$$\begin{aligned} u(\xi,\eta) &= \phi(\eta) + \psi(\xi) \\ &\Downarrow\ \xi = x + ct,\ \eta = x - ct \\ u(x,t) &= \phi(x - ct) + \psi(x + ct) \end{aligned}$$

NOTE This is the general solution of the wave equation, and it is interesting in that it physically represents the sum of any two moving waves, each moving in opposite directions with velocity c

STEP 4 Substituting the General Solution into the ICs

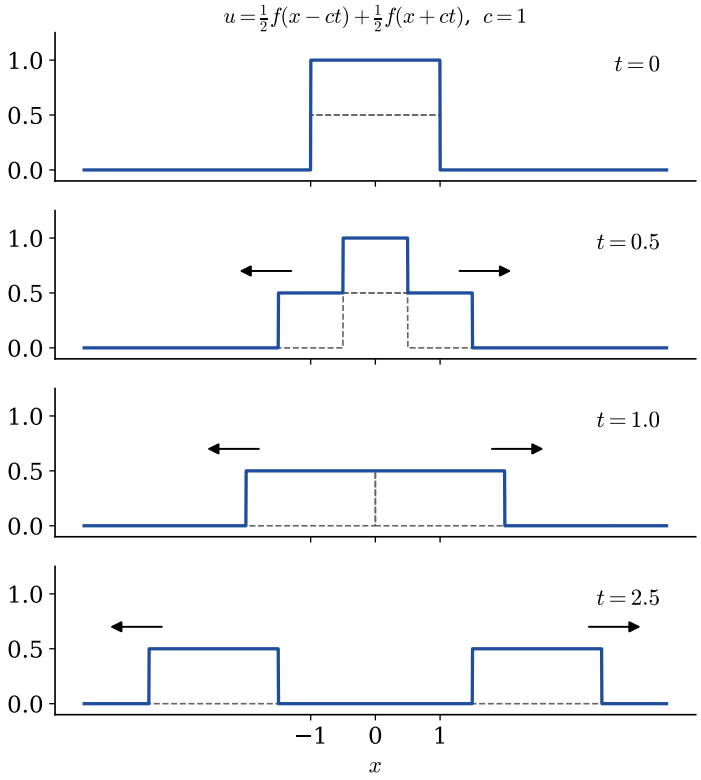
$$
\begin{aligned}
u(x,t) &= \phi(x-ct) + \psi(x+ct) \\
&\Downarrow\ u(x,0) = f(x),\ u_t(x,0) = g(x) \\
\phi(x) + \psi(x) &= f(x) \\
-c\phi_x(x) + c\psi_x(x) &= g(x) \\
&\Downarrow \text{ integrating the } 2^{nd} \text{equation from } x_0 \text{ to } x \\
\phi(x) + \psi(x) &= f(x) \\
-c\phi(x) + c\psi(x) &= \int_{x_0}^{x} g(\alpha)\,d\alpha + C \\
&\Downarrow \\
\phi(x) &= \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\alpha)\,d\alpha - \frac{C}{2c} \\
\psi(x) &= \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\alpha)\,d\alpha + \frac{C}{2c} \\
&\Downarrow
\end{aligned}
$$

$$
u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\alpha)\,d\alpha \qquad \text{(DAS)}
$$

This is what we were aiming for, and it is called the **D'Alembert solution** to (DA)

- **Motion of a Simple Square Wave**

$$
\begin{aligned}
u(x,0) &= \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{everywhere else} \end{cases} \\
u_t(x,0) &= 0
\end{aligned}
$$



- **Initial Velocity Given**

  Suppose now the initial position of the string is at equilibrium and we impose an initial velocity (as in a piano string) of $\sin x$

$$\begin{aligned} u(x,0) &= 0 \\ u_t(x,0) &= \sin x \end{aligned}$$

  Here, the solution would be

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct} \sin\xi \, d\xi \\ &= \frac{1}{2c}\left[\cos(x-ct) - \cos(x+ct)\right] \\ &= \frac{1}{c}\sin x \cdot \sin ct \end{aligned}$$

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib import animation, rc
from IPython.display import HTML

plt.rcParams['font.size'] = 12
plt.rcParams['xtick.direction'] = 'in'
plt.rcParams['ytick.direction'] = 'in'
```

```
fig = plt.figure(figsize=(6, 4))
ax = plt.axes(xlim=(-6.0*np.pi, 6.0*np.pi), ylim=(-1.5, 1.5))

ax.set_xticks([-6.0*np.pi,-3.0*np.pi, 0, 3.0*np.pi, 6.0*np.pi])
ax.set_xticklabels([r'$-6\pi$',r'$-3\pi$','$0$',r'$3\pi$',r'$6\pi$'])
ax.set_yticks([-1.2, -0.6, 0, 0.6, 1.2])
ax.set_xlabel('$x$')
ax.set_ylabel('$u(x,t)$')

plt.close()
```

```
time_text = ax.text(-1.5, 1.3, '')
line, = ax.plot([], [], lw=2)
def init():
    time_text.set_text('t = 0.0')
    line.set_data([], [])
    return (line,)

c = 1
def animate(t):
    time_text.set_text(f't = {t:3.1f}')
    x = np.linspace(-6.0*np.pi, 6.0*np.pi, 300)
    u = 1.0/(2.0*c)*(np.cos(x -c*t) -np.cos(x +c*t))
    line.set_data(x, u)
```

```
    return (line,)

tt = list(np.linspace(0, 2.0*np.pi/c, 100))
anim = animation.FuncAnimation(fig, animate,
        init_func=init, frames=tt, interval=200, blit=True)
HTML('<center>' + anim.to_html5_video() + '</center>')
```

Initial Velocity Given: $u_t(x,0) = \sin x$

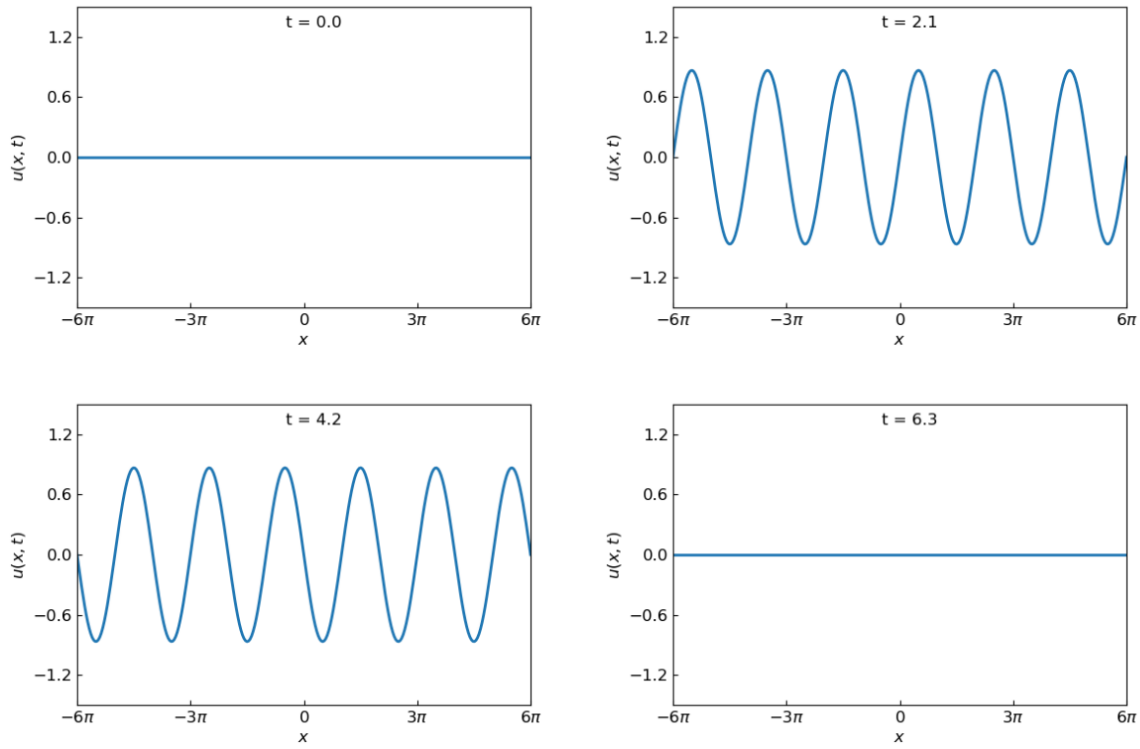


Figure 13.1: Four frames of the animation; play it on the web page

13.3 More on the D'Alembert Solution

- We proved that in the last section the solution of the pure initial-value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} && -\infty < x < \infty,\ 0 < t < \infty \\ u(x,0) &= f(x) && -\infty < x < \infty \\ u_t(x,0) &= g(x) \end{aligned}$$

  is given by

$$u(x,t) = \frac{1}{2}\left[\, f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

  We now present an interpretation of this solution in the xt-plane at the two specific cases

CASE 1 Initial Position Given; Initial Velocity Zero

- Let's consider the following initial condition

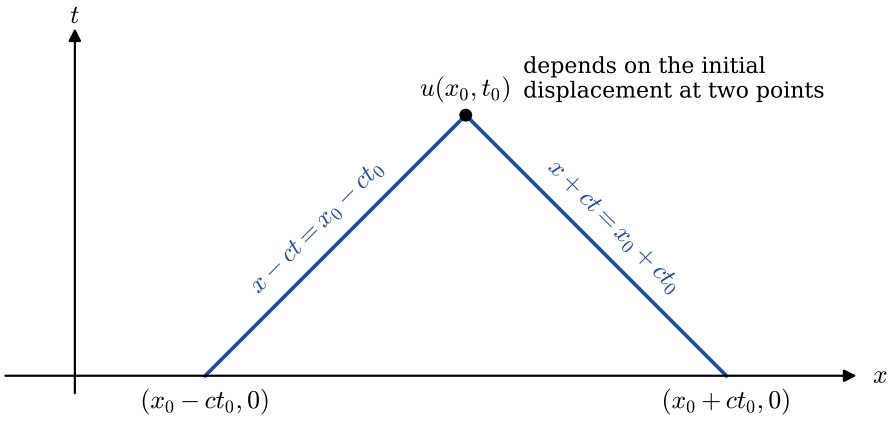
$$\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= 0 \end{aligned}, \quad -\infty < x < \infty$$

the D'Alembert solution is

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]$$

and the solution $u(x_0,t_0)$ can be interpreted as being the average of the initial displacement $f(x)$ at the points $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$ found by backtracking along the lines

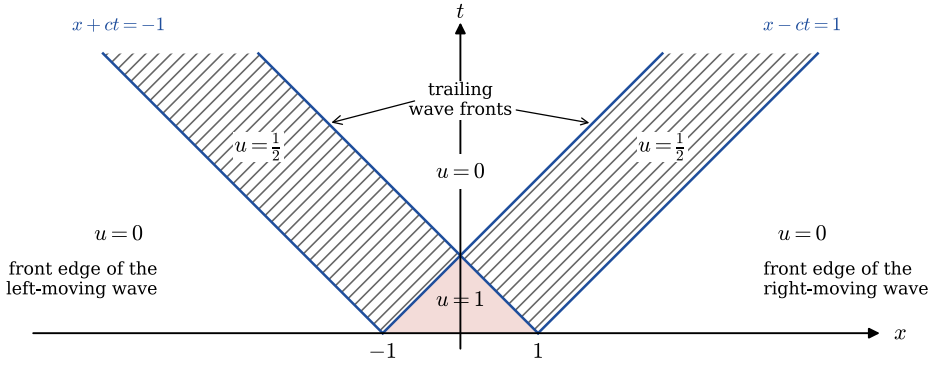
$$\begin{aligned} x - ct &= x_0 - ct_0 \\ x + ct &= x_0 + ct_0 \end{aligned} \quad \text{characteristic curves}$$



- For example, using this interpretation, the initial condition

$$\begin{aligned} u(x,0) &= \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{everywhere else} \end{cases} \\ u_t(x,0) &= 0 \end{aligned}$$

would give us the solution in the xt-plane shown in figure



CASE 2 Initial Displacement Zero; Velocity Arbitrary

- Consider now the IC

$$\begin{matrix} u(x,0) = 0 \\ u_t(x,0) = g(x) \end{matrix}, \quad -\infty < x < \infty$$

the solution is

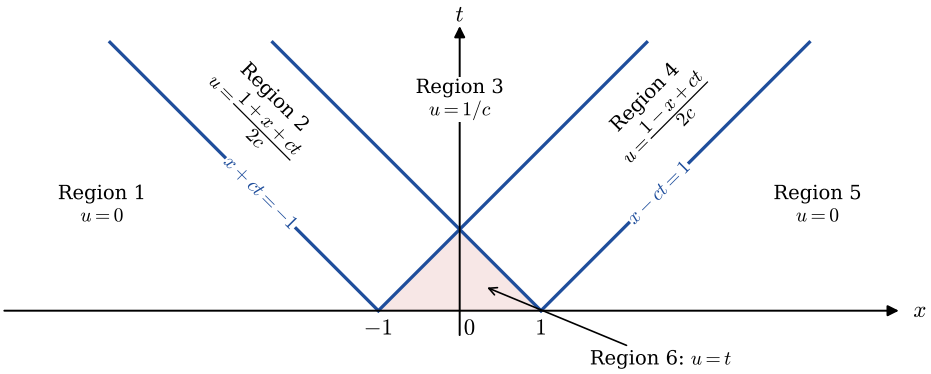
$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

and, hence, the solution $u(x_0,t_0)$ can be interpreted as integrating the initial velocity between $x_0 - ct_0$ and $x_0 + ct_0$ on the initial line $t = 0$

- Again, using this interpretation, the solution to the initial-value problem

$$u(x,0) = 0 \quad -\infty < x < \infty$$
$$u_t(x,0) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{everywhere else} \end{cases}$$

has a solution in the xt-plane in figure



- To find the displacement, we compute the D'Alembert solution

$$
\begin{aligned}
u(x,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi \\
&= \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,d\xi = 0, && (x,t) \in \text{Region } 1 \\
&= \frac{1}{2c}\int_{-1}^{x+ct} 1\,d\xi = \frac{1+x+ct}{2c}, && (x,t) \in \text{Region } 2 \\
&= \frac{1}{2c}\int_{-1}^{1} 1\,d\xi = \frac{1}{c}, && (x,t) \in \text{Region } 3 \\
&= \frac{1}{2c}\int_{x-ct}^{1} 1\,d\xi = \frac{1-x+ct}{2c}, && (x,t) \in \text{Region } 4 \\
&= \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,d\xi = 0, && (x,t) \in \text{Region } 5 \\
&= \frac{1}{2c}\int_{x-ct}^{x+ct} 1\,d\xi = t, && (x,t) \in \text{Region } 6
\end{aligned}
$$

```
fig = plt.figure(figsize=(6, 4))
ax = plt.axes(xlim=(-15, 15), ylim=(-0.1, 1.5))

ax.set_xticks([-15, -10, -5, 0, 5, 10, 15])
ax.set_yticks([0, 0.5, 1.0, 1.5])
ax.set_xlabel('$x$')
ax.set_ylabel('$u(x,t)$')

plt.close()

time_text = ax.text(-2, 1.3, '')
line, = ax.plot([], [], lw=2)
def init():
    time_text.set_text('t = 0.0')
    line.set_data([], [])
    return (line,)
```

```
c = 1
def animate(t):
    time_text.set_text(f't = {t:3.1f}')
    xx = np.linspace(-15, 15, 300)
    uu = np.zeros_like(xx)

    for i, x in enumerate(xx):
        ch1 = x +c*t
        ch2 = x -c*t
        if ch1 <-1.0 or ch2 > 1.0:
            uu[i] = 0.0
        elif t < 1.0 /c:
            if ch2 <-1.0:
                uu[i] = (1.0 +ch1) /(2.0*c)
            elif ch1 < 1.0:
```

```
                uu[i] = t
            else:
                uu[i] = (1.0 -ch2) /(2.0*c)
        else:
            if ch1 < 1.0:
                uu[i] = (1.0 +ch1) /(2.0*c)
            elif ch2 <-1.0:
                uu[i]=1.0 /c
            else:
                uu[i] = (1.0 -ch2) /(2.0*c)

    line.set_data(xx, uu)
    return (line,)
```

```
tt = list(np.linspace(0, 20, 50))
anim = animation.FuncAnimation(fig, animate, init_func=init,
    frames=tt, interval=300, blit=True)

HTML('<center>' + anim.to_html5_video() + '</center>')
```

Initial Velocity Given: $u_t(x,0) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{everywhere else} \end{cases}$

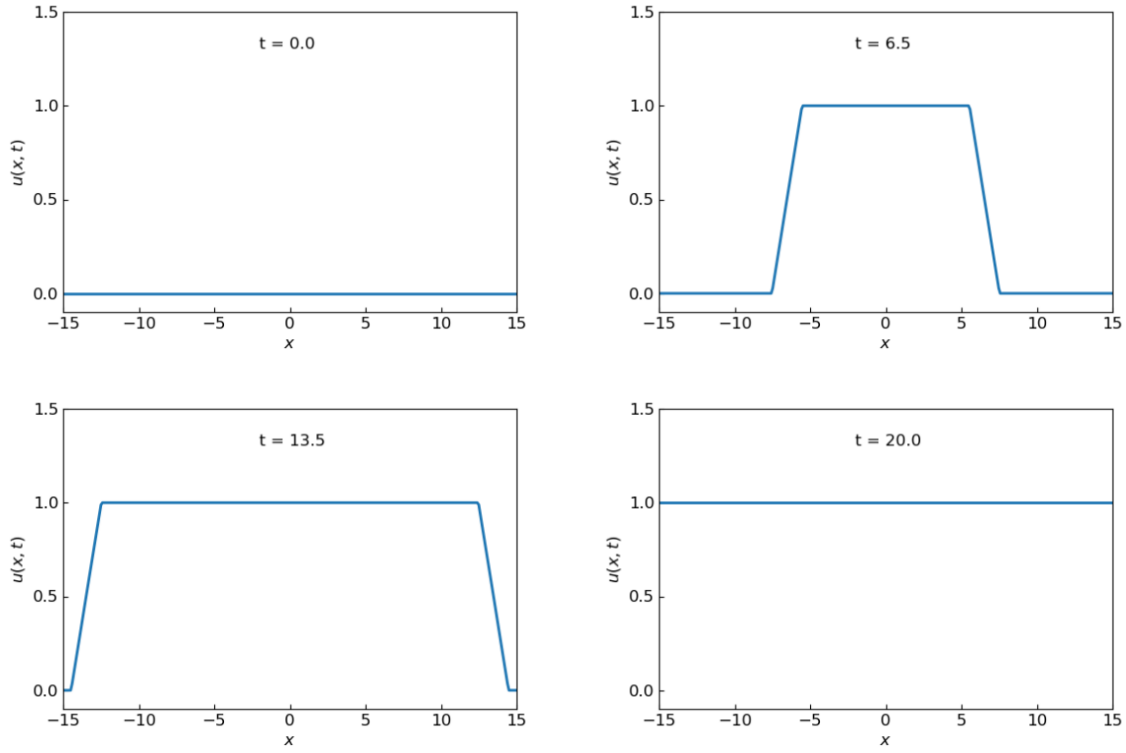


Figure 13.2: Four frames of the animation; play it on the web page

Solution of the Semi-Infinite String via the D'Alembert Formula

- In the remainder of the section, we will solve the initial-boundary-value problem for the semi-infinite string

$$\begin{aligned} u_{tt} &= c^2 u_{xx} && 0 < x < \infty,\ 0 < t < \infty \\ u(0,t) &= 0 && 0 < t < \infty \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) && 0 < x < \infty \end{aligned}$$

- We proceed in a manner similar to that used with the infinite string, which is to find the general solution to the PDE

$$u(x,t) = \phi(x - ct) + \psi(x + ct) \tag{GS}$$

If we now substitute this general solution into the initial conditions, we arrive at

$$\begin{aligned} \phi(x - ct) &= \frac{1}{2} f(x - ct) - \frac{1}{2c} \int_{x_0}^{x-ct} g(\xi)\, d\xi - \frac{C}{2c} \\ \psi(x + ct) &= \frac{1}{2} f(x + ct) + \frac{1}{2c} \int_{x_0}^{x+ct} g(\xi)\, d\xi + \frac{C}{2c} \end{aligned} \tag{IM}$$

- We now have a problem we didn't encounter when dealing with the infinite string. Since we are looking for the solution $u(x,t)$ everywhere in the first quadrant $(x > 0,\ t > 0)$ of the xt-plane, it is obvious that we must find

$$\begin{aligned} &\phi(x - ct) && \text{for all} && -\infty < x - ct < \infty \\ &\psi(x + ct) && \text{for all} && 0 < x + ct < \infty \end{aligned}$$

- Unfortunately, the first equation only gives us $\phi(x - ct)$ for $x - ct \geq 0$, since our initial data $f(x)$ and $g(x)$ are only known for positive arguments
- As long as $x - ct \geq 0$, we have no problem, since we can substitute (IM) into the general solution (GS) to get

$$u(x,t) = \frac{1}{2}\left[f(x - ct) + f(x + ct)\right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi)\, d\xi, \quad x \geq ct$$

The question is, what to do when $x < ct$?

- When $x < ct$, substituting the general solution (GS) into the BC $u(0,t) = 0$ gives

$$\phi(-ct) = -\psi(ct)$$

and, hence, by functional substitution

$$\phi(x - ct) = -\frac{1}{2} f(ct - x) - \frac{1}{2c} \int_{x_0}^{ct-x} g(\xi)\, d\xi - \frac{C}{2c}$$

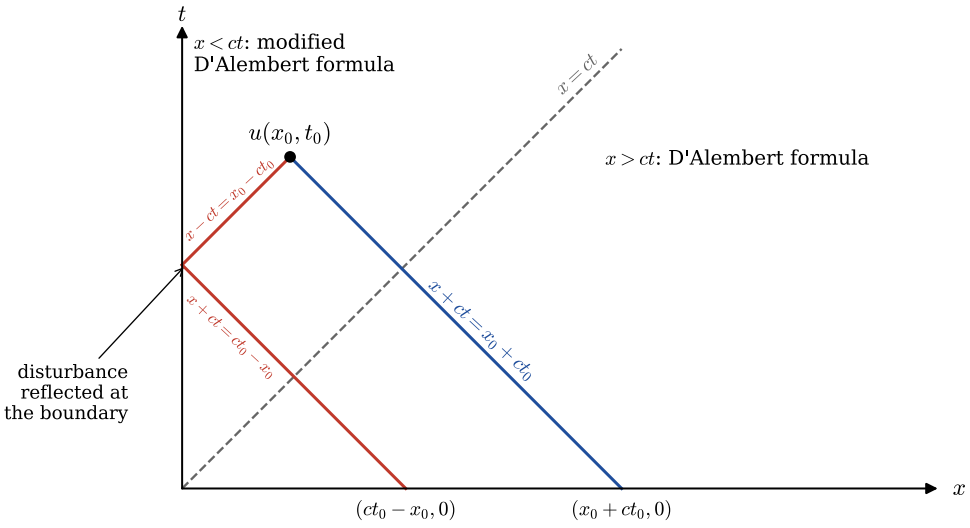
- Substituting this value of ϕ into the general solution (GS) gives

$$u(x,t) = \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\xi)\,d\xi, \quad 0 < x < ct$$

- For $x \geq ct$, the solution is the same as the D'Alembert solution for the infinite wave, while for $x < ct$, the solution $u(x,t)$ is modified as a result of the wave reflecting from the boundary (The sign of the wave is changed when it's reflected)
- The straight lines

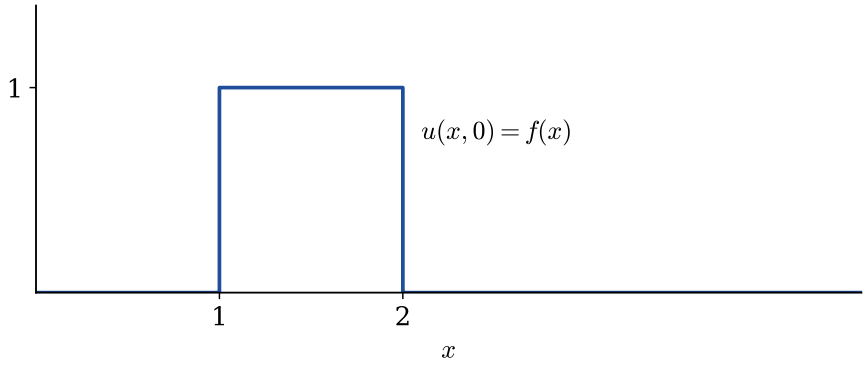
$$x + ct = \text{constant}$$
$$x - ct = \text{constant}$$

are known as **characteristics**, and it is along these lines that disturbances are propagated. Characteristics are generally associated with hyperbolic equations



Example Solve the semi-infinite string problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} && 0 < x < \infty,\ 0 < t < \infty \\ u(0,t) &= 0 && 0 < t < \infty \\ u(x,0) &= f(x) \\ u_t(x,0) &= 0 && 0 < x < \infty \end{aligned}$$



Answer 1 (end of chapter) · Solution

```
fig = plt.figure(figsize=(6, 4))
ax = plt.axes(xlim=(0, 5), ylim=(-1.0, 1.5))

ax.set_xticks([0, 2.5, 5])
ax.set_yticks([-1.0, -0.5, 0.0, 0.5, 1.0, 1.5])
ax.set_xlabel('$x$')
ax.set_ylabel('$u(x,t)$')

plt.close()

time_text = ax.text(2.2, 1.3, '')
line, = ax.plot([], [], lw=2)
def init():
    time_text.set_text('t = 0.0')
    line.set_data([], [])
    return (line,)
```

```
c = 1
def f_i(x):
    if x >= 1 and x <= 2:
        u = 1.0
    else:
        u = 0.0
    return u

def animate(t):
    time_text.set_text(f't = {t:3.1f}')
    xx = np.linspace(0, 5, 400)
    uu = np.zeros_like(xx)

    for i, x in enumerate(xx):
        ch1 = x +c*t
        ch2 = x -c*t
        if ch2 >= 0:
            uu[i] = 0.5*(f_i(ch1) +f_i(ch2))
        else:
            uu[i] = 0.5*(f_i(ch1) -f_i(-ch2))
```

```
    line.set_data(xx, uu)
    return (line,)
```

```
tt = list(np.linspace(0, 7, 100))
anim = animation.FuncAnimation(fig, animate,
        init_func=init, frames=tt, interval=750, blit=True)
HTML('<center>' + anim.to_html5_video() + '</center>')
```

Semi-infinite String

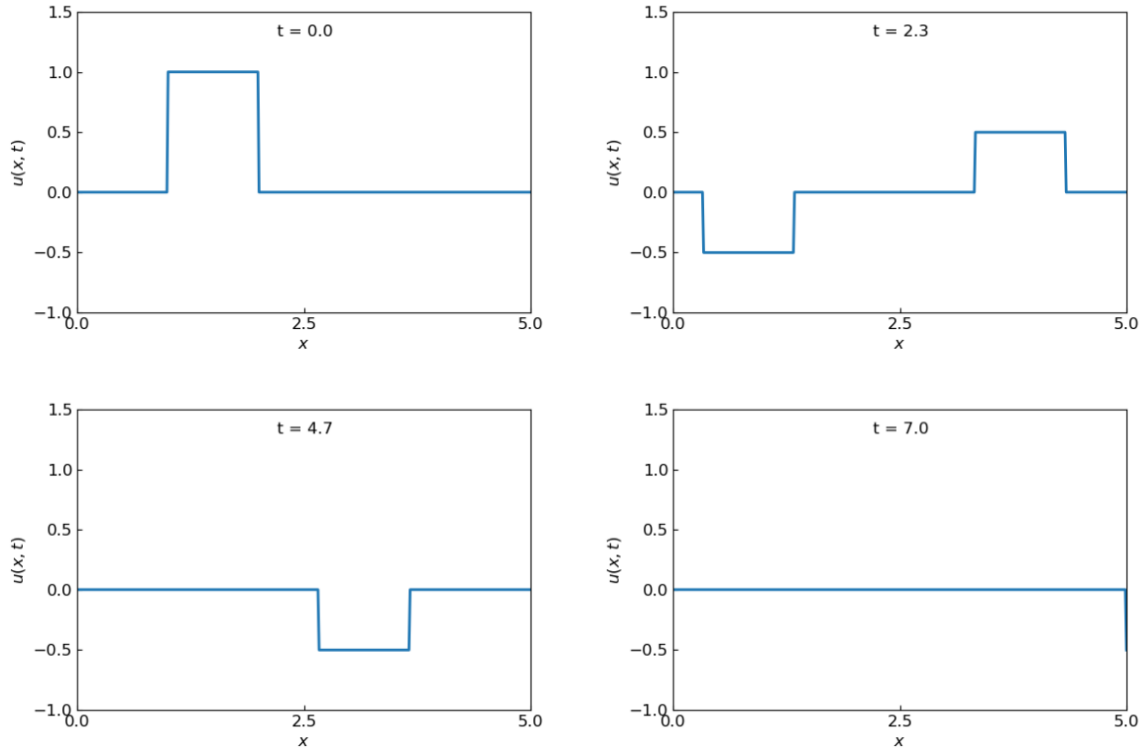


Figure 13.3: Four frames of the animation; play it on the web page

Example Solve the semi-infinite string problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} && 0 < x < \infty,\ 0 < t < \infty \\ u_x(0,t) &= 0 && 0 < t < \infty \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) && 0 < x < \infty \end{aligned}$$

following an approach analogous to the one used to solve the semi-infinite string problem in the lesson

Solution For $x \geq ct$, the solution is the same as the D'Alembert solution for the infinite wave, while for $x < ct$, the solution $u(x,t)$ is modified as a result of the wave reflecting from the boundary

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(ct-x)\right] + \frac{1}{2c}\left[\int_0^{ct-x} g(\xi)\,d\xi + \int_0^{x+ct} g(\xi)\,d\xi\right], \quad 0 < x < ct$$

Example Simulate the semi-infinite string problem with the initial conditions:

$$u(x,0) = \exp\left[-\frac{(x-3)^2}{0.5}\right]$$
$$u_t(x,0) = 0$$

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation

# ================================================================
#  Parameters and Grid
# ================================================================
c = 1.0                         # Wave speed
x_max = 10.0                    # Half of spatial domain
nx = 800                        # Number of spatial points
x = np.linspace(-x_max, x_max, nx)  # Include negative side

# ================================================================
#  Initial Conditions
# ================================================================
def f(x):
    """Initial displacement: Gaussian pulse centered at x=3"""
    return np.exp(-(x - 3)**2 /0.5)

def g(x):
    """Initial velocity: zero everywhere."""
    return np.zeros_like(x)

# ================================================================
#  Extensions for boundary reflection behavior
# ================================================================
def f_even(x):
    """Even extension for Neumann BC (symmetric reflection)"""
    return f(abs(x))

def f_odd(x):
    """Odd extension for Dirichlet BC (sign-flipped reflection)"""
    return np.sign(x) *f(abs(x))

# ================================================================
#  Analytic solutions (using extended wave functions)
# ================================================================
def u_neumann(x, t):
    """Solution for Neumann BC (u_x(0,t)=0)"""
```

```
    return 0.5 * (f_even(x -c*t) +f_even(x +c*t))

def u_dirichlet(x, t):
    """Solution for Dirichlet BC (u(0,t)=0)"""
    return 0.5 * (f_odd(x -c*t) +f_odd(x +c*t))

# ================================================================
#  Visualization Setup
# ================================================================
fig, ax = plt.subplots(figsize=(8, 4))
# Main curves (physical side)
line_neu, = ax.plot([], [], lw=2, color='royalblue', label='Neumann (uₓ(0,t)=0)')
line_dir, = ax.plot([], [], lw=2, color='crimson',  label='Dirichlet (u(0,t)=0)')
# Ghost (mirror) curves for visual symmetry
ghost_neu, = ax.plot([], [], lw=1, color='royalblue', alpha=0.3, ls='--')
ghost_dir, = ax.plot([], [], lw=1, color='crimson',  alpha=0.3, ls='--')

ax.axvline(0, color='gray', ls='--', lw=1.2, label='Boundary x=0')
ax.set_xlim(-x_max, x_max)
ax.set_ylim(-1.2, 1.2)
ax.set_xlabel("x")
ax.set_ylabel("u(x,t)")
ax.legend(loc='upper right')
ax.set_title("Wave Reflection Symmetry (Neumann vs Dirichlet)")

# ================================================================
#  Animation Functions
# ================================================================
def init():
    """Initialize empty lines for animation frames"""
    for ln in [line_neu, line_dir, ghost_neu, ghost_dir]:
        ln.set_data([], [])
    return line_neu, line_dir, ghost_neu, ghost_dir

def animate(frame):
    """Update frame at each time step"""
    t = frame * 0.05
    # Full-domain solutions
    y_neu = u_neumann(x, t)
    y_dir = u_dirichlet(x, t)

    # Restrict main waves to x > 0 (physical region)
    mask_pos = x >= 0
    # "Ghost" mirrors for x < 0 to visualize symmetry
    mask_neg = x <= 0

    # Main (physical side)
    line_neu.set_data(x[mask_pos], y_neu[mask_pos])
    line_dir.set_data(x[mask_pos], y_dir[mask_pos])
    # Mirror (imaginary side)
    ghost_neu.set_data(x[mask_neg], y_neu[mask_neg])
    ghost_dir.set_data(x[mask_neg], y_dir[mask_neg])
```

```
    ax.set_title(f"t = {t:.2f} — Blue: Neumann (symmetric) | Red: Dirichlet (sign-flipped)")
    return line_neu, line_dir, ghost_neu, ghost_dir

# ============================================================
#  Run the animation
# ============================================================
ani = FuncAnimation(fig, animate, init_func=init, frames=200, interval=50, blit=True)
ani.save("figures/reflection.mp4", writer="ffmpeg", fps=30)

plt.close()
```

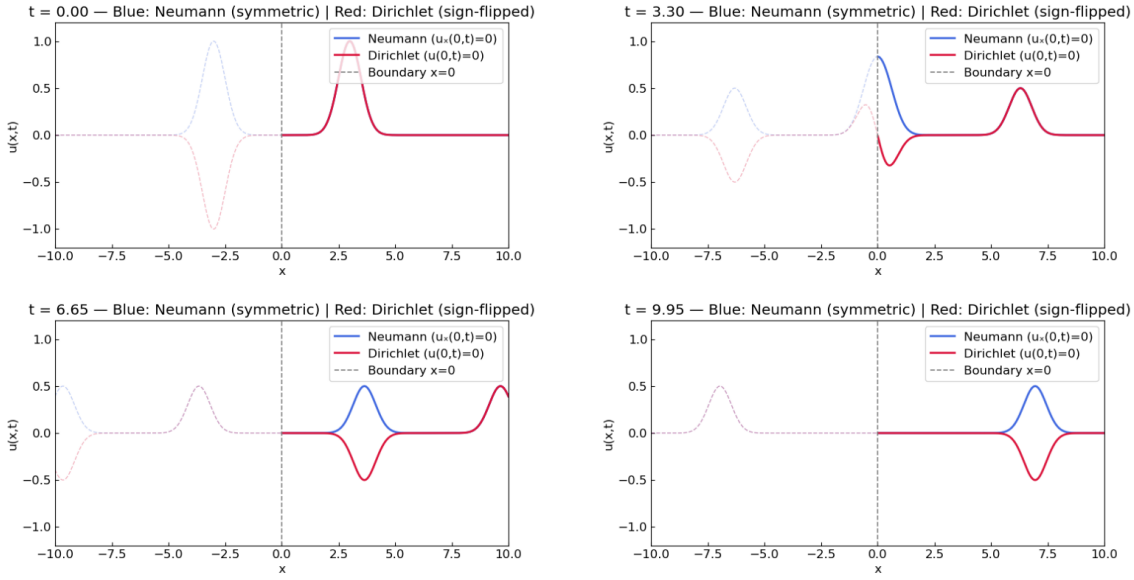


Figure 13.4: Four frames of the animation; play it on the web page

Example Solve the wave equation on the semi-infinite string $x \geq 0$:

$$u_{tt} = c^2 u_{xx}, \quad x > 0,\ t > 0$$

with initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad (x \geq 0)$$

and **a nonhomogeneous boundary condition**

$$u(0,t) = h(t), \quad t \geq 0$$

Thus, the difficulty comes from the fact that the boundary value at $x = 0$ is not zero, so the standard odd-reflection D'Alembert method cannot be applied directly

STEP 1: Constructing a Boundary-Generating Wave

To handle the nonhomogeneous boundary condition, we first construct a function $w(x,t)$ that:

- satisfies the wave equation,
- satisfies the boundary condition $w(0,t) = h(t)$,
- and does not affect the given initial data

A convenient choice is a right-traveling wave of the form

$$w(x,t) = H\Big(t - \frac{x}{c}\Big)$$

where $H(s)$ is the causal extension of h:

$$H(s) = \begin{cases} h(s), & s \geq 0, \\ 0, & s < 0 \end{cases}$$

This function meets the boundary condition $w(0,t) = h(t)$ and has zero initial values:

$$w(x,0) = 0, \quad w_t(x,0) = 0$$

STEP 2: Reduction to a Homogeneous Boundary Problem

Define $v(x,t) = u(x,t) - w(x,t)$. Then v satisfies:

- the same wave equation,
$$v_{tt} = c^2 v_{xx}$$
- homogeneous Dirichlet boundary condition
$$v(0,t) = u(0,t) - w(0,t) = h(t) - h(t) = 0$$
- unchanged initial data,
$$v(x,0) = f(x), \quad v_t(x,0) = g(x)$$

Thus, the original nonhomogeneous boundary-value problem is transformed into a standard homogeneous Dirichlet problem on a half-line, which can then be solved by odd reflection and the usual D'Alembert formula

Example Solve the wave equation on the half-line $x \geq 0$:

$$u_{tt} = c^2 u_{xx}, \quad x > 0,\ t > 0$$

with initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

and a nonhomogeneous Neumann boundary condition

$$u_x(0,t) = p(t), \quad t \geq 0$$

STEP 1: Constructing a Boundary-Induced Wave

We construct a right-traveling wave of the form

$$w(x,t) = -c\,P\Big(t - \frac{x}{c}\Big)$$

where $P(s)$ is the causal antiderivative of $p(t)$:

$$P(s) = \begin{cases} \displaystyle\int_0^s p(\tau)\,d\tau, & s \geq 0 \\ 0, & s < 0 \end{cases}$$

This function satisfies $w_x(0,t) = p(t)$ and has zero initial values,

$$w(x,0) = 0, \quad w_t(x,0) = 0$$

so it produces exactly the incoming wave required by the boundary without disturbing the initial conditions

STEP 2: Reduction to a Homogeneous Neumann Problem

Define $v(x,t) = u(x,t) - w(x,t)$. Then v satisfies

- the same wave equation,
$$v_{tt} = c^2 v_{xx}$$
- a homogeneous Neumann boundary condition,
$$v_x(0,t) = u_x(0,t) - w_x(0,t) = p(t) - p(t) = 0$$
- and unchanged initial data,
$$v(x,0) = f(x), \qquad v_t(x,0) = g(x)$$

Thus the original nonhomogeneous Neumann problem is transformed into a standard homogeneous Neumann half-line problem, which can then be solved by an even extension of the initial data and the classical D'Alembert formula

Solution of the Wave Equation on $x \geq 0$ with a Robin Boundary Condition

We consider the wave equation on the half-line

$$u_{tt} = c^2 u_{xx}, \quad x > 0,\ t > 0$$

with initial data

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x)$$

and a homogeneous Robin boundary condition

$$\alpha u(0,t) + \beta u_x(0,t) = 0, \qquad t \geq 0$$

where $(\alpha, \beta) \neq (0,0)$

To solve this problem, we apply the Laplace transform in time and derive an explicit formula for the transformed solution, which then leads to a time-domain representation involving a delayed convolution kernel

STEP 1: Laplace Transform in t

Let $U(x,s) = \mathcal{L}\{u(x,t)\}$. Transforming the wave equation yields the ODE

$$c^2 U_{xx}(x,s) - s^2 U(x,s) = -(sf(x) + g(x))$$

The general solution of this inhomogeneous ODE is

$$U(x,s) = A(s)e^{-sx/c} + V_p(x,s)$$

where the first term is the decaying homogeneous solution and

$$V_p(x,s) = \frac{1}{2cs}\int_0^\infty e^{-\frac{s}{c}|x-y|}\,(sf(y) + g(y))\,dy$$

is a standard particular solution for the half-line

STEP 2: Enforcing the Robin Boundary Condition

Transforming the boundary condition gives

$$\alpha U(0,s) + \beta U_x(0,s) = 0$$

Since

$$U(0,s) = A(s) + V_p(0,s), \quad U_x(0,s) = -\frac{s}{c}A(s) + V_{p,x}(0,s),$$

we substitute these and solve for $A(s)$. Using the identities

$$V_{p,x}(0,s) = \frac{s}{c}\,V_p(0,s)$$

we obtain the closed formula

$$A(s) = -\frac{\alpha c + \beta s}{\alpha c - \beta s}\,V_p(0,s)$$

Thus the transformed solution becomes

$$\boxed{U(x,s) = V_p(x,s) - \frac{\alpha c + \beta s}{\alpha c - \beta s}\,V_p(0,s)\,e^{-sx/c}.}$$

The factor

$$M(s) = \frac{\alpha c + \beta s}{\alpha c - \beta s}$$

acts as the Laplace-domain reflection coefficient at the Robin boundary

STEP 3: Inverse Transform and Time-Domain Representation

Write the inverse Laplace transform of

$$M(s) = \frac{\alpha c + \beta s}{\alpha c - \beta s} = -1 - \frac{2\alpha c}{\beta s - \alpha c}$$

Its inverse transform is

$$m(t) = \mathcal{L}^{-1}\{M(s)\} = -\delta(t) - \frac{2\alpha c}{\beta}\, e^{\frac{\alpha c}{\beta} t} H(t),$$

a causal kernel consisting of

- an instantaneous reflection $(-\delta)$,
- an exponentially weighted memory term

Let

$$v_{\text{free}}(x,t) = \mathcal{L}^{-1}\{V_p(x,s)\}$$

be the free solution (the D'Alembert wave produced by the initial data)

Applying the convolution–delay rules of the Laplace transform gives the time-domain solution

$$\boxed{\begin{aligned} u(x,t) &= v_{\text{free}}(x,t) - (m * v_{\text{free}}(0,\cdot))\Big(t - \frac{x}{c}\Big) \\ &= v_{\text{free}}(x,t) + v_{\text{free}}\,(0, t - \tfrac{x}{c}) \\ &\quad + \frac{2\alpha c}{\beta} \int_0^{t-\frac{x}{c}} e^{\frac{\alpha c}{\beta}(t-\frac{x}{c}-s)} v_{\text{free}}(0,s)\, ds \end{aligned}}$$

valid for $t \geq x/c$, extended by causality for $t < x/c$

This formula expresses the solution as:

- the direct wave from the initial data, plus
- a reflected wave generated by the boundary, described by a convolution kernel that incorporates the Robin boundary's impedance

The Nonhomogeneous Wave Equation

- We consider now the pure nonhomogeneous wave equation:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + F(x,t) && -\infty < x < \infty,\ 0 < t < \infty \\ u(x,0) &= 0 \\ u_t(x,0) &= 0 && -\infty < x < \infty \end{aligned}$$

  which includes the forcing term $F(x,t)$, but zero initial conditions

- We again make the change of variables $\xi = x + ct$ and $\eta = x - ct$. The differential equation then becomes

$$u_{\xi\eta} = -\frac{1}{4c^2} F(\xi, \eta)$$

$$\left.\begin{matrix} u = 0 \\ u_\xi = u_\eta = 0 \end{matrix}\right\} \text{ at } \xi = \eta \quad \leftarrow \quad \left.\begin{matrix} u = 0 \\ u_t = 0 \end{matrix}\right\} \text{ at } t = 0$$

$$\Downarrow \text{ integrating with respect to } \xi \text{ from } \eta \text{ to } \xi$$

$$u_\eta(\xi, \eta) = -\frac{1}{4c^2} \int_\eta^\xi F\left(\bar{\xi}, \eta\right)\, d\bar{\xi} + \underbrace{u_\eta(\eta, \eta)}_{=0}$$

$$\Downarrow \quad u(\xi,\eta)=u(\xi,\xi)-\textstyle\int_\eta^\xi u_\eta(\xi,\bar{\eta})\, d\bar{\eta},\ u(\xi,\xi)=0$$

$$u = \frac{1}{4c^2} \int_\eta^\xi \left[\int_{\bar{\eta}}^\xi F\left(\bar{\xi}, \bar{\eta}\right)\, d\bar{\xi} \right] d\bar{\eta}$$

- We let $\bar{\xi} = \beta + c\tau$ and $\bar{\eta} = \beta - c\tau$, $(t \geq \tau,\ \beta = x)$. The domain of integration $\eta \leq \bar{\eta} \leq \bar{\xi} \leq \xi$ becomes

$$\begin{gathered} \eta \leq \beta - c\tau \leq \beta + c\tau \leq \xi \\ \Downarrow \\ \eta + c\tau \leq \beta \leq \xi - c\tau \\ 0 \leq \tau \leq \frac{1}{2c}(\xi - \eta) \end{gathered}$$

- The transformation from $(\bar{\xi},\ \bar{\eta})$ to $(\beta,\ \tau)$ gives the solution formula

$$u = \frac{1}{4c^2} \int_\eta^\xi \left[\int_{\bar{\eta}}^\xi F\left(\bar{\xi}, \bar{\eta}\right)\, d\bar{\xi} \right] d\bar{\eta}$$

$$\Downarrow\ d\bar{\xi}\, d\bar{\eta} = \left\| \begin{matrix} \frac{\partial\bar{\xi}}{\partial\beta} & \frac{\partial\bar{\xi}}{\partial\tau} \\ \frac{\partial\bar{\eta}}{\partial\beta} & \frac{\partial\bar{\eta}}{\partial\tau} \end{matrix} \right\| d\beta\, d\tau = |-2c|\, d\beta\, d\tau = 2c\, d\beta\, d\tau$$

$$= \frac{1}{2c} \int_0^{(\xi-\eta)/2c} \left[\int_{\eta+c\tau}^{\xi-c\tau} F\left(\beta, \tau\right)\, d\beta \right] d\tau$$

$$\Downarrow\ \xi = x + ct,\ \eta = x - ct$$

$$u = \frac{1}{2c} \int_0^t \left[\int_{x-c(t-\tau)}^{x+c(t-\tau)} F\left(\beta, \tau\right)\, d\beta \right] d\tau$$

This expression shows how every part of the forcing function $F(x, t)$ contributes to the displacement $u(x, t)$ through the wave propagation mechanism described by D'Alembert's principle

- The Green's function of the one-dimensional wave equation is given by

$$G(x,t) = \frac{1}{2c} H\left(t - \tfrac{|x|}{c}\right)$$

where $H(t)$ is the Heaviside step function. According to Duhamel's principle, the solution $u(x,t)$ can be written as a space–time convolution of the forcing term $F(x,t)$ with the Green's function $G(x,t)$:

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-\beta, t-\tau)\, F(\beta,\tau)\, d\beta\, d\tau$$

Substituting the form of $G(x,t)$ and simplifying, we obtain the explicit D'Alembert-type integral solution

- The inner integral

$$\int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\beta,\tau)\, d\beta$$

represents the total effect of the forcing term F over the spatial region that influences the point x after time $t-\tau$ It accounts for the wave propagation with speed c

- The outer integral $\int_0^t (\cdots)\, d\tau$ sums up all the contributions over time from the initial moment $\tau = 0$ to the current time t
- The factor $\frac{1}{2c}$ arises from the Green's function normalization in one dimension

Example Solve the infinite string problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + e^{-ax^2-bt} && -\infty < x < \infty,\ 0 < t < \infty \\ u(x,0) &= 0 && -\infty < x < \infty \\ u_t(x,0) &= 0 \end{aligned}$$

Answer 2 (end of chapter) · Solution

13.4 The Wave Equation in Two and Three Dimensions (Free Space)

The problem of this section is to generalize the D'Alembert solution to two and three dimensions

- **Waves in Three Dimensions**

We start by considering spherical waves in three dimensions that have given ICs; that is, we would like to solve the initial value problem

$$\begin{aligned} u_{tt} &= c^2(u_{xx} + u_{yy} + u_{zz}), \quad \begin{cases} -\infty < x < \infty \\ -\infty < y < \infty \\ -\infty < z < \infty \end{cases} \\ u(x,y,z,0) &= \phi(x,y,z) \\ u_t(x,y,z,0) &= \psi(x,y,z) \end{aligned} \tag{3D}$$

– To solve this problem, we first solve the simpler one (set $\phi = 0$)

$$\begin{aligned} u_{tt} &= c^2(u_{xx} + u_{yy} + u_{zz}), \quad (x, y, z) \in \mathbb{R}^3 \\ u(x, y, z, 0) &= 0 \\ u_t(x, y, z, 0) &= \psi(x, y, z) \end{aligned} \tag{3Dv}$$

This problem can be solved by the Fourier transform

$$\mathcal{F}(u) = \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} u(x, y, z, t) e^{-i(\omega_1 x + \omega_2 y + \omega_3 z)}\, dx\, dy\, dz = U(\omega_1, \omega_2, \omega_3, t)$$
$$\mathcal{F}(\psi) = \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} \psi(x, y, z) e^{-i(\omega_1 x + \omega_2 y + \omega_3 z)}\, dx\, dy\, dz = \Psi(\omega_1, \omega_2, \omega_3)$$

$$\Downarrow \quad \mathcal{F}^{-1}$$

$$u(x, y, z, t) = \frac{1}{(2\pi)^{3/2}} \lim_{L\to\infty} \iiint_{\omega_1^2+\omega_2^2+\omega_3^2<L^2} \Psi(\omega_1, \omega_2, \omega_3) \frac{\sin ct\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}}{c\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}} e^{i(\omega_1 x + \omega_2 y + \omega_3 z)}\, d\omega_1\, d\omega_2\, d\omega_3$$

$$= \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} \psi(\xi, \eta, \zeta) \left[\frac{1}{(2\pi)^{3/2}} \lim_{L\to\infty} \iiint_{\omega_1^2+\omega_2^2+\omega_3^2<L^2} e^{-i[\omega_1(\xi-x)+\omega_2(\eta-y)+\omega_3(\zeta-z)]} \frac{\sin ct\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}}{c\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}}\, d\omega_1 \right.$$

– We introduce spherical coordinates $(\varrho, \varphi, \vartheta)$ in the $(\omega_1, \omega_2, \omega_3)$ space with the north pole $\varphi = 0$ in the direction of the vector $\langle \xi - x, \eta - y, \zeta - z \rangle$. Then the integral in $(\omega_1, \omega_2, \omega_3)$ becomes

$$\frac{1}{(2\pi)^{3/2}} \lim_{L\to\infty} \iiint_{\omega_1^2+\omega_2^2+\omega_3^2<L^2} e^{-i[\omega_1(\xi-x)+\omega_2(\eta-y)+\omega_3(\zeta-z)]} \frac{\sin ct\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}}{c\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}}\, d\omega_1\, d\omega_2\, d\omega_3$$

$$\Downarrow \; r = |\mathbf{r}| = \sqrt{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2}, \quad \varrho = |\varrho| = \sqrt{\omega_1^2+\omega_2^2+\omega_3^2}$$

$$= \frac{1}{(2\pi)^{3/2}} \lim_{L\to\infty} \int_0^L \int_0^{2\pi} \int_0^{\pi} e^{-ir\varrho\cos\varphi} \frac{\sin ct\varrho}{c\varrho} \varrho^2 \sin\varphi\, d\varphi\, d\vartheta\, d\varrho$$

$$\Downarrow \; \alpha = \cos\varphi, \; d\alpha = -\sin\varphi d\varphi$$

$$= \frac{1}{(2\pi)^{3/2}} \lim_{L\to\infty} \frac{2\pi}{c} \int_0^L \varrho \sin ct\varrho \underbrace{\left[\frac{e^{-ir\varrho\alpha}}{-ir\varrho} \right]_{-1}^{1}}_{\frac{2}{r\varrho}\sin r\varrho} d\varrho$$

$$= \frac{1}{(2\pi)^{3/2}} \lim_{L\to\infty} \frac{4\pi}{cr} \int_0^L \sin ct\varrho \cdot \sin r\varrho\, d\varrho$$

– To treat the integration with respect to ξ, η, and ζ, we introduce spherical coordinates (r, ϕ, θ) in the (ξ, η, ζ) space with their origin at (x, y, z), so that

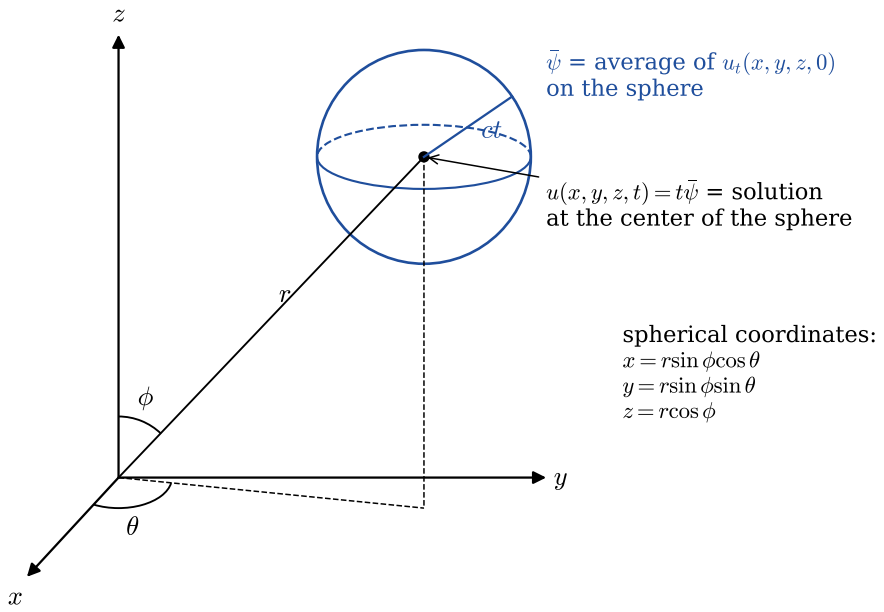
$$\begin{aligned} \xi - x &= r\sin\phi\cos\theta \\ \eta - y &= r\sin\phi\sin\theta \\ \zeta - z &= r\cos\phi \end{aligned}$$

Then the solution becomes

$$
\begin{aligned}
u(x,y,z,t) &= \frac{4\pi}{(2\pi)^3 c}\lim_{L\to\infty}\int_0^\infty\int_0^{2\pi}\int_0^{\pi}\psi(x+r\sin\phi\cos\theta, y+r\sin\phi\sin\theta, z+r\cos\phi)\,\frac{1}{r}\left[\int_0^L \sin ct\varrho\right.\\
&= \frac{1}{2\pi^2 c}\lim_{L\to\infty}\int_0^L \sin ct\varrho\left[\int_0^\infty \sin r\varrho\,\underbrace{\left(\int_0^{2\pi}\int_0^{\pi}\psi(x+r\sin\phi\cos\theta, y+r\sin\phi\sin\theta, z+r\cos\right.}_{g(r)}\right.\\
&= \frac{1}{2\pi^2 c}\frac{\pi}{2}\int_0^\infty \sin ct\varrho\,\underbrace{\left[\frac{2}{\pi}\int_0^\infty \sin r\varrho\, g(r)\,dr\right]}_{G(\varrho)}d\varrho = \frac{1}{4\pi c}\underbrace{\int_0^\infty \sin ct\varrho\, G(\varrho)\,d\varrho}_{g(ct)}\\
&= \frac{t}{4\pi(ct)^2}\int_0^{2\pi}\int_0^{\pi}\psi(x+ct\sin\phi\cos\theta, y+ct\sin\phi\sin\theta, z+ct\cos\phi)\,(ct)^2\sin\phi\,d\phi\,d\theta = t\bar{\psi}
\end{aligned}
$$

where $\bar{\psi}$ is the average of the initial disturbance ψ over the surface of the sphere of radius ct centered at (x,y,z)

– The interpretation of this solution is that the initial disturbance ψ radiates outward spherically (velocity c) at each point, so that after so many seconds, the point (x,y,z) will be influenced by those initial disturbances on a sphere (of radius ct) around that point



– Now, to finish the problem, what about the other half; that is

$$
\begin{aligned}
&u_{tt} = c^2(u_{xx}+u_{yy}+u_{zz}), \quad (x,y,z)\in\mathbb{R}^3\\
&u(x,y,z,0) = \phi(x,y,z)\\
&u_t(x,y,z,0) = 0
\end{aligned}
\tag{3Dp}
$$

This is easy: A famous theorem developed by *Stokes* says all we have to do to solve this problem is to change the ICs to $v=0$, $v_t=\phi$, and then differentiate this solution with

respect to time. Indeed, if v solves the wave equation with $v = 0$, $v_t = \phi$ at $t = 0$, then $u = v_t$ also solves the wave equation and

$$u(x, y, z, 0) = v_t(x, y, z, 0) = \phi, \qquad u_t(x, y, z, 0) = v_{tt}(x, y, z, 0) = c^2 \nabla^2 v(x, y, z, 0) = 0$$

– In other words, we solve

$$\begin{aligned} v_{tt} &= c^2(v_{xx} + v_{yy} + v_{zz}), \quad (x, y, z) \in \mathbb{R}^3 \\ v(x, y, z, 0) &= 0 \\ v_t(x, y, z, 0) &= \phi(x, y, z) \end{aligned}$$

to get $v = t\bar{\phi}$ and then differentiate with respect to time. This gives us the solution

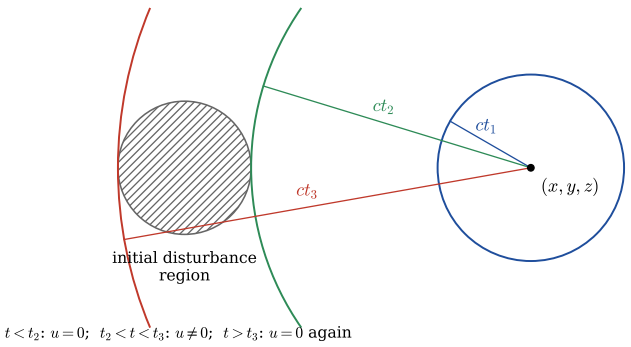
$$u = \frac{\partial}{\partial t}\left[t\bar{\phi}\right]$$

to problem (3Dp)

– We now have the solution to our general three-dimensional problem (3D). It's just

$$u(x, y, z, t) = \frac{\partial}{\partial t}\left[t\bar{\phi}\right] + t\bar{\psi}$$

where $\bar{\phi}$ and $\bar{\psi}$ are the averages of the functions ϕ and ψ over the sphere of radius ct centered at (x, y, z)



– Suppose now the initial disturbances ϕ and ψ are zero except for a small sphere. As time increases, the radius of the sphere around (x, y, z) increases with velocity c

– and so after t_2 seconds, it will finally intersect the initial disturbance region, and, hence, $u(x, y, z, t)$ becomes nonzero

– For $t_2 < t < t_3$, the solution at (x, y, z) will be nonzero, since the sphere intersects the disturbance region

– but when $t = t_3$, the solution at (x, y, z) abruptly becomes zero again. In other words, the wave disturbance originating from the initial-disturbance region has a sharp trailing edge. This general principle is known as **Huygens' principle** for the three dimensions, and it is the reason why sound waves in three dimensions stimulate our ears but die off instantaneously when the wave has passed

Example Illustrate by picture and words the spherical wave solution of the three-dimensional problem

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}), \quad \begin{cases} -\infty < x < \infty \\ -\infty < y < \infty \\ -\infty < z < \infty \end{cases}$$

$$u(x, y, z, 0) = 0$$

$$u_t(x, y, z, 0) = \begin{cases} 1 & x^2 + y^2 + z^2 \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Answer 3 (end of chapter) · Solution

- **Two-Dimensional Wave Equation**

  To solve the two-dimensional problem

$$\begin{aligned} u_{tt} &= c^2(u_{xx} + u_{yy}), \quad \begin{cases} -\infty < x < \infty \\ -\infty < y < \infty \end{cases} \\ u(x, y, 0) &= \phi(x, y) \\ u_t(x, y, 0) &= \psi(x, y) \end{aligned} \tag{2D}$$

  we merely let the initial disturbances ϕ and ψ in the three-dimensional problem depend on only the two variables x and y. Doing this, the three-dimensional formula

$$u = \frac{\partial}{\partial t}[t\bar{\phi}] + t\bar{\psi}$$

  for u will describe cylindrical waves and, hence, give us the solution for the two-dimensional problem

  – This technique is called the method of descent. Carrying out the computations, we get

$$\bar{\psi}(x, y, t) = \frac{1}{4\pi(ct)^2} \int_0^{2\pi} \int_0^{\pi} \psi(x + ct\sin\phi\cos\theta, y + ct\sin\phi\sin\theta)\,(ct)^2 \sin\phi\, d\phi\, d\theta \quad \leftarrow \text{spherical surface}$$

$$\Downarrow \quad r = ct\,\sin\phi, \;\; dr = ct\,\cos\phi\, d\phi \quad \rightarrow \quad \int_0^{\pi} ct\sin\phi\, d\phi = 2\int_0^{ct} \frac{r}{\sqrt{(ct)^2 - r^2}} dr$$

$$= \frac{1}{2\pi ct} \int_0^{2\pi} \int_0^{ct} \psi(x + r\cos\theta, y + r\sin\theta) \frac{1}{\sqrt{(ct)^2 - r^2}}\, r\, dr\, d\theta \quad \leftarrow \text{circle interior}$$

$$\bar{\phi}(x, y, t) = \frac{1}{2\pi ct} \int_0^{2\pi} \int_0^{ct} \phi(x + r\cos\theta, y + r\sin\theta) \frac{1}{\sqrt{(ct)^2 - r^2}}\, r\, dr\, d\theta$$

Note that in this solution, the two integrals of the initial conditions ϕ and ψ are integrated over the interior of a circle with center (x, y) and radius ct. We see that initial disturbances give rise to sharp leading waves, but not to sharp trailing waves. Thus, Huygens' principle doesn't hold in two dimensions

- **One-Dimensional Wave Equation**

  Finally, if we assume the initial conditions ϕ and ψ depend only on one variable, this gives rise to plane waves and, hence, the preceding equation descends one more dimension

$$\begin{aligned}
\bar{\psi}(x,t) &= \frac{1}{2\pi ct}\int_0^{2\pi}\int_0^{ct}\frac{\psi(x+r\cos\theta)}{\sqrt{(ct)^2-r^2}}\,r\,dr\,d\theta \\
&\Downarrow r^2 = x^2+y^2,\ (ct)^2-x^2=\alpha^2 \\
&= \frac{1}{2\pi ct}\int_{x-ct}^{x+ct}\underbrace{\int_{-\alpha}^{\alpha}\frac{1}{\sqrt{\alpha^2-y^2}}\,dy}_{\pi}\,\psi(x)\,dx = \frac{1}{2ct}\int_{x-ct}^{x+ct}\psi(x)\,dx \\
\bar{\phi}(x,t) &= \frac{1}{2ct}\int_{x-ct}^{x+ct}\phi(x)\,dx
\end{aligned}$$

  to the well-known D'Alembert solution. Note in the D'Alembert solution, the initial position ϕ gives rise to sharp trailing edges, but the initial velocity ψ does not. In other words, one dimension is a little unusual in that the initial position satisfies Huygens' principle, but the initial velocity does not

Example What is the two dimensional solution of the analogous cylindrical-wave problem

$$\begin{aligned}
u_{tt} &= c^2(u_{xx}+u_{yy}), \quad \begin{cases} -\infty < x < \infty \\ -\infty < y < \infty \end{cases} \\
u(x,y,0) &= 0 \\
u_t(x,y,0) &= \begin{cases} 1 & x^2+y^2 \le 1 \\ 0 & \text{elsewhere} \end{cases}
\end{aligned}$$

Answer 4 (end of chapter) · Solution

The Dimensional Geometry Summary

Dimension	Region of Influence	Type of Surface	Formula Name	Behavior
1D	Interval $x \pm ct$	2 points	D'Alembert Formula	Waves move left/right
2D	Disk $r \le ct$	Circle	Poisson Formula	Influence from entire disk
3D	Sphere $r = ct$	Spherical shell	Kirchhoff Formula	Influence from surface only

The difference comes from the geometry of the wavefront and how energy spreads:

- In **1D**, energy stays in two pulses (no spreading)

- In **2D**, energy spreads along the circular front → amplitude decays like $1/\sqrt{t}$
- In **3D**, energy spreads over a sphere → amplitude decays like $1/t$

These scaling differences explain why **sound intensity decreases faster in 3D** than in 2D or 1D

Green's Functions for the Wave Equation (1D, 2D, 3D)

The wave equation considered:

$$\partial_t^2 G(x,t) - c^2 \Delta_x G(x,t) = \delta(x)\delta(t), \quad G(x,t) = 0 \text{ for } t < 0$$

Dimension	Green's function G	Support / where nonzero	Key properties / remarks
1D	$G(x,t) = \frac{1}{2c} H\big(t - \frac{\lvert x\rvert}{c}\big)$	Inside the characteristic cone $\lvert x\rvert \le ct$	Constant $\frac{1}{2c}$ behind both fronts: no Huygens' principle. $\partial_t G = \frac{1}{2}[\delta(x-ct) + \delta(x+ct)]$ gives two undistorted pulses
2D	$G(r,t) = \frac{H(t-\frac{r}{c})}{2\pi\, c\sqrt{c^2t^2 - r^2}}$, $r = \lvert x\rvert$	Inside the light cone: $t > r/c$	Wave influence fills the interior of the light cone; singularity at $t = r/c$
3D	$G(r,t) = \frac{1}{4\pi c^2 r}\,\delta(t - \frac{r}{c})$, $r = \lvert x\rvert$	On the light-cone surface $t = r/c$	Huygens' principle: response lies exactly on the spherical wavefront; amplitude $\propto 1/r$

Notes: - $H(t)$: Heaviside step function - All functions above are distributions - Formulas correspond to the operator $\partial_t^2 - c^2\Delta$

13.5 Boundary Conditions Associated with the Wave Equation

The purpose of this section is to discuss some of the various types of BCs that are associated with physical problems of wave motions. Here, we will stick to one-dimensional problems where the BCs (linear ones) are generally grouped into one of three kinds

- **First Kind: Controlled End Points**

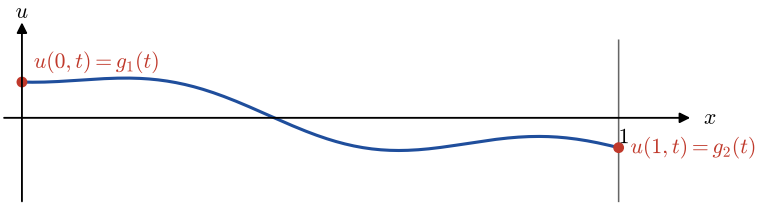
  We are now involved with problems like

$$u_{tt} = c^2 u_{xx} \quad 0 < x < 1, \;\; 0 < t < \infty$$

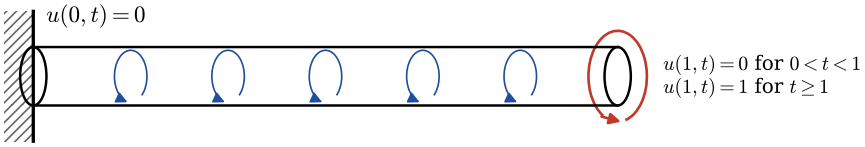
$$\begin{aligned} u(0,t) &= g_1(t) \\ u(1,t) &= g_2(t) \end{aligned} \quad 0 < t < \infty$$

$$\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned} \quad 0 \le x \le 1$$

where we control the end points so that they move in a given manner



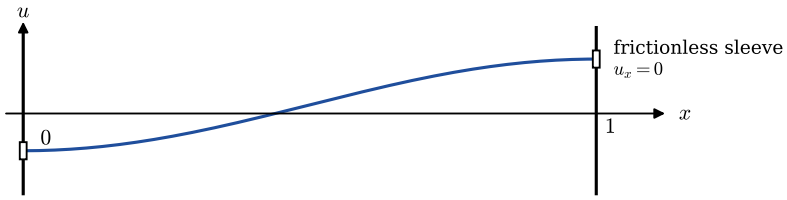
A typical problem of this kind would involve suddenly twisting (at $t = 1$) the right end of a fastened rod so many degrees and observing the resulting torsional vibration



- **Second Kind: Force Given on the Boundaries**

  Inasmuch as the vertical forces on the string at the left and right ends are given by $-Tu_x(0,t)$ and $Tu_x(1,t)$, respectively, by allowing the ends of the string to slide vertically on frictionless sleeves, the boundary conditions become

$$\begin{matrix} u_x(0,t) = 0 \\ u_x(1,t) = 0 \end{matrix}, \quad 0 < t < \infty$$

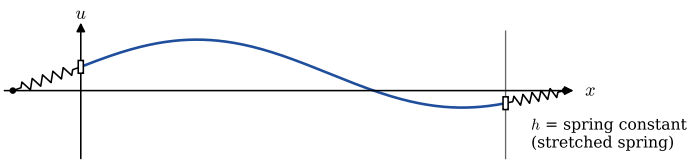


If a vertical force $f(t)$ is applied at the end $x = 1$, then the BC would be

$$u_x(1,t) = \frac{1}{T} f(t)$$

- **Third Kind: Elastic Attachment on the Boundaries**

  Consider finally a violin string whose ends are attached to an elastic arrangement like the one shown in figure



Here, the spring attachments at each end give rise to vertical forces proportional to the displacements $-u(0,t)$ and $-u(1,t)$

Setting the vertical tensions of the spring at the two ends $-Tu_x(0,t)$ and $Tu_x(1,t)$ equal to these displacements (multiplied by the spring constant h) gives us our desired BCs

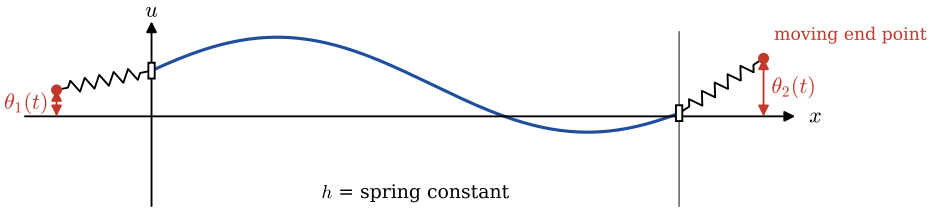
$$\begin{aligned} -Tu_x(0,t) &= -hu(0,t) \\ Tu_x(1,t) &= -hu(1,t) \end{aligned}$$

We can rewrite these two homogeneous BCs as

$$\begin{aligned} u_x(0,t) - \frac{h}{T}u(0,t) &= 0 \\ u_x(1,t) + \frac{h}{T}u(1,t) &= 0 \end{aligned}$$

If the two spring attachments are displaced according to the functions $\theta_1(t)$ and $\theta_2(t)$, we would have the nonhomogeneous BCs

$$\begin{aligned} u_x(0,t) &= \frac{h}{T}\left[u(0,t) - \theta_1(t)\right] \\ u_x(1,t) &= -\frac{h}{T}\left[u(1,t) - \theta_2(t)\right] \end{aligned}$$



NOTE What is the general nature of BC

$$u_x(0,t) = \frac{h}{T}\left[u(0,t) - \theta_1(t)\right]$$

when $h \to \infty$ and $h \to 0$

13.6 The Finite Vibrating String (Standing Waves)

- So far, we have studied the wave equation $u_{tt} = c^2 u_{xx}$ for the unbounded domain and have found D'Alembert solutions to be certain *traveling waves* (moving in opposite directions)
- When we study the same wave equation in a bounded region of space $0 < x < L$, we find that the waves no longer appear to be moving due to their repeated interaction with the boundaries and, in fact, often appear to be what are known as *standing waves*

- **Separation of Variables Solution to the Finite Vibrating String**

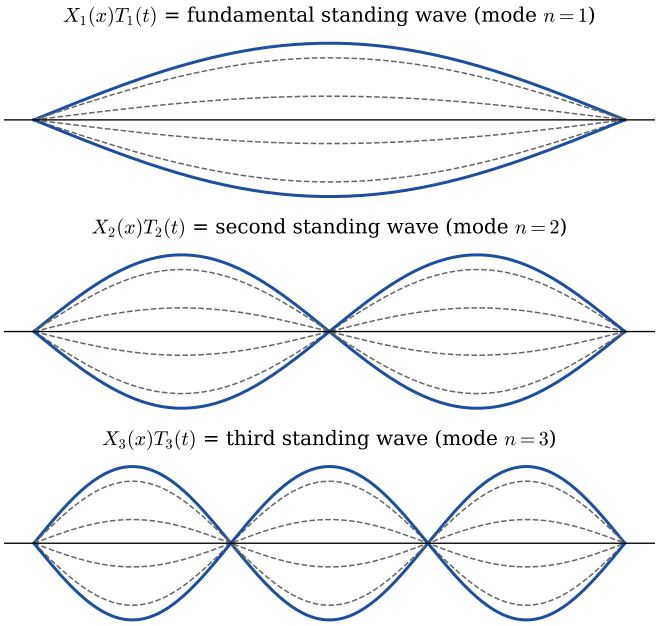
  Consider what happens when a guitar string (fixed at both ends $x = 0,\ L$) described by the simple hyperbolic IBVP is set in motion

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < L, \quad 0 < t < \infty$$

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned} \qquad 0 < t < \infty \tag{FS}$$

$$\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned} \qquad 0 \le x \le L$$

  – What happens is that the traveling wave solution to the PDE and IC keeps reflecting from the boundaries in such a way that the wave motion does not appear to be moving, but, in fact, appears to be vibrating in one position:



  – We start by seeking standing-wave solutions to the PDE; that is solutions of the form

$$u(x,t) = X(x)T(t)$$

  Substituting this expression into the wave equation and separating variables gives us the two ODEs

$$\begin{gathered} \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda < 0 \\ \Downarrow \\ T'' + c^2 \lambda T = 0 \\ X'' + \lambda X = 0 \end{gathered}$$

  in which only positive values of λ give feasible (nonzero and bounded) solutions

– The solutions of these two ODEs yield

$$T(t) = a\cos c\sqrt{\lambda}t + b\sin c\sqrt{\lambda}t$$
$$X(x) = d\cos\sqrt{\lambda}x + e\sin\sqrt{\lambda}x$$

Plugging this equation into $u(0,t) = u(L,t) = 0$ gives

$$\begin{aligned} u(0,t) &= d\cdot T(t) = 0 \quad \Rightarrow \quad T(t) \neq 0, \quad d = 0 \\ u(L,t) &= e\sin\sqrt{\lambda}L\cdot T(t) = 0 \quad \Rightarrow T(t) \neq 0, \quad e \neq 0, \ \sin\sqrt{\lambda}L = 0 \\ &\Rightarrow \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1,2,3,\cdots \end{aligned}$$

Hence, we have now found a sequence of simple vibrations (which we subscript with n)

$$\begin{aligned} u_n(x,t) &= T_n(t)X_n(x) \\ &= \left[a_n\cos\frac{n\pi ct}{L} + b_n\sin\frac{n\pi ct}{L}\right]\sin\frac{n\pi x}{L}, \quad n = 1,2,3,\cdots \end{aligned}$$

– All of which satisfy the wave equation and the BCs, and constitute a family of standing waves. Since any sum of these vibrations is also a solution to the PDE and BCs (since the PDE and BCs are linear and homogeneous), we add them together in such a way that the resulting sum also agrees with the ICs. Substituting the sum

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi ct}{L} + b_n\sin\frac{n\pi ct}{L}\right]\sin\frac{n\pi x}{L}$$

into the ICs

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

gives the two equations

$$\sum_{n=1}^{\infty} a_n\sin\frac{n\pi x}{L} = f(x), \quad \sum_{n=1}^{\infty} b_n\frac{n\pi c}{L}\sin\frac{n\pi x}{L} = g(x)$$

– and using the orthogonality condition

$$\int_0^L \sin\frac{m\pi x}{L}\,\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & m \neq n \\ L/2 & m = n \end{cases}$$

we can find the coefficients a_n and b_n

$$a_n = \frac{2}{L}\int_0^L f(x)\,\sin\frac{n\pi x}{L}\,dx$$
$$b_n = \frac{2}{n\pi c}\int_0^L g(x)\,\sin\frac{n\pi x}{L}\,dx$$

NOTE

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi ct}{L}+b_n\sin\frac{n\pi ct}{L}\right]\sin\frac{n\pi x}{L}\\
&= \frac{1}{2}\sum_{n=1}^{\infty}a_n\left[\sin\frac{n\pi}{L}(x+ct)+\sin\frac{n\pi}{L}(x-ct)\right]\\
&\quad -\frac{1}{2}\sum_{n=1}^{\infty}b_n\left[\cos\frac{n\pi}{L}(x+ct)-\cos\frac{n\pi}{L}(x-ct)\right]\\
&\Downarrow\ f,\ g \text{ extended as odd } 2L\text{-periodic functions, } \sum_{n=1}^{\infty}a_n\sin\frac{n\pi x}{L}=f(x),\ \sum_{n=1}^{\infty}b_n\frac{n\pi c}{L}\sin\frac{n\pi x}{L}=g(x)\ \to\ -\\
&= \frac{1}{2}\left[f(x+ct)+f(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(\alpha)\,d\alpha
\end{aligned}
$$

Example Find the solution to the vibrating-string problem (FS) if the ICs are given by

$$
\begin{aligned}
u(x,0) &= \sin\frac{\pi x}{L}+0.5\sin\frac{3\pi x}{L}\\
u_t(x,0) &= 0
\end{aligned}
$$

Answer 5 (end of chapter) · Solution

Example What is the solution of the vibrating-string problem (FS) if the ICs are

$$
\begin{aligned}
u(x,0) &= 0\\
u_t(x,0) &= \sin\frac{3\pi x}{L}
\end{aligned}
$$

Answer 6 (end of chapter) · Solution

Example Solve the damped vibrating-string problem

$$u_{tt}=c^2u_{xx}-\beta u_t \quad 0<x<1,\ \ 0<t<\infty$$

$$
\begin{aligned}
u(0,t) &= 0\\
u(1,t) &= 0
\end{aligned}
\qquad 0<t<\infty
$$

$$
\begin{aligned}
u(x,0) &= f(x)\\
u_t(x,0) &= 0
\end{aligned}
\qquad 0\le x\le 1
$$

Answer 7 (end of chapter) · Solution

13.7 Solution of Nonhomogeneous Wave Equation via the Finite Transform

- Consider the nonhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} + \sin \pi x \quad 0 < x < 1, \;\; 0 < t < \infty$$

$$\begin{aligned} u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned} \qquad 0 < t < \infty$$

$$\begin{aligned} u(x,0) &= 1 \\ u_t(x,0) &= 0 \end{aligned} \qquad 0 \le x \le 1$$

STEP 1 Determine the transform

Since the x-variable ranges from 0 to 1, we use a finite transform. Also, you will see why, in this case, we use the sine transform. We could solve this problem with the Laplace transform by transforming t

STEP 2 Carry out the transformation

$$\begin{aligned} \mathcal{F}_s[u_{tt}] &= c^2 \mathcal{F}_s[u_{xx}] + \mathcal{F}_s[\sin \pi x] \\ &\Downarrow \\ \frac{d^2 U_n(t)}{dt^2} &= -(n\pi c)^2 U_n(t) + 2n\pi c^2 \left[u(0,t) - (-1)^n u(1,t)\right] + S_n(t) \\ &\Downarrow \; u(0,t) = 0, \;\; u(1,t) = 0 \\ \frac{d^2 U_n(t)}{dt^2} &= -(n\pi c)^2 U_n(t) + S_n(t) \end{aligned}$$

where

$$S_n(t) = 2 \int_0^1 \sin \pi x \, \sin n\pi x \, dx = \begin{cases} 1 & n = 1 \\ 0 & n = 2, 3, \cdots \end{cases}$$

If we now transform the initial conditions, we will arrive at the initial conditions for our ODE

$$\mathcal{F}_s[u(x,0)] = U_n(0) = \begin{cases} \dfrac{4}{n\pi} & n = 1, 3, \cdots \\ 0 & n = 2, 4, \cdots \end{cases}$$

$$\mathcal{F}_s[u_t(x,0)] = \frac{dU_n(0)}{dt} = 0$$

So, solving our new initial-value problem, we have

$$U_1(t) = \left(\frac{4}{\pi} - \frac{1}{\pi^2 c^2}\right) \cos \pi c t + \frac{1}{\pi^2 c^2}$$

$$U_n(t) = \begin{cases} 0 & n = 2, 4, \cdots \\ \dfrac{4}{n\pi} \cos n\pi c t & n = 3, 5, \cdots \end{cases}$$

Hence, the solution $u(x,t)$ of the problem is

$$u(x,t) = \left[\left(\frac{4}{\pi} - \frac{1}{\pi^2 c^2}\right)\cos \pi ct + \frac{1}{\pi^2 c^2}\right]\sin \pi x + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n+1}\cos(2n+1)\pi ct\ \sin(2n+1)\pi x$$

13.8 The Vibrating Drumhead (Wave Equation in Polar Coordinates)

- The problem is to find $u(r,\theta,t)$ (which stands for the height of the drumhead from the plane) that satisfies

$$\begin{aligned} u_{tt} &= c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) \\ u(1,\theta,t) &= 0 && 0 < t < \infty \\ u(r,\theta,0) &= f(r,\theta) && 0 < r < 1 \\ u_t(r,\theta,0) &= g(r,\theta) \end{aligned}$$

  We will look for solutions of the form

$$u(r,\theta,t) = U(r,\theta)T(t)$$

  This gives the shape $U(r,\theta)$ of the vibrations times the oscillatory factor $T(t)$

- Carrying out this substitution, we arrive at the two equations

$$\begin{aligned} u_{tt} &= c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) \\ &\Downarrow \\ UT'' &= c^2\left(U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta}\right)T \\ &\Downarrow \\ \frac{T''}{c^2T} &= \frac{U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta}}{U} = -\lambda < 0 \\ &\Downarrow \\ T'' + c^2\lambda T &= 0 && \text{Simple Harmonic Motion} \\ U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} + \lambda U &= 0 && \text{Helmholtz Equation} \end{aligned}$$

- For the next step, we want to solve the Helmholtz equation, but, first, it needs a boundary condition. To find it, we substitute $u(r,\theta,t) = U(r,\theta)T(t)$ into the boundary condition of the drumhead to get

$$u(1,\theta,t) = U(1,\theta)T(t) = 0 \quad \Rightarrow \quad U(1,\theta) = 0$$

- **Solution of the Helmholtz Eigenvalue Problem**
  - To solve

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} + \lambda U = 0$$
$$U(1,\theta) = 0$$

    we let

$$U(r,\theta) = R(r)\Theta(\theta)$$

    and plug it into the Helmholtz BVP

$$\begin{aligned}
U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} + \lambda U &= 0 \\
\Downarrow\, U(r,\theta) &= R(r)\Theta(\theta) \\
R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' + \lambda R\Theta &= 0 \\
\Downarrow & \\
-\frac{r^2R'' + rR' + \lambda r^2 R}{R} = \frac{\Theta''}{\Theta} &= -\mu \le 0 \\
\Downarrow & \\
\Theta'' + \mu\Theta &= 0 \\
r^2R'' + rR' + (\lambda r^2 - \mu)R &= 0
\end{aligned}$$

    We get a new separation constant μ

  - It's obvious that the drumhead is periodic with period 2π in θ. Thus

$$\begin{aligned}
\Theta'' + \mu\Theta &= 0 \\
\Downarrow & \\
\Theta(\theta) &= a\cos\sqrt{\mu}\theta + b\sin\sqrt{\mu}\theta \\
\Downarrow\ \Theta(0)=\Theta(2\pi),\ \Theta'(0)=\Theta'(2\pi) & \\
\begin{bmatrix} 1-\cos 2\pi\sqrt{\mu} & -\sin 2\pi\sqrt{\mu} \\ \sqrt{\mu}\sin 2\pi\sqrt{\mu} & \sqrt{\mu}\,(1-\cos 2\pi\sqrt{\mu}) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\xrightarrow[\text{non-trivial solution}]{} \quad \sqrt{\mu}(1-\cos 2\pi\sqrt{\mu})^2 + \sqrt{\mu}\sin^2 2\pi\sqrt{\mu} = 0 \ \to\ 1 &= \\
\Downarrow & \\
\mu = n^2,\ \ n = 0,1,2,\cdots & \\
\Downarrow & \\
\Theta_n(\theta) = a_n\cos n\theta + b_n\sin n\theta & \\
= \sqrt{a_n^2 + b_n^2}\,(\cos n\theta\cos\phi + \sin n\theta\sin\phi)\,,\ \ \tan\phi = \frac{b_n}{a_n} & \\
= c_n\cos(n\theta - \phi) &
\end{aligned}$$

– So, in order to solve the Helmholtz equation, we must solve the following differential equation

$$
\begin{aligned}
r^2 R_n'' + r R_n' + (\lambda r^2 - n^2) R_n &= 0, \quad 0 < r < 1 \\
|R_n(0)| &< \infty \\
R_n(1) &= 0
\end{aligned}
$$

It is the parameterized Bessel's equation and has two linearly independent solutions $J_n(\sqrt{\lambda} r)$ and $Y_n(\sqrt{\lambda} r)$

```
from scipy.special import jv, yv, jn_zeros

fig = plt.figure(figsize=(6, 8))
ax1 = fig.add_subplot(211)
ax2 = fig.add_subplot(212)

ax1.axis((0, 10, -0.6, 1))
ax1.set_ylabel('$J_n$')

ax2.axis((0, 10, -3, 1))
ax2.set_xlabel(r'$\sqrt{\lambda}$')
ax2.set_ylabel('$Y_n$')

ax1.plot([0, 10], [0, 0], 'k:')
ax2.plot([0, 10], [0, 0], 'k:')

x = np.linspace(0, 10, 200)
for n in range(4):
    y1 = jv(n, x)
    y2 = yv(n, x)
    ax1.plot(x, y1, label=rf'$J_{n:d}(\sqrt{{\lambda}})$')
    ax2.plot(x, y2, label=rf'$Y_{n:d}(\sqrt{{\lambda}})$')

ax1.legend(bbox_to_anchor=(1.01, 1), loc='upper left')
ax2.legend(bbox_to_anchor=(1.01, 1), loc='upper left')

plt.show()
```

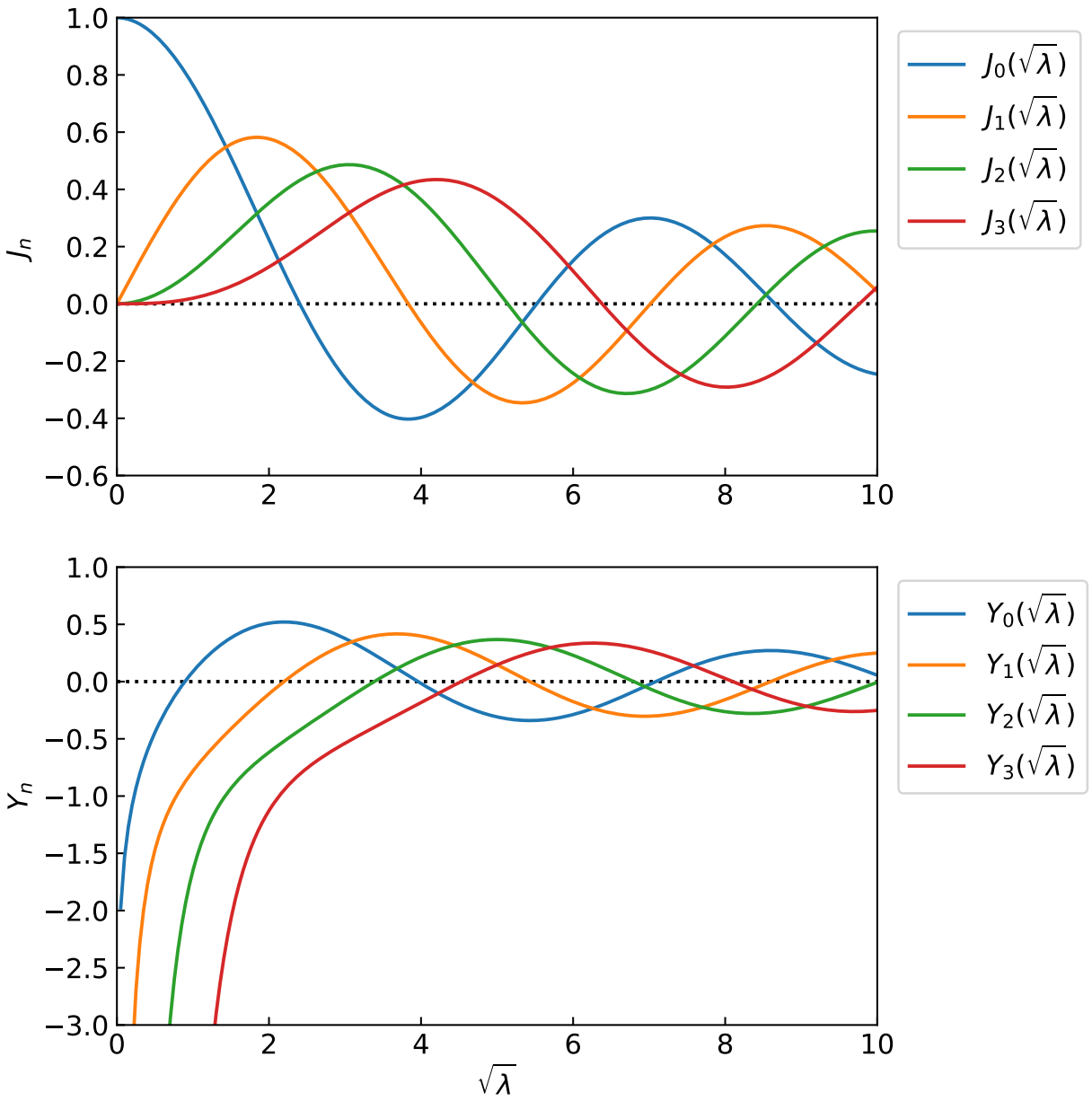


Figure 13.5: Bessel's functions

- – Since the functions $Y_n(\sqrt{\lambda}r)$ are unbounded at $r = 0$, we choose as our solution $J_n(\sqrt{\lambda}r)$. The last step in finding $R_n(r)$ is to use the boundary condition $R_n(1) = 0$ to find λ. We must pick λ to be one of the roots of $J_n(\sqrt{\lambda}) = 0$; that is

$$\sqrt{\lambda_{nm}} = \alpha_{nm}$$

  where α_{nm} is the m-th root of $J_n(r) = 0$, $m = 1, 2, 3, \cdots$. With these roots, we have just solved the Helmholtz eigenvalue problem

  – The eigenvalues λ_{nm} are α_{nm}^2; and the corresponding eigenfunctions are

$$U_{nm}(r,\theta) = c_n J_n(\alpha_{nm} r)\cos(n\theta - \phi), \quad n = 0,1,2,\cdots, \quad m = 1,2,3,\cdots$$

The general shape of $U_{nm}(r,\theta)$ is the same for different values of the constants c_n and ϕ. We plot these functions for the different values of n and m with $c_n = 1$ and $\phi = 0$

```
ncols = 4
nrows = 4

fig, axs = plt.subplots(ncols=ncols,
    nrows=nrows, figsize=(7, 7),
    subplot_kw=dict(projection='polar'))

for ax in axs.flat:
    ax.set(xticks=[], yticks=[])

azimuths = np.radians(np.linspace(0, 360, 100))
zeniths = np.linspace(0, 1, 100)

theta, r = np.meshgrid(azimuths, zeniths)

for n in range(ncols):
    alpha_nm = jn_zeros(n, nrows)
    for m in range(nrows):
        U_nm = jv(n, alpha_nm[m] *r) *np.cos(n *theta)
        axs[n, m].contourf(theta, r, U_nm, levels=[0, 10])
        if m == 0:
            axs[n, 0].set_ylabel(f'n = {n}')
        if n == 0:
            axs[nrows -1, m].set_xlabel(f'm = {m +1}')

plt.show()
```

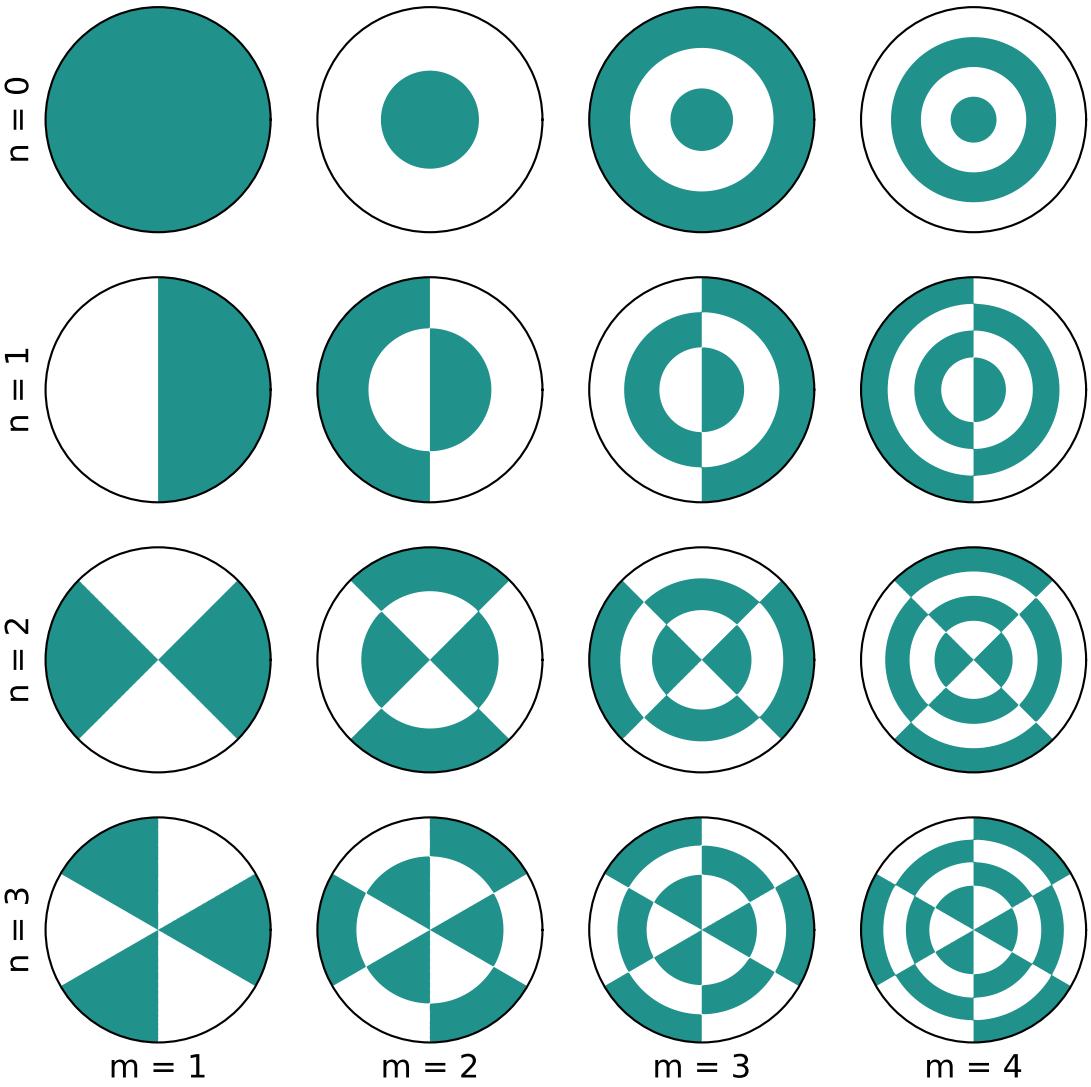


Figure 13.6: Helmholtz Eigenfunctions: $U_{nm}(r,\theta)$

- Now that we have solved the Helmholtz equation for the basic shapes $U_{nm}(r,\theta)$, the final step is to multiply by the time factor

$$T_{nm}(t) = A_{nm}\cos c\alpha_{nm}t + B_{nm}\sin c\alpha_{nm}t$$

and add the products in such a way that the initial conditions are satisfied. That is, the solution to our problem will be

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\alpha_{nm}r)\cos(n\theta-\phi_{nm})\left[A_{nm}\cos c\alpha_{nm}t + B_{nm}\sin c\alpha_{nm}t\right]$$

Rather than going through the complicated process of finding A_{nm} and B_{nm} for the general case, we will find the solution for the situation where u is independent of θ (very common, $n=0$). In other words, we will assume that the initial position of the drumhead depends only on r

- In particular, we consider

$$u(r,\theta,0) = f(r)$$
$$u_t(r,\theta,0) = 0$$

(It's just as easy to do the case where $u_t \neq 0$). With these assumptions, the solution now becomes

$$u(r,t) = \sum_{m=1}^{\infty} A_{0m} J_0(\alpha_{0m} r) \cos c\alpha_{0m} t$$

and our goal is to find A_{0m} so that

$$f(r) = \sum_{m=1}^{\infty} A_{0m} J_0(\alpha_{0m} r) \qquad \text{(IC)}$$

- To find the constants A_{0m}, we use the orthogonality condition of the Bessel functions:

$$\int_0^1 r J_0(\alpha_{0i} r)\, J_0(\alpha_{0j} r)\, dr = \begin{cases} 0 & i \neq j \\ \frac{1}{2} J_1^2(\alpha_{0i}) & i = j \end{cases}$$

Hence, we multiply each side of equation (IC) by $rJ_0(\alpha_{0j} r)$ and integrate from 0 to 1, giving

$$\int_0^1 r f(r) J_0(\alpha_{0j} r)\, dr = A_{0j} \int_0^1 r J_0^2(\alpha_{0j} r)\, dr$$

from which we can solve for A_{0j}

$$A_{0j} = \frac{2}{J_1^2(\alpha_{0j})} \int_0^1 r f(r) J_0(\alpha_{0j} r)\, dr, \quad j = 1, 2, 3, \cdots$$

- **For example** the solution to the vibrating drumhead with initial conditions

$$u(r,\theta,0) = J_0(2.405r) + 0.5 J_0(8.654r)$$
$$u_t(r,\theta,0) = 0$$

would be

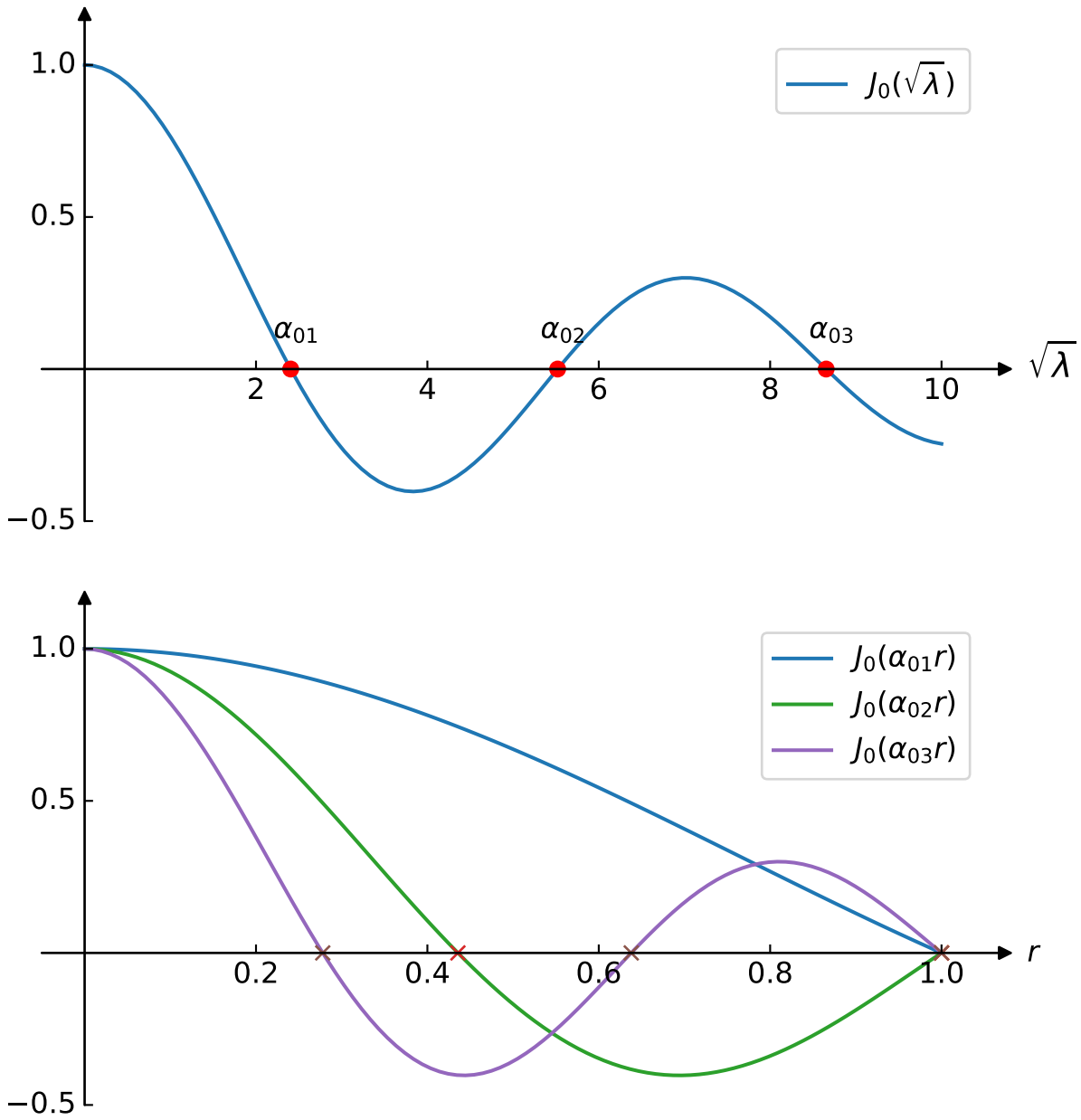
$$u(r,t) = J_0(2.405r)\cos(2.405ct) + 0.5 J_0(8.654r)\cos(8.654ct)$$

- We start by drawing $J_0(\sqrt{\lambda})$. Now, in order to graph the functions

$$J_0(\alpha_{01} r),\ J_0(\alpha_{02} r),\ \cdots,\ J_0(\alpha_{0m} r)$$

we rescale the r-axis so that m-th root passes through $r = 1$

```
from mpl_toolkits.axisartist.axislines import SubplotZero

fig = plt.figure(figsize=(7, 8))
ax1 = SubplotZero(fig, 211)
ax2 = SubplotZero(fig, 212)

fig.add_subplot(ax1)
ax1.axis["left", "right", "bottom", "top"].set_visible(False)
ax1.axis["xzero", "yzero"].set_visible(True)
ax1.axis["xzero", "yzero"].set_axisline_style("-|>")

ax1.set_xticks([2, 4, 6, 8, 10])
ax1.set_ylim(-0.5, 1.1)
ax1.set_yticks([-0.5, 0.5, 1.0])
ax1.text(11, -0.03, r'$\sqrt{\lambda}$', fontsize=14)

fig.add_subplot(ax2)
ax2.axis["left", "right", "bottom", "top"].set_visible(False)
ax2.axis["xzero", "yzero"].set_visible(True)
ax2.axis["xzero", "yzero"].set_axisline_style("-|>")

ax2.set_xticks([0.2, 0.4, 0.6, 0.8, 1.0])
ax2.set_ylim(-0.5, 1.1)
ax2.set_yticks([-0.5, 0.5, 1.0])
ax2.text(1.1, -0.03, '$r$', fontsize=12)

mm = 3
alpha_0m =jn_zeros(0, mm)

r1 = np.linspace(0, 10, 100)
y1 = jv(0, r1)
ax1.plot(r1, y1, label=r'$J_0(\sqrt{\lambda})$')
ax1.plot(alpha_0m, np.zeros(mm), 'ro')

for m in range(mm):
    ax1.text(alpha_0m[m] -0.2, 0.1, rf'$\alpha_{{0{m +1}}}$', fontsize=12)
ax1.legend()

r2 = np.linspace(0, 1, 100)
for m in range(mm):
    y2 = jv(0, alpha_0m[m]*r2)
    ax2.plot(r2, y2, label=rf'$J_0(\alpha_{{0{m +1}}}r)$')
    ax2.plot(alpha_0m[:m +1] /alpha_0m[m], np.zeros(m +1), 'x')
ax2.legend()

plt.show()
```



- In general, what we would do is to expand the initial position $f(r)$ into the sum of these basic shapes $J_0(\alpha_{0m}r)$, observe the vibration for each one and then add them up

```
fig = plt.figure(figsize=(7, 5))
ax = plt.axes(xlim=(0, 1), ylim=(-1.5, 1.5))

ax.set_xticks([0.0, 0.2, 0.4, 0.6, 0.8, 1.0])
ax.set_yticks([-1.5, -1.0, -0.5, 0, 0.5, 1.0, 1.5])
ax.tick_params(pad=10)
ax.set_xlabel('$r$')
ax.set_ylabel('$u(r,t)$')

plt.close()
```

```
line1, = ax.plot([], [], lw=1, ls=':')
line2, = ax.plot([], [], lw=1, ls=':')
line_sum, = ax.plot([], [], lw=4)

time_text = ax.text(0.45, 1.3, '')
def init():
    time_text.set_text('t = 0.0')
    line1.set_data([], [])
    line2.set_data([], [])
    line_sum.set_data([], [])
    return (line1, line2, line_sum)

c = 1
alpha_0m =jn_zeros(0, 3)

def animate(t):
    time_text.set_text(f't = {t:.2f}')
    r = np.linspace(0, 1, 100)

    u1 = jv(0, alpha_0m[0] *r) *np.cos(c *alpha_0m[0] *t)
    u2 = 0.5 *jv(0, alpha_0m[2] *r) *np.cos(c *alpha_0m[2] *t)
    u_sum = u1 +u2

    line1.set_data(r, u1)
    line2.set_data(r, u2)
    line_sum.set_data(r, u_sum)

    return (line1, line2, line_sum)

tt = np.linspace(0, 10, 100)
anim = animation.FuncAnimation(fig, animate,
         init_func=init, frames=tt, interval=200, blit=True)

HTML('<center>' + anim.to_html5_video() + '</center>')
```

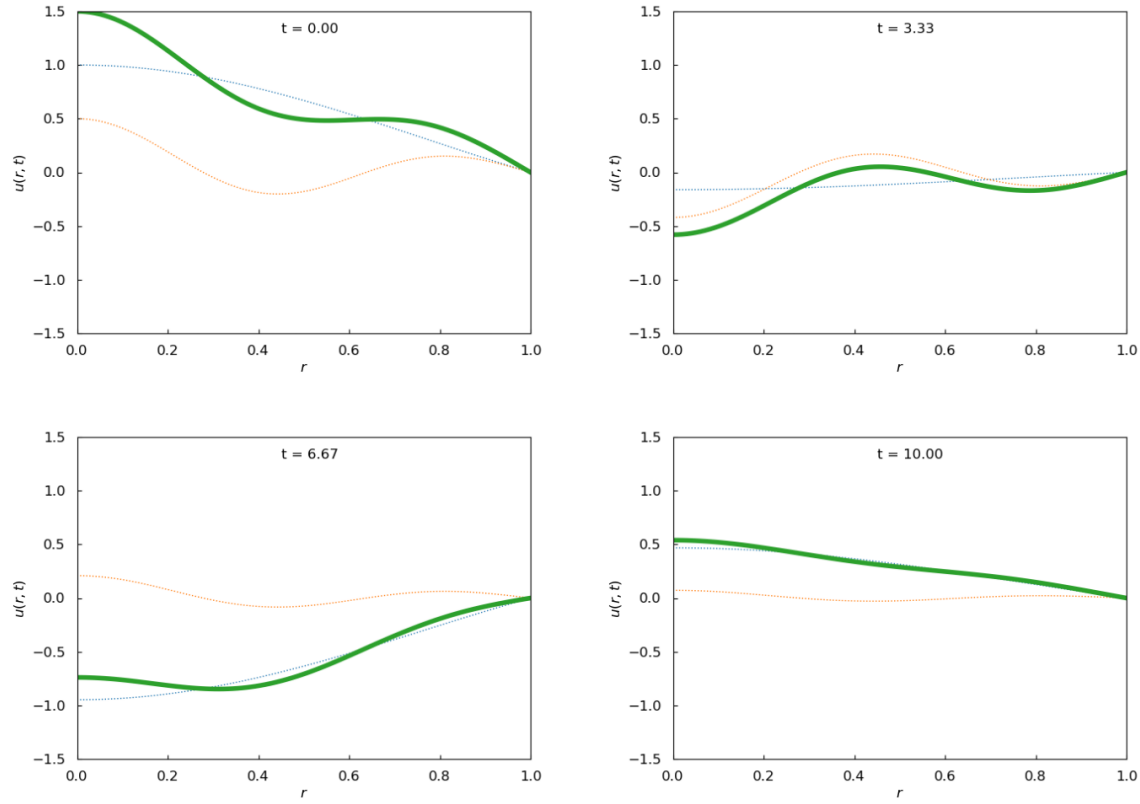


Figure 13.7: Four frames of the animation; play it on the web page

Example Solve the vibrating drum head problem

$$u_{tt} = \nabla^2 u \qquad 0 < r < 1, \;\; 0 < t < \infty$$

$$u(1, \theta, t) = 0 \qquad 0 < t < \infty$$

$$\begin{aligned} u(r, \theta, 0) &= 1 - r^2 \\ u_t(r, \theta, 0) &= 0 \end{aligned} \qquad 0 \le r \le 1$$

Answer 8 (end of chapter) · Solution

13.9 Dimensionless Problems

- The basic idea behind dimensionless analysis is that by introducing new (dimensionless) variables in a problem, the problem becomes **purely mathematical** and contains none of the physical constants that originally characterized it
- In this way, many different equations in physics, biology, engineering, and chemistry that contain special nuances via physical parameters are all transformed into the same simple form

Converting a Diffusion Problem to Dimensionless Form

- Suppose we start with the initial-boundary-value problem:

$$\begin{aligned} u_t &= \alpha u_{xx} && 0 < x < L,\ \ 0 < t < \infty \\ u(0,t) &= T_1 \\ u(L,t) &= T_2 && 0 < t < \infty \\ u(x,0) &= \sin\frac{\pi x}{L} && 0 \le x \le L \end{aligned}$$

  Our goal is to change this problem to a new equivalent formulation that has the properties

  1. No physical parameters (like α) in the new equation
  2. The initial and boundary conditions are simpler

- **Transforming the Dependent Variable** $u \to U = \dfrac{u(x,t) - T_1}{T_2 - T_1}$

$$\begin{aligned} U_t &= \alpha U_{xx} && 0 < x < L,\ \ 0 < t < \infty \\ U(0,t) &= 0 \\ U(L,t) &= 1 && 0 < t < \infty \\ U(x,0) &= \frac{\sin\dfrac{\pi x}{L} - T_1}{T_2 - T_1} && 0 \le x \le L \end{aligned}$$

- **Transforming the Space Variable** $x \to \xi = {}^{x}/_{L}$

$$\begin{aligned} U_t &= \frac{\alpha}{L^2} U_{\xi\xi} && 0 < \xi < 1,\ 0 < t < \infty \\ U(0,t) &= 0 \\ U(1,t) &= 1 && 0 < t < \infty \\ U(\xi,0) &= \frac{\sin\pi\xi - T_1}{T_2 - T_1} && 0 \le \xi \le 1 \end{aligned}$$

- **Transforming the Time Variable** $t \to \tau$

  Since our goal is to eliminate the constant $\dfrac{\alpha}{L^2}$ from the PDE, we proceed as follows:

  1. Try a transformation of the form $\tau = ct$, where c is an unknown constant
  2. Compute $U_t = U_\tau \tau_t = cU_\tau$
  3. Substitute this derivative into the PDE to obtain

$$cU_\tau = \frac{\alpha}{L^2} U_{\xi\xi}$$

  and, hence, pick $c = \dfrac{\alpha}{L^2}$. This gives us our new time

$$\tau = \frac{\alpha}{L^2} t$$

- We now have the completely dimensionless problem

$$\begin{aligned} U_\tau &= U_{\xi\xi} && 0<\xi<1,\ 0<\tau<\infty \\ U(0,\tau) &= 0 \\ U(1,\tau) &= 1 && 0<\tau<\infty \\ U(\xi,0) &= \frac{\sin\pi\xi - T_1}{T_2 - T_1} && 0\le\xi\le 1 \end{aligned}$$

Transforming a Hyperbolic Problem to Dimensionless Form

- Consider the vibrating string

$$\begin{aligned} u_{tt} &= c^2 u_{xx} && 0<x<L,\ \ 0<t<\infty \\ u(0,t) &= 0 \\ u(L,t) &= 0 && 0<t<\infty \\ u(x,0) &= \sin\tfrac{\pi x}{L} + 0.5\sin\tfrac{3\pi x}{L} \\ u_t(x,0) &= 0 && 0<x<L \end{aligned}$$

- By transforming the independent variables (no need to transform u) into new ones

$$\xi = \frac{x}{L} \quad \text{and} \quad \tau = \frac{c}{L}t$$

we get the new problem

$$\begin{aligned} u_{\tau\tau} &= u_{\xi\xi} && 0<\xi<1,\ \ 0<\tau<\infty \\ u(0,\tau) &= 0 \\ u(1,\tau) &= 0 && 0<\tau<\infty \\ u(\xi,0) &= \sin\pi\xi + 0.5\sin 3\pi\xi \\ u_\tau(\xi,0) &= 0 && 0<\xi<1 \end{aligned}$$

which has the solution

$$u(\xi,\tau) = \cos\pi\tau\sin\pi\xi + 0.5\cos 3\pi\tau\sin 3\pi\xi$$

Example How could you pick a new space variable ξ so that v is eliminated in the equation

$$u_t + vu_x = 0$$

Answer 9 (end of chapter) · Solution

13.10 Classification of PDEs (Canonical Form of the Hyperbolic Equation)

- The purpose here is to classify the second-order linear PDE

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \qquad \text{(SL)}$$

in which A, B, C, D, E, F, and G are, in general, functions of x and y in two independent variables and could be constants, as

  - Hyperbolic if $B^2 - 4AC > 0$ at (x, y)
  - Parabolic if $B^2 - 4AC = 0$ at (x, y)
  - Elliptic if $B^2 - 4AC < 0$ at (x, y)

- and depending on which is true, the equation can be transformed into a corresponding canonical (simple) form by introducing new coordinates $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$

$$u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta) \quad \text{or} \quad u_{\xi\xi} - u_{\eta\eta} = \Psi(\xi, \eta, u, u_\xi, u_\eta)$$
$$u_{\eta\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$$
$$u_{\xi\xi} + u_{\eta\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$$

The student should note that whether equation (SL) is hyperbolic, parabolic, or elliptic depends *only* on the coefficients of the second derivatives. We now come to the major portion of this section; rewriting hyperbolic equations in their canonical form

- **The Canonical Form of the Hyperbolic Equation**

The objective here is to introduce new coordinates $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ so that the general PDE contains only one second derivative $u_{\xi\eta}$. First of all, we compute the partial derivatives

$$\begin{aligned}
&Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \\
&\qquad\qquad \Downarrow \quad \xi = \xi(x, y),\ \eta = \eta(x, y) \\
u_x &= u_\xi \xi_x + u_\eta \eta_x \\
u_y &= u_\xi \xi_y + u_\eta \eta_y \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy} \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy} \\
&\qquad\qquad \Downarrow
\end{aligned}$$

$$\bar{A}u_{\xi\xi} + \bar{B}u_{\xi\eta} + \bar{C}u_{\eta\eta} + \bar{D}u_{\xi} + \bar{E}u_{\eta} + \bar{F}u = \bar{G}$$

in which

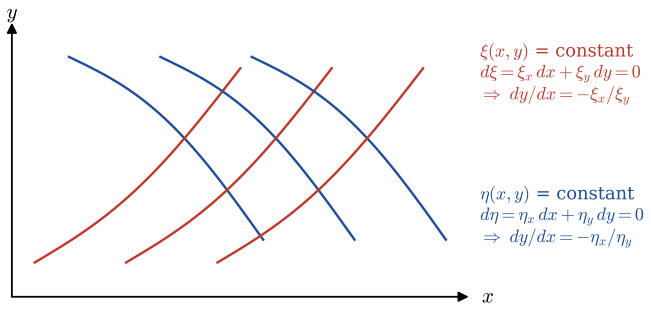
$$\begin{aligned}
\bar{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
\bar{B} &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
\bar{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
&\Downarrow \\
\begin{pmatrix} 2\bar{A} & \bar{B} \\ \bar{B} & 2\bar{C} \end{pmatrix} &= \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} 2A & B \\ B & 2C \end{pmatrix} \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}^T \\
&\Downarrow \\
\bar{B}^2 - 4\bar{A}\bar{C} &= (\xi_x\eta_y - \xi_y\eta_x)^2(B^2 - 4AC) \\
&= J^2(B^2 - 4AC)
\end{aligned}$$

in which J is the Jacobian of the transformation and we select the transformation (ξ, η) such that $J \neq 0$

- The next step in our process is to set the coefficients $\bar{A}$ and $\bar{C}$ to zero and solve for the transformation ξ and η. This will give us the coordinates that reduce the original PDE to canonical form:

$$\begin{aligned}
\bar{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \\
\bar{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0 \\
&\Downarrow \\
A\left[\frac{\xi_x}{\xi_y}\right]^2 + B\left[\frac{\xi_x}{\xi_y}\right] + C = 0, &\quad A\left[\frac{\eta_x}{\eta_y}\right]^2 + B\left[\frac{\eta_x}{\eta_y}\right] + C = 0 \\
&\Downarrow \\
\left[\frac{\xi_x}{\xi_y}\right] = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, &\quad \left[\frac{\eta_x}{\eta_y}\right] = \frac{-B - \sqrt{B^2 - 4AC}}{2A}
\end{aligned}$$

- We have now reduced the problem to finding the two functions $\xi(x, y)$ and $\eta(x, y)$ so that their ratio $[\xi_x/\xi_y]$ and $[\eta_x/\eta_y]$ satisfy the above equation. Finding functions satisfying these conditions is really quite easy once we look for a few moments at figure



$$\Downarrow \ \xi = \text{constant}, \eta = \text{constant}$$

$$\frac{dy}{dx} = -\left[\frac{\xi_x}{\xi_y}\right] = \frac{B - \sqrt{B^2 - 4AC}}{2A} = g_1(x, y)$$

$$\frac{dy}{dx} = -\left[\frac{\eta_x}{\eta_y}\right] = \frac{B + \sqrt{B^2 - 4AC}}{2A} = g_2(x, y)$$

$$\Downarrow \ \text{integration and arrangement}$$

$$G_1(x, y) = c_1$$

$$G_2(x, y) = c_2$$

$$\Downarrow$$

$$\xi = G_1(x, y)$$

$$\eta = G_2(x, y)$$

Example Consider the simple equation

$$u_{xx} - 4u_{yy} + u_x = 0, \quad B^2 - 4AC = 16 > 0$$

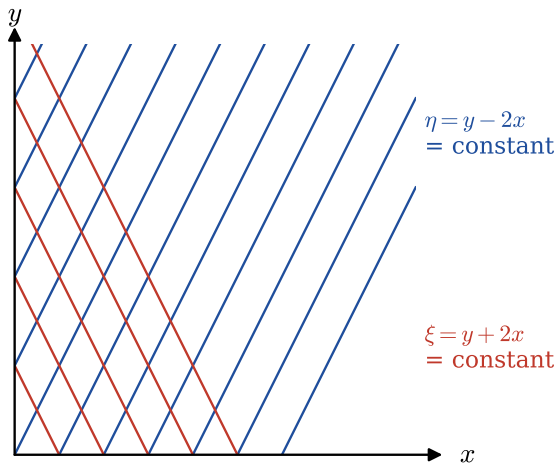
whose characteristic equations are

$$\frac{dy}{dx} = -\left[\frac{\xi_x}{\xi_y}\right] = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -2$$

$$\frac{dy}{dx} = -\left[\frac{\eta_x}{\eta_y}\right] = \frac{B + \sqrt{B^2 - 4AC}}{2A} = 2$$

To find ξ and η, we first integrate for x and solve for the constants c_1 and c_2, getting

$$\begin{array}{l} y = -2x + c_1 \\ y = 2x + c_2 \end{array} \quad \Rightarrow \quad \begin{array}{l} \xi = y + 2x = c_1 \\ \eta = y - 2x = c_2 \end{array}$$



Example Suppose we start with the equation

$$y^2 u_{xx} - x^2 u_{yy} = 0, \quad x > 0,\ y > 0$$

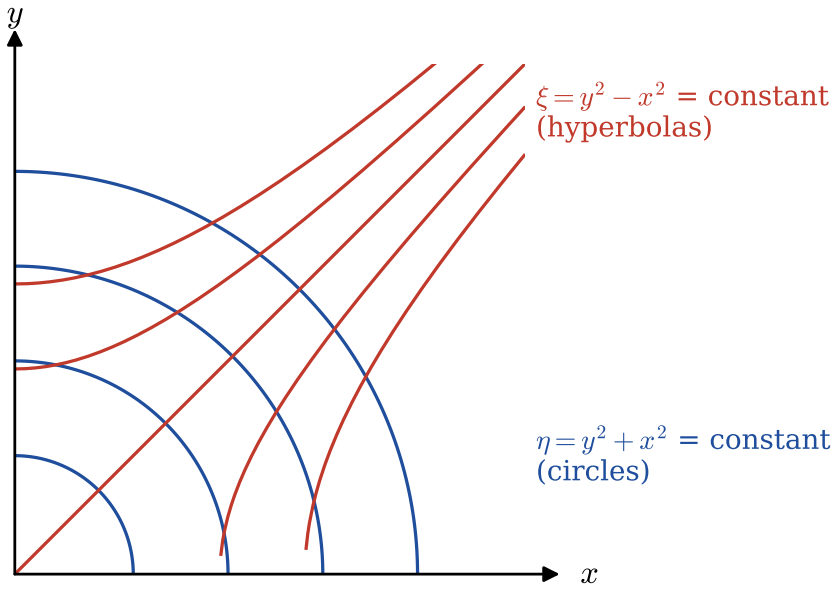
which is a hyperbolic equation in the first quadrant. We consider the problem of finding new coordinates that will change the original equation to canonical form for x and y in the first quadrant

STEP 1 Solve the two characteristic equations

$$\frac{dy}{dx} = -\left[\frac{\eta_x}{\eta_y}\right] = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -\frac{x}{y}$$
$$\frac{dy}{dx} = -\left[\frac{\xi_x}{\xi_y}\right] = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \frac{x}{y}$$

Integrating these two equations by the ODE technique of separating variables gives the implicit relationship

$$\begin{aligned} y^2 + x^2 &= c_2 \\ y^2 - x^2 &= c_1 \\ &\Downarrow \\ \xi &= y^2 - x^2 \\ \eta &= y^2 + x^2 \end{aligned}$$



We substitute this new coordinate into the equation to find the new equation

$$\begin{gathered} y^2 u_{xx} - x^2 u_{yy} = 0 \\ \Downarrow \quad \xi = y^2 - x^2,\ \eta = y^2 + x^2 \\ -16x^2 y^2 u_{\xi\eta} - 2(x^2 + y^2)u_\xi + 2(y^2 - x^2)u_\eta = 0 \end{gathered}$$

STEP 2 Finally, solving for x and y in terms of ξ and η, we get

$$u_{\xi\eta} = \frac{\eta u_\xi - \xi u_\eta}{2(\xi^2 - \eta^2)}$$

Example Find the new characteristic coordinates for

$$u_{xx} + 4u_{xy} = 0$$

Solve the transformed equation in the new coordinate system and then transform back to the original coordinates to find the solution to the original problem

Answer 10 (end of chapter) · Solution

NOTES

- The general hyperbolic equation actually has two canonical forms; the other one can be found by making yet another transformation

  $$\alpha = \xi + \eta$$
  $$\beta = \xi - \eta$$

  and rewriting the first canonical form in terms of α and β. The partial derivatives are transformed as follows:

  $$u_\xi = u_\alpha \alpha_\xi + u_\beta \beta_\xi = u_\alpha + u_\beta$$
  $$u_\eta = u_\alpha \alpha_\eta + u_\beta \beta_\eta = u_\alpha - u_\beta$$
  $$u_{\xi\eta} = u_{\alpha\alpha}\alpha_\eta + u_{\alpha\beta}\beta_\eta + u_{\beta\alpha}\alpha_\eta + u_{\beta\beta}\beta_\eta = u_{\alpha\alpha} - u_{\beta\beta}$$

- In the general parabolic equation, we obtain only a single canonical transformation as

  $$\bar{A} = 0, \quad \frac{dy}{dx} = -\left[\frac{\xi_x}{\xi_y}\right], \quad \text{or} \quad \bar{C} = 0, \quad \frac{dy}{dx} = -\left[\frac{\eta_x}{\eta_y}\right]$$

  Suppose ξ is calculated. What could be the possible choice for η? We can select η as any arbitrary function of x and y such that η is independent of ξ $(J \neq 0)$

- In the general elliptic case, $A\lambda^2 + B\lambda + C = 0$ leads to complex conjugate canonical transformation ξ and η. Since ξ and η are complex, we introduce new real variables

  $$\alpha = \frac{1}{2}(\xi + \eta)$$
  $$\beta = \frac{1}{2i}(\xi - \eta)$$

  Under the transformation $(x, y) \to (\alpha, \beta)$, the canonical form is given by

  $$u_{\alpha\alpha} + u_{\beta\beta} = \Phi(\alpha, \beta, u, u_\alpha, u_\beta)$$

- The three major classifications of linear PDEs as hyperbolic, parabolic, and elliptic equations essentially classify physical problems into three basic physical types:
  - **Wave propagation**
  - **Diffusion**
  - **Steady-state problems**

  The mathematical solutions of these three types of equations are quite different

13.11 First-Order Equations (Method of Characteristics)

- If the student recalls, when we solved the diffusion-convection equation

$$u_t = \alpha u_{xx} - v u_x, \quad -\infty < x < \infty, \quad 0 < t < \infty$$

  the constant α stood for the amount of diffusion, while v stood for the velocity of the medium
- Hence if $\alpha = 0$ (no diffusion), it is clear (since we only have convection) that the solution will travel along the x-axis with velocity v. In other words, if the initial solution is $u(x, 0) = \phi(x)$, then the solution to

$$u_t = -v u_x$$

  would be $u(x, t) = \phi(x - vt)$
- So we can think of the first-order equation

$$a(x, t)u_x + b(x, t)u_t = 0$$

  as the concentration along a stream where the velocity is given by

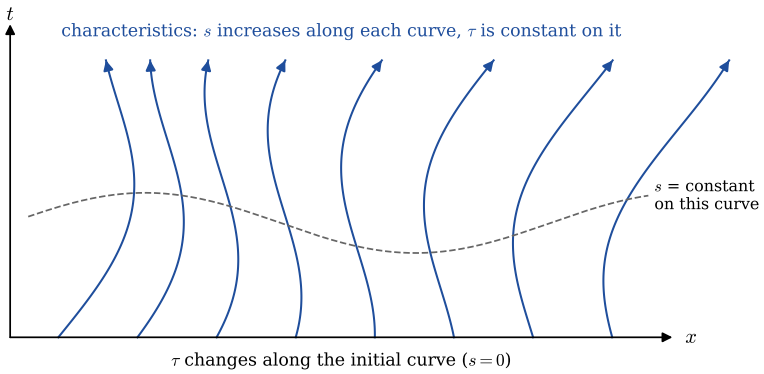
$$v = \frac{a(x, t)}{b(x, t)}$$

- The purpose of this section is to introduce the notion of first-order partial differential equations and method of characteristics. The problem we will solve is the initial-value problem

$$a(x, t)u_x + b(x, t)u_t + c(x, t)u = 0$$

$$u(x, 0) = \phi(x), \quad -\infty < x < \infty, \quad 0 < t < \infty$$

  The solution to this equation is based on a physical fact, that an initial disturbance at some point x propagates along a line (curve) in the xt-plane (called a characteristic)



- With this in mind, the idea is to introduce two new coordinates s and τ (to replace x and t) that have the properties
  - s will change along the characteristic curves
  - τ will change along the initial curve (most likely the line $t = 0$)

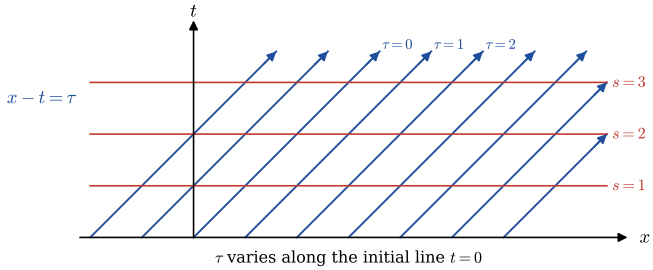
$$
\begin{aligned}
a(x,t)u_x + b(x,t)u_t + c(x,t)u &= 0 \\
&\Downarrow \text{ characteristic curves } \{(x(s),t(s)) : 0 < s < \infty\} \\
\frac{dx}{ds} = a(x,t),\ \frac{dt}{ds} &= b(x,t) \\
&\Downarrow \\
\frac{du}{ds} = u_x\frac{dx}{ds} + u_t\frac{dt}{ds} &= a(x,t)u_x + b(x,t)u_t \\
&\Downarrow \\
\frac{du}{ds} + c(x,t)u &= 0
\end{aligned}
$$

Example Solve the following IVP with constant coefficients:

$$
\begin{aligned}
u_x + u_t + 2u = 0 \qquad & -\infty < x < \infty,\ \ 0 < t < \infty \\
u(x,0) = \sin x \qquad & -\infty < x < \infty
\end{aligned}
$$

STEP 1 Find the characteristics (along which the initial data propagate)

$$
\begin{aligned}
\frac{dx}{ds} = 1,\ \ \frac{dt}{ds} &= 1 \quad 0 < s < \infty \\
&\Downarrow \\
x(s) &= s + c_1 \\
t(s) &= s + c_2 \\
&\Downarrow \ \ x(0) = \tau,\ \ t(0) = 0 \\
x &= s + \tau \\
t &= s \\
&\Downarrow \\
x - t = \tau, \quad & -\infty < \tau < \infty
\end{aligned}
$$

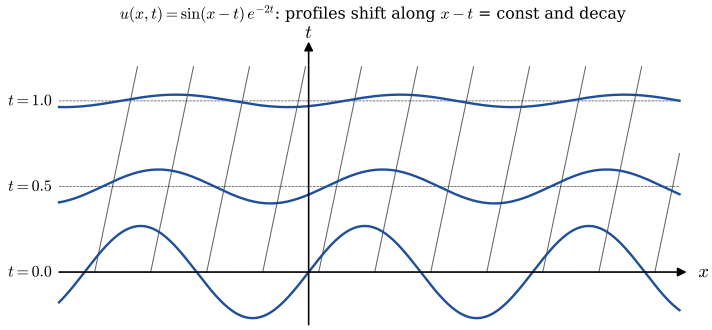


STEP 2 Using this coordinate system, we change the PDE to the ODE

$$\frac{du}{ds} + 2u = 0, \quad u(0) = \sin\tau$$
$$\Downarrow$$
$$u(s,\tau) = \sin\tau \cdot e^{-2s}$$

STEP 3 We now solve the transformation for s and τ in terms of x and t:

$$u(x,t) = \sin(x-t) \cdot e^{-2t}$$



Example Solve the following IVP with variable coefficients:

$$\begin{aligned} xu_x + u_t + tu = 0 \qquad & -\infty < x < \infty,\ 0 < t < \infty \\ u(x,0) = F(x) \qquad & -\infty < x < \infty, \text{ an arbitrary initial wave} \end{aligned}$$

STEP 1 Find the characteristics (along which the initial data propagate)

$$\frac{dx}{ds} = x, \quad \frac{dt}{ds} = 1 \quad 0 < s < \infty$$
$$\Downarrow$$
$$\begin{aligned} x(s) &= c_1 e^s \\ t(s) &= s + c_2 \\ &\Downarrow \quad x(0) = \tau, \quad t(0) = 0 \\ x &= \tau e^s \\ t &= s \end{aligned}$$

STEP 2 Using this coordinate system, we change the PDE to the ODE

$$\frac{du}{ds} + su = 0, \quad u(0) = F(\tau)$$
$$\Downarrow$$
$$u(s,\tau) = F(\tau) \cdot e^{-s^2/2}$$

STEP 3 We now solve the transformation for s and τ in terms of x and t

$$u(x,t) = F(xe^{-t}) \cdot e^{-t^2/2}$$

Example Solve the problem in higher dimensions (surface waves)

$$\begin{aligned} au_x + bu_y + cu_t + du = 0 \qquad & -\infty < x < \infty,\ -\infty < y < \infty,\ 0 < t < \infty \\ u(x,y,0) = e^{-(x^2+y^2)} \qquad & -\infty < x < \infty,\ -\infty < y < \infty \end{aligned}$$

Answer 11 (end of chapter) · Solution

13.12 Nonlinear First-Order Equations (Conservation Equations)

- One of the most useful partial differential equations in mathematics is the conservation equation

$$u_t + f_x = 0$$

This equation says that the increase of a physical quantity u_t is equal to the change in flux $-f_x$ of that quantity across the boundary

- **Derivation of the Conservation Equation**

Suppose we have a stretch of highway on which cars are moving from left to right, and we assume there are no exit and entrance ramps. We let

$$u(x,t) = \text{density of cars at } x \text{ (cars per unit length at } x)$$
$$f(x,t) = \text{flux of cars at } x \text{ (cars per minute passing the point } x)$$

Then, it is fairly obvious that for a road segment $[a,b]$, the change in the number of cars (with respect to time) is given by both of the following expressions:

$$\text{change in the number of cars in } [a,b] = \frac{d}{dt}\int_a^b u(x,t)\,dx$$
$$\text{change in the number of cars in } [a,b] = f(a,t) - f(b,t) = -\int_a^b \frac{\partial f}{\partial x}(x,t)\,dx$$

Setting these two integrals equal to each other, we have

$$\int_a^b \frac{\partial u}{\partial t}(x,t)\,dx = -\int_a^b \frac{\partial f}{\partial x}(x,t)\,dx$$

Finally, since the integral $[a,b]$ was arbitrary, we have the one-dimensional conservation equation

$$u_t + f_x = 0$$

If we carried out a similar analysis in two or three dimensions, we would have

$$u_t + \nabla \cdot \mathbf{f} = 0: \quad u_t + (f_1)_x + (f_2)_y = 0, \;\; u_t + (f_1)_x + (f_2)_y + (f_3)_z = 0$$

- **Conservation Equation Applied to the Traffic Problem**
  - We are now ready to see how the conservation equation $u_t + f_x = 0$ can be applied to the traffic problem
  - To do this, we rewrite the change in flux f_x as

    $$f_x = \frac{df}{du}u_x, \;\; \text{chain rule}$$

    and substitute it in the conservation equation. Doing this, we arrive at

    $$u_t + \frac{df}{du}u_x = u_t + g(u)u_x = 0$$

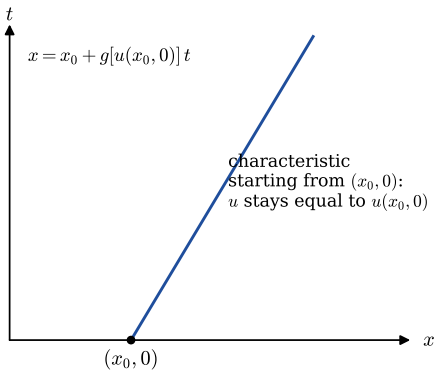
    where we can imagine a disturbance (wave) in the traffic density moving upstream or downstream with velocity $g(u)$
  - Hence, after t minutes, the position x of the disturbance will be

    $$x = x_0 + g(u)t, \quad \text{characteristic equation}$$

  - Remembering that the density $u(x,t)$ will not change along each characteristic, if we know the initial density $u(x_0,0)$, then the characteristic equation for x can be written

    $$x = x_0 + g[u(x_0,0)]t$$

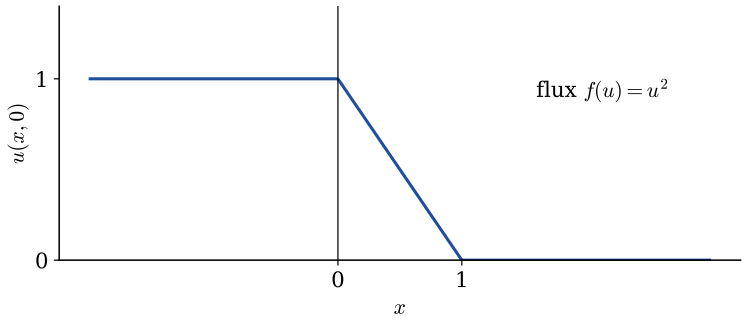
  - It is clear that by knowing these characteristic curves starting from each point (and knowing the solution doesn't change along these curves), we can piece together the solution $u(x,t)$ for all time t. We won't actually find the explicit equation for the solution $u(x,t)$ in terms of x and t but will use our knowledge of the characteristics of the equation to solve some interesting problems



- **Traffic-Flow Problem**
  - We will now solve a traffic-flow problem where the flux is given by $f(u) = u^2$ and the initial density of cars is given. In other words, we have

$$u_t + 2uu_x = 0 \quad -\infty < x < \infty,\ 0 < t < \infty$$

$$u(x,0) = \begin{cases} 1 & x \le 0 \\ 1 - x & 0 < x < 1 \\ 0 & 1 \le x \end{cases}$$



STEP 1

We begin by finding the characteristics starting from each initial point $(x_0, 0)$. For $x_0 < 0$, the characteristics are

$$\begin{aligned} x &= x_0 + g[u(x_0, 0)]t \\ &= x_0 + g[1]t \\ &= x_0 + 2t \\ &\Downarrow \\ t &= \frac{1}{2}(x - x_0) \end{aligned}$$

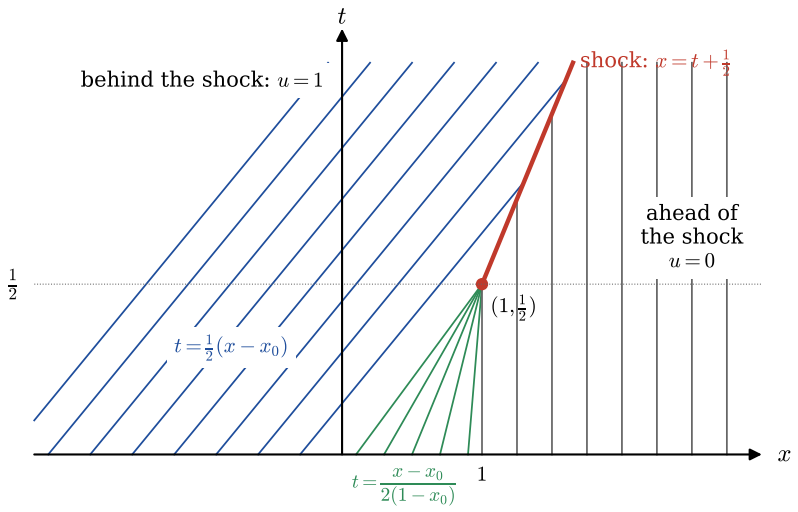
Now consider those initial points $0 < x_0 < 1$. Here, the characteristic curves are

$$
\begin{aligned}
x &= x_0 + g[u(x_0, 0)]t \\
&= x_0 + g[1 - x_0]t \\
&= x_0 + 2(1 - x_0)t \\
&\Downarrow \\
t &= \frac{x - x_0}{2(1 - x_0)}
\end{aligned}
$$

Finally, for $1 \leq x_0 < \infty$, we have the characteristics

$$
\begin{aligned}
x &= x_0 + g[u(x_0, 0)]t \\
&= x_0 + g[0]t \\
&= x_0
\end{aligned}
$$

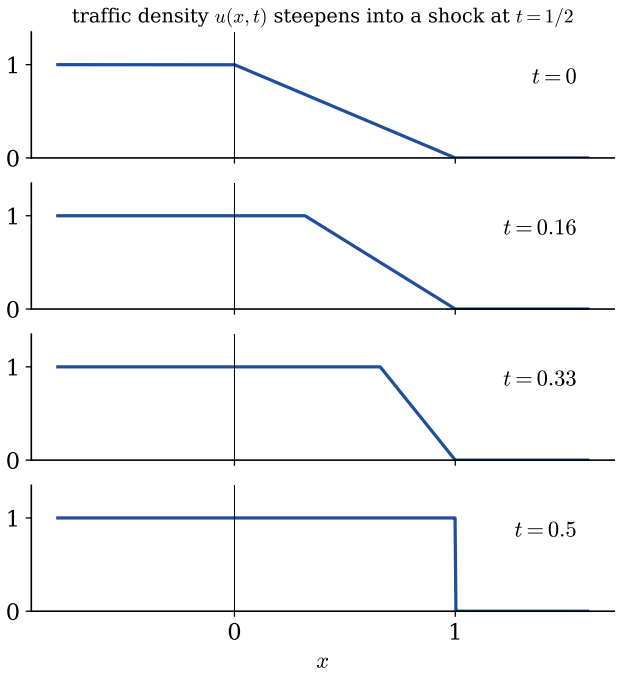
which represents vertical lines starting at x_0



The solution remains constant along each characteristic line. Now the solution for $x \in \mathbb{R},\ 0 \leq t < \frac{1}{2}$ is given by

$$
u(x,t) = \begin{cases} 1 & x \leq 2t, \quad 0 \leq t < \frac{1}{2} \\ \dfrac{1-x}{1-2t} & 2t \leq x < 1, \quad 0 \leq t < \frac{1}{2} \\ 0 & 1 \leq x, \quad 0 \leq t < \frac{1}{2} \end{cases}
$$

The difficulty in this example is that some characteristic lines will intersect each other. In other words, at the intersection point $(1, \frac{1}{2})$ of the two characteristic lines, the solution becomes multi-valued. This pathology means that the solution in the usual sense fails to exist after $t \geq \frac{1}{2}$



STEP 2

- Hence, to find the solution past $t = \frac{1}{2}$, we must use another type of analysis. When characteristics run together, we have the phenomenon of shock waves (discontinuous solutions), and what we must ask is, how fast does the leading edge of the shock wave propagate along the road?
- Before proceeding further, it is necessary to introduce the concept of a jump condition: a condition that holds at a discontinuity or abrupt change. Consider one dimensional problem where there is a jump in the conserved quantity u:

$$u_t + f_x = 0$$

Let the solution exhibit a jump (or shock) at $x = x_s(t)$ and integrate over the partial domain, $x \in (x_1, x_2)$, where $x_1 < x_s(t) < x_2$,

$$\frac{d}{dt}\left(\int_{x_1}^{x_s(t)} u\,dx + \int_{x_s(t)}^{x_2} u\,dx\right) = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} f(u)\,dx$$

$$\Downarrow$$

$$\int_{x_1}^{x_s(t)} u_t\,dx + u^-(x_s(t),t)\frac{dx_s}{dt} + \int_{x_s(t)}^{x_2} u_t\,dx - u^+(x_s(t),t)\frac{dx_s}{dt} = -\,f(u)\Big|_{x_1}^{x_2}$$

$$x_1 \to x_s(t),\ x_2 \to x_s(t) \quad \Downarrow$$

$$[u^-(x_s(t),t) - u^+(x_s(t),t)]\frac{dx_s}{dt} = f(u^-(x_s(t),t)) - f(u^+(x_s(t),t))$$

$$\Downarrow$$

$$s = \frac{dx_s}{dt} = \frac{f(u^-(x_s(t),t)) - f(u^+(x_s(t),t))}{u^-(x_s(t),t) - u^+(x_s(t),t)}$$

in which s is the speed of the discontinuity and represents the jump condition for the conservation equation

- A shock situation arises in a system where its characteristics intersect, and under these conditions a requirement for a unique single-valued solution is that the solution should satisfy the Lax entropy condition

$$f'(u^+(x_s(t),t)) < s < f'(u^-(x_s(t),t))$$

where $f'(u^+(x_s(t),t))$ and $f'(u^-(x_s(t),t))$ represent characteristic speeds at downstream and upstream conditions, respectively

- So, in our example, the speed of the wave front would be(remembering that $f(u) = u^2$)

$$s = \frac{1-0}{1-0} = 1$$

- Therefore, a shock is formed at $(x,t) = (1, \frac{1}{2})$:
  - After that, it moves with constant speed 1, along the line $x = t + \frac{1}{2}$, for $t \geq \frac{1}{2}$.
  - In conclusion, the solution takes different forms in the three different regions in the space-time plane

$$u(x,t) = \begin{cases} 1 & x \leq 2t, \quad 0 \leq t < \frac{1}{2} \text{ or } x - \frac{1}{2} \leq t, \quad t \geq \frac{1}{2} \\ \dfrac{1-x}{1-2t} & 2t \leq x < 1, \quad 0 \leq t < \frac{1}{2} \\ 0 & x \geq 1, \quad 0 \leq t < \frac{1}{2} \text{ or } x - \frac{1}{2} > t, \quad t \geq \frac{1}{2} \end{cases}$$

Example Solve

$$u_t + uu_x = 0 \qquad -\infty < x < \infty,\ 0 < t < \infty$$
$$u(x,0) = \begin{cases} 3(1-|x|) & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

Answer 12 (end of chapter) · Solution

Example Verify that $u = \phi[x - g(u)t]$ is an implicit solution of the nonlinear problem

$$u_t + g(u)u_x = 0 \qquad -\infty < x < \infty,\ 0 < t < \infty$$
$$u(x,0) = \phi(x)$$

Answer 13 (end of chapter) · Solution

13.13 Systems of PDEs

- In many areas of science, several unknown quantities (and their derivatives) are modeled by a system of interlocking equations. And we seek to find all of these functions simultaneously
- There are other reasons for studying systems of equations. Although things are not quite so simple in PDE theory, it is often possible to write a single higher-order PDE as a system of first-order equations
- For example, the telegraph equation

$$u_{tt} = c^2 u_{xx} + a u_t + b u$$

can be rewritten in equivalent form as three first-order PDEs by introducing the three variables:

$$\begin{aligned} u_1 &= u \\ u_2 &= u_x \\ u_3 &= u_t \end{aligned}$$

- It is a simple matter to see that the telegraph equation is equivalent to the three equations:

$$\begin{aligned} \frac{\partial u_1}{\partial x} &= u_2 \\ \frac{\partial u_1}{\partial t} &= u_3 \\ \frac{\partial u_3}{\partial t} &= c^2 \frac{\partial u_2}{\partial x} + a u_3 + b u_1 \end{aligned}$$

Solution of the Linear System $\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = \mathbf{0}$

- Consider the initial-value problem consisting of two PDEs and two ICs

$$\begin{aligned} \frac{\partial u_1}{\partial t} + 8\frac{\partial u_2}{\partial x} &= 0 \\ \frac{\partial u_2}{\partial t} + 2\frac{\partial u_1}{\partial x} &= 0 \\ u_1(x,0) &= f(x) \\ u_2(x,0) &= g(x) \end{aligned}$$

We start by writing the two PDEs in matrix form

$$\begin{bmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \end{bmatrix} + \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$
\begin{aligned}
&\Downarrow \\
&\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = \mathbf{0} \\
&\Downarrow \quad \mathbf{u} = \mathbf{P}\mathbf{v} \\
&\mathbf{v}_t + \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{v}_x = \mathbf{0} \\
&\Downarrow \\
&\mathbf{v}_t + \Gamma\mathbf{v}_x = \mathbf{0}
\end{aligned}
$$

where

$$
\mathbf{P} = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}, \ \Gamma = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}
$$

- These are the two uncoupled equations that we can solve independently

$$
\begin{aligned}
\frac{\partial v_1}{\partial t} + 4\frac{\partial v_1}{\partial x} &= 0 \\
\frac{\partial v_2}{\partial t} - 4\frac{\partial v_2}{\partial x} &= 0
\end{aligned}
$$

These equations have travelling-wave solutions

$$
\begin{aligned}
v_1(x,t) &= \phi(x - 4t) \\
v_2(x,t) &= \psi(x + 4t)
\end{aligned}
$$

where ϕ and ψ are arbitrary differentiable functions

- To obtain our general solution u, we merely compute

$$
\mathbf{u} = \mathbf{P}\mathbf{v} = \begin{bmatrix} 2\phi(x - 4t) - 2\psi(x + 4t) \\ \phi(x - 4t) + \psi(x + 4t) \end{bmatrix}
$$

We now substitute general solution into the ICs to get

$$
\begin{aligned}
2\phi(x) - 2\psi(x) &= f(x) \\
\phi(x) + \psi(x) &= g(x) \\
&\Downarrow \\
\phi(x) &= \frac{1}{4}\left[f(x) + 2g(x)\right] \\
\psi(x) &= \frac{1}{4}\left[2g(x) - f(x)\right]
\end{aligned}
$$

and, hence, the solution to IVP is

$$
\begin{aligned}
u_1(x,t) &= \frac{1}{2}\left[f(x - 4t) + 2g(x - 4t)\right] - \frac{1}{2}\left[2g(x + 4t) - f(x + 4t)\right] \\
u_2(x,t) &= \frac{1}{4}\left[f(x - 4t) + 2g(x - 4t)\right] + \frac{1}{4}\left[2g(x + 4t) - f(x + 4t)\right]
\end{aligned}
$$

Example What is the general solution of the system

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} = 0$$
$$\frac{\partial u_2}{\partial t} + 4\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} = 0$$

Answer 14 (end of chapter) · Solution

Summary

Topic	Key result
D'Alembert	$u_{tt} = c^2 u_{xx},\ u(x,0) = f,\ u_t(x,0) = g$: $u = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\alpha)\,d\alpha$
Vibrating string	fixed ends: $u = \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L}\right]\sin\frac{n\pi x}{L}$
Drumhead	radially symmetric: $u = \sum A_m J_0(\alpha_{0m} r)\cos(\alpha_{0m} ct),\ J_0(\alpha_{0m}) = 0$
Canonical form	characteristics $\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$ give ξ, η and $u_{\xi\eta} + \cdots = 0$
First-order linear	$u_t + v u_x = 0$: $u = f(x - vt)$, constant along the characteristics $x - vt = \text{const}$
Conservation law	$u_t + f(u)_x = 0$; $u_t + g(u)u_x = 0,\ u(x,0) = \phi$: $u = \phi[x - g(u)t]$
Shock speed	$s = \frac{dx_s}{dt} = \frac{f(u^-) - f(u^+)}{u^- - u^+}$, $f'(u^+) < s < f'(u^-)$

Worked Exercises

1. Find $u(0,0,0,t)$

$$u_{tt} = \nabla^2 u,\ \ t > 0$$
$$u(x,y,z,0) = 0,\ u_t(x,y,z,0) = x^2 + y^2 + z^2$$

Solution

Step 1: Use the Kirchhoff's formula (solution to 3D wave equation)

For the 3D wave equation (with wave speed $c = 1$), the solution with initial conditions $u(\mathbf{x}, 0) = f(\mathbf{x})$, $u_t(\mathbf{x}, 0) = g(\mathbf{x})$ is:

$$u(\mathbf{x},t) = \frac{1}{4\pi t}\int_{|\xi-\mathbf{x}|=t} g(\xi)\,dS(\xi) + \frac{1}{4\pi t^2}\int_{|\xi-\mathbf{x}|=t} f(\xi)\,dS(\xi)$$

where the integrals are over the sphere of radius t centered at $\mathbf{x}$

In our problem:

- $f(x,y,z) = 0$
- $g(x,y,z) = x^2+y^2+z^2 = |\xi|^2$
- Point of interest: $\mathbf{x} = (0,0,0)$

Thus:

$$u(0,0,0,t) = \frac{1}{4\pi t}\int_{|\xi|=t} |\xi|^2\,dS(\xi)$$

Step 2: Evaluate the integral

On the sphere $|\xi| = t$, we have $|\xi|^2 = t^2$ constant

Then:

$$u(0,0,0,t) = \frac{1}{4\pi t}\int_{|\xi|=t} t^2\,dS(\xi) = \frac{t^2}{4\pi t}\cdot \text{Surface area of sphere of radius } t$$

The surface area of a sphere of radius t is $4\pi t^2$, so:

$$u(0,0,0,t) = \frac{t^2}{4\pi t}\cdot 4\pi t^2 = t^2\cdot t = \boxed{t^3}$$

2. Find $u(x,y,z,t)$

$$\begin{aligned} u_{tt} &= \nabla^2 u, \quad t>0 \\ u(x,y,z,0) &= 0,\ u_t(x,y,z,0) = x \end{aligned}$$

Solution

Step 1: Use Kirchhoff's formula (3D wave equation)

For the 3D wave equation $u_{tt} = \nabla^2 u$ with wave speed $c = 1$, the general solution with initial data

$$u(\mathbf{x},0) = f(\mathbf{x}), \quad u_t(\mathbf{x},0) = g(\mathbf{x})$$

is given by:

$$u(\mathbf{x},t) = \frac{1}{4\pi t}\int_{|\xi-\mathbf{x}|=t} g(\xi)\,dS(\xi) + \frac{1}{4\pi t^2}\int_{|\xi-\mathbf{x}|=t} f(\xi)\,dS(\xi)$$

In our case:

- $f(x,y,z) = 0$
- $g(x,y,z) = x$ (we'll write this as ξ_1 to distinguish from the field point $\mathbf{x} = (x_1,x_2,x_3)$)

- Therefore:

$$u(\mathbf{x},t) = \frac{1}{4\pi t}\int_{|\xi-\mathbf{x}|=t} \xi_1\, dS(\xi)$$

Step 2: Use coordinate shift

Let's simplify by performing a change of variables to center the sphere

Let $\mathbf{x} = (x_1, x_2, x_3)$, and let $\xi = \mathbf{x} + t\omega$, where $\omega \in S^2$ is a unit vector. Then $|\xi - \mathbf{x}| = t$, and $dS(\xi) = t^2\, d\omega$. So:

$$u(\mathbf{x},t) = \frac{1}{4\pi t}\int_{|\xi-\mathbf{x}|=t} \xi_1\, dS(\xi) = \frac{1}{4\pi t}\int_{S^2} (x_1 + t\omega_1)\cdot t^2\, d\omega$$

Simplify:

$$u(\mathbf{x},t) = \frac{t}{4\pi}\int_{S^2} (x_1 + t\omega_1)\, d\omega = \frac{t}{4\pi}\left[x_1\int_{S^2} d\omega + t\int_{S^2}\omega_1\, d\omega\right]$$

Now compute the integrals:

- $\displaystyle\int_{S^2} d\omega = 4\pi$
- $\displaystyle\int_{S^2} \omega_1\, d\omega = 0$ (odd function over symmetric sphere)

So:

$$u(\mathbf{x},t) = \frac{t}{4\pi}\cdot x_1 \cdot 4\pi = x_1 t = xt$$

3. Solve the vibrations of a membrane which is in the xy plane:

$$\begin{aligned} u_{tt} &= \frac{\rho}{T}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right],\ \ 0 < x < \pi,\ 0 < y < \pi \\ &u(0,y,t) = 0,\ u(\pi,y,t) = 0 \\ &u(x,0,t) = 0,\ u(x,\pi,t) = 0 \\ &u(x,y,0) = \sin 2x \sin 3y + \sin 3x \sin 2y.\ u_t(x,y,0) = 0 \end{aligned}$$

Solution

Step 1: Separation of Variables

Assume a solution of the form:

$$u(x,y,t) = X(x)Y(y)T(t)$$

Plugging into the wave equation:

$$XYT'' = c^2\,(X''YT + XY''T)$$

Divide both sides by XYT:

$$\frac{T''}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

We separate variables:

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda \Rightarrow \begin{cases} X'' + \mu X = 0 \\ Y'' + (\lambda - \mu)Y = 0 \end{cases}$$

With the boundary conditions:

$$\begin{aligned} &X(0) = X(\pi) = 0 \Rightarrow X_n(x) = \sin(nx), \quad \mu = n^2 \\ &Y(0) = Y(\pi) = 0 \Rightarrow Y_m(y) = \sin(my), \quad \lambda - \mu = m^2 \Rightarrow \lambda = n^2 + m^2 \end{aligned}$$

Then the time part:

$$\begin{aligned} &T'' + c^2(n^2 + m^2)T = 0 \\ &\quad \Rightarrow T_{n,m}(t) = A_{n,m}\cos(c\sqrt{n^2 + m^2}t) + B_{n,m}\sin(c\sqrt{n^2 + m^2}t) \end{aligned}$$

Step 2: General Solution

$$\begin{aligned} u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Big[&A_{n,m}\cos(c\sqrt{n^2 + m^2}t) \\ &+ B_{n,m}\sin(c\sqrt{n^2 + m^2}t)\Big]\sin(nx)\sin(my) \end{aligned}$$

Step 3: Apply Initial Conditions

Initial displacement:

$$u(x, y, 0) = \sum_{n,m} A_{n,m}\sin(nx)\sin(my) = \sin 2x \sin 3y + \sin 3x \sin 2y$$

So $A_{2,3} = 1$, $A_{3,2} = 1$, all others zero

Initial velocity:

$$u_t(x, y, 0) = -c\sum_{n,m} c\sqrt{n^2 + m^2}B_{n,m}\sin(nx)\sin(my) = 0 \Rightarrow B_{n,m} = 0$$

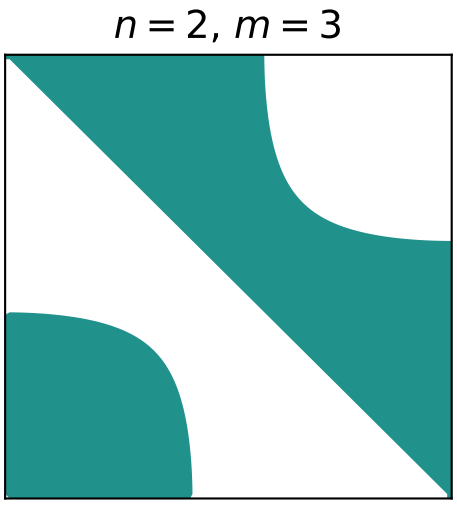
Final Answer

$$u(x, y, t) = \cos\left(c\sqrt{13}\,t\right)\sin(2x)\sin(3y) + \cos\left(c\sqrt{13}\,t\right)\sin(3x)\sin(2y)$$

where $c = \sqrt{\rho/T}$

```
import numpy as np
import matplotlib.pyplot as plt

def plot_rectangle_membrane_normal_modes(n, m):

    fig, ax = plt.subplots(figsize=(3, 3))

    ax.set(xticks=[], yticks=[])

    x_ = np.linspace(0, np.pi, 100)
    y_ = np.linspace(0, np.pi, 100)

    X_, Y_ = np.meshgrid(x_, y_)
    U_ = np.sin(n*X_) *np.sin(m*Y_) +np.sin(m*X_) *np.sin(n*Y_)

    ax.contourf(X_, Y_, U_, levels=[0, 10])

    ax.set_title(f'$n={n}$, $m={m}$')

    plt.show()
```

```
plot_rectangle_membrane_normal_modes(2, 3)
```



4. Solve the vibrations of a membrane which is in the xy plane:

$$
\begin{aligned}
&u_{tt} = \frac{\rho}{T}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right], \quad 0 < x < \pi,\ 0 < y < \pi \\
&u(0, y, t) = 0,\ u(\pi, y, t) = 0 \\
&u(x, 0, t) = 0,\ u(x, \pi, t) = 0 \\
&u(x, y, 0) = \sin x \sin 4y + \sin 4x \sin y,\ u_t(x, y, 0) = 0
\end{aligned}
$$

Solution

Step 1: Use Separation of Variables

Assume a solution of the form:

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[A_{nm}\cos(c\sqrt{n^2+m^2}\,t) + B_{nm}\sin(c\sqrt{n^2+m^2}\,t)\Big]\sin(nx)\sin(my)$$

From the initial displacement:

$$u(x,y,0) = \sum_{n,m} A_{nm}\sin(nx)\sin(my) = \sin x\sin 4y + \sin 4x\sin y$$

So the only nonzero coefficients are:

$$A_{1,4} = 1, \quad A_{4,1} = 1$$

From the initial velocity condition:

$$u_t(x,y,0) = 0 \Rightarrow B_{nm} = 0$$

Step 2: Final Solution

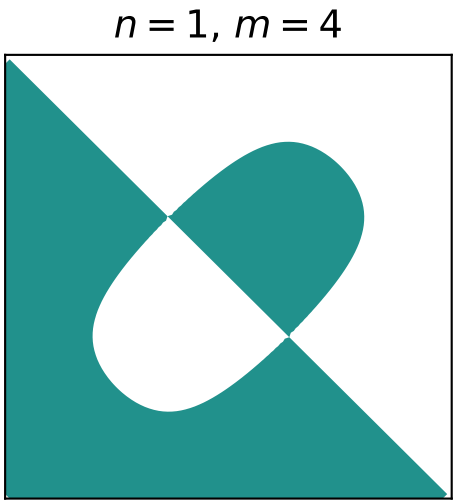
$$\begin{aligned} u(x,y,t) &= A_{1,4}\cos(c\sqrt{1^2+4^2}\,t)\sin x\sin 4y \\ &\quad + A_{4,1}\cos(c\sqrt{4^2+1^2}\,t)\sin 4x\sin y \\ &= \cos(c\sqrt{17}\,t)\sin x\sin 4y + \cos(c\sqrt{17}\,t)\sin 4x\sin y \end{aligned}$$

Thus, the solution is:

$$u(x,y,t) = \cos(c\sqrt{17}\,t)\,[\sin x\sin 4y + \sin 4x\sin y]$$

where $c = \sqrt{\frac{\rho}{T}}$

```
plot_rectangle_membrane_normal_modes(1, 4)
```



5. Suppose the vibration of a string is described by $u_{tt} = u_{xx}$ and has an initial displacement as given by the following diagram:

(a)

(b)

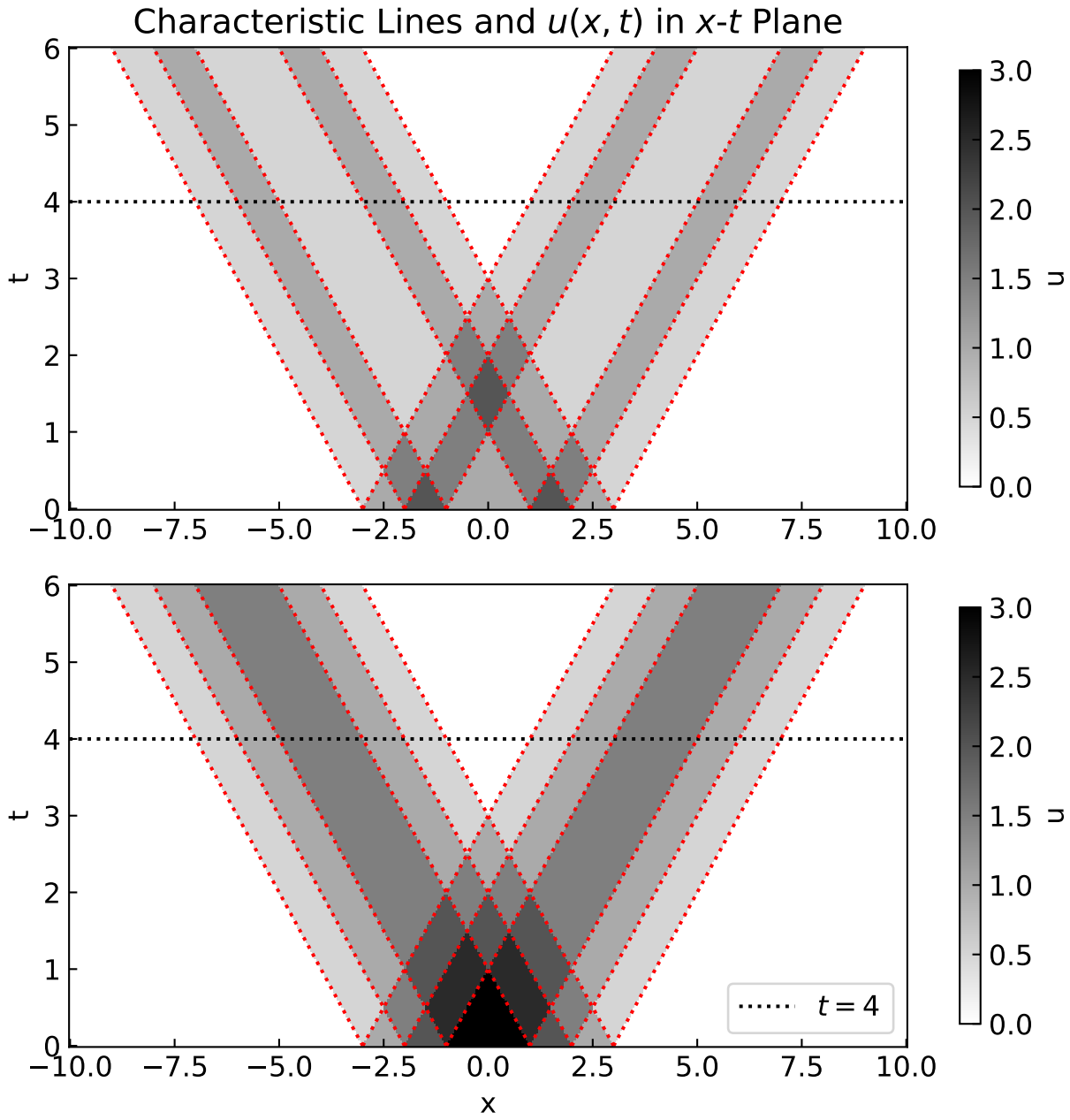
Assuming the initial velocity $u_t(x, 0) = 0$, describe the solution of this problem in the xt-plane and plot the solution at $t = 4$ in the xu-plane

Solution

```
# Re-import libraries due to code state reset
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.animation as animation
from IPython.display import HTML

# Define the piecewise function f(x)
def fa(x):
    x = np.array(x)
    result = np.zeros_like(x)
    result[(x >= -3) & (x < -2)] = 1
    result[(x > -2) & (x < -1)] = 2
    result[(x >= -1) & (x < 1)] = 1
    result[(x >= 1) & (x < 2)] = 2
    result[(x >= 2) & (x < 3)] = 1
    # result is already 0 elsewhere
    return result

def fb(x):
    x = np.array(x)
    result = np.zeros_like(x)
    result[(x >= -3) & (x < -2)] = 1
    result[(x > -2) & (x < -1)] = 2
    result[(x >= -1) & (x < 1)] = 3
    result[(x >= 1) & (x < 2)] = 2
    result[(x >= 2) & (x < 3)] = 1
    # result is already 0 elsewhere
    return result
```

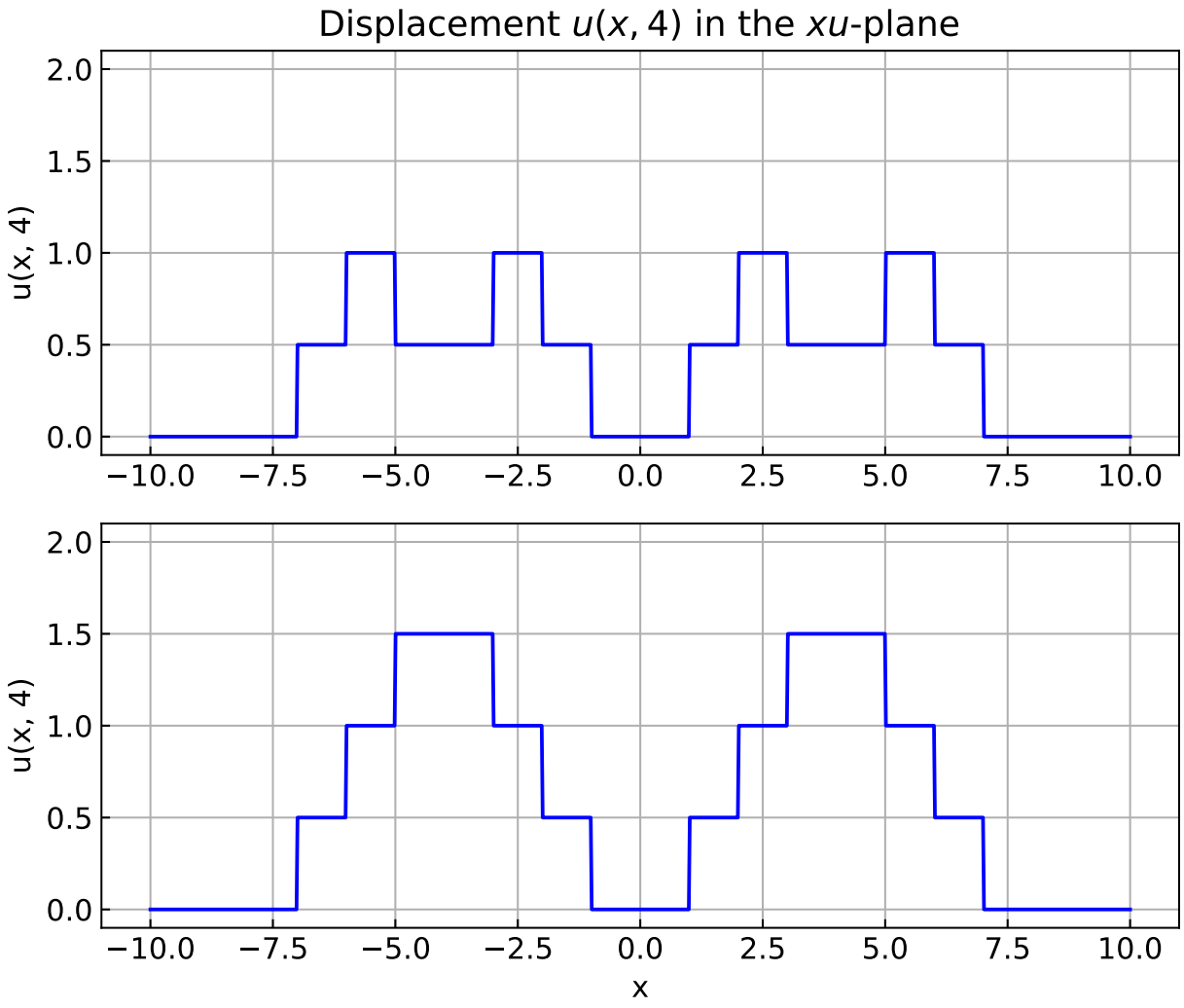
```
# Create meshgrid
x = np.linspace(-10, 10, 400)
t = np.linspace(0, 6, 200)
X, T = np.meshgrid(x, t)

Ua = 0.5 *(fa(X -T) +fa(X +T))
Ub = 0.5 *(fb(X -T) +fb(X +T))

# Plot u
fig, axes = plt.subplots(2, 1, figsize=(7, 7))
```

```
cba = axes[0].pcolormesh(x, t, Ua, vmin=0, vmax=3, cmap='gray_r')
cbb = axes[1].pcolormesh(x, t, Ub, vmin=0, vmax=3, cmap='gray_r')
plt.colorbar(cba, shrink=0.9, ticks=[0, 0.5, 1, 1.5, 2, 2.5, 3], label='u')
plt.colorbar(cbb, shrink=0.9, ticks=[0, 0.5, 1, 1.5, 2, 2.5, 3], label='u')

axes[0].axhline(y=4, color="black", linestyle=":", label='$t=4$')
axes[1].axhline(y=4, color="black", linestyle=":", label='$t=4$')

# Plot characteristics
for x0 in [-3, -2, -1, 1, 2, 3]:
    axes[0].plot(x0 +t, t, 'red', linestyle=':')
    axes[0].plot(x0 -t, t, 'red', linestyle=':')
    axes[1].plot(x0 +t, t, 'red', linestyle=':')
    axes[1].plot(x0 -t, t, 'red', linestyle=':')

axes[0].set_title('Characteristic Lines and $u(x,t)$ in $x$-$t$ Plane')
axes[1].set_xlabel('x')
axes[0].set_ylabel('t')
axes[1].set_ylabel('t')

plt.legend(loc=4)
plt.tight_layout()
plt.show()
```




```
# Define u(x, t) = 1/2 [f(x -t) +f(x +t)] at t = 4

t = 4
x_vals = np.linspace(-10, 10, 1000)
ua_vals = 0.5 *(fa(x_vals -t) +fa(x_vals +t))
ub_vals = 0.5 *(fb(x_vals -t) +fb(x_vals +t))

# Plot u(x, 4)
fig, axes = plt.subplots(2, 1, figsize=(7, 6))

axes[0].plot(x_vals, ua_vals, color='blue')
axes[1].plot(x_vals, ub_vals, color='blue')

axes[0].set_title('Displacement $u(x, 4)$ in the $xu$-plane')
axes[1].set_xlabel('x')
```

```
axes[0].set_ylabel('u(x, 4)')
axes[1].set_ylabel('u(x, 4)')
axes[0].grid(True)
axes[1].grid(True)

axes[0].set_ylim([-0.1, 2.1])
axes[1].set_ylim([-0.1, 2.1])

plt.tight_layout()
plt.show()
```




```
# Create grid
x = np.linspace(-10, 10, 400)
t_vals_light = np.linspace(0, 6, 40)

# Set up the figure
fig, axes = plt.subplots(2, 1, figsize=(7, 6))

line0, = axes[0].plot([], [], lw=2)
line1, = axes[1].plot([], [], lw=2)
axes[0].set_xlim(-10, 10)
```

```
axes[1].set_xlim(-10, 10)

axes[0].set_ylim(-0.1, 3.1)
axes[1].set_ylim(-0.1, 3.1)

axes[1].set_xlabel('x')
axes[0].set_ylabel('u(x,t)')
axes[1].set_ylabel('u(x,t)')
#axes[0].set_title('Wave Propagation Over Time')

plt.close()

# Animation functions
def init():
    line0.set_data([], [])
    line1.set_data([], [])
    return line0, line1

def animate(i):
    t = t_vals_light[i]
    ua = 0.5 *(fa(x -t) +fa(x +t))
    ub = 0.5 *(fb(x -t) +fb(x +t))
    line0.set_data(x, ua)
    line1.set_data(x, ub)
    axes[0].set_title(f'Wave at t = {t:.2f}')
    return line0, line1

# Create and save animation
ani = animation.FuncAnimation(fig, animate, frames=len(t_vals_light),
                              init_func=init, blit=True, interval=400)

HTML('<center>' + ani.to_html5_video() + '</center>')
```

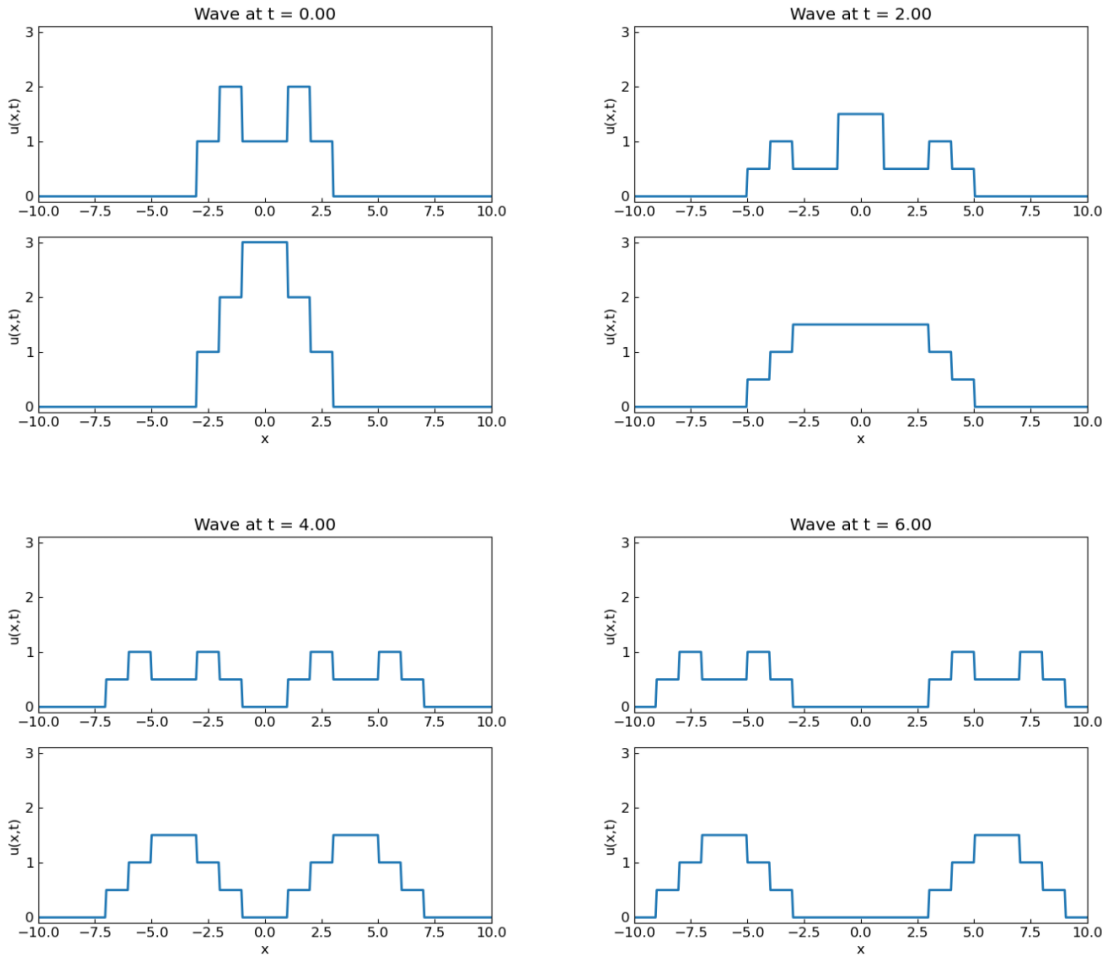


Figure 13.8: Four frames of the animation; play it on the web page

6. Find $u\left(\frac{3}{8}, 3\right)$

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < 1, \ 0 < t < \infty \\ u_x(0,t) &= 0 \\ u(1,t) &= 0 \\ u(x,0) &= (1-x)^3 \\ u_t(x,0) &= (1-x)^2 \end{aligned}$$

Solution

Method of Solution: D'Alembert's Formula with Function Extension

Step 1: D'Alembert Solution on the Infinite Domain

The general D'Alembert solution for the wave equation on the entire real line is:

$$u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\, ds$$

This solution assumes that u is defined on the entire real line with no boundary conditions. To use this method on the interval $0 < x < 1$, we extend $f(x)$ and $g(x)$ to the whole real line by reflecting them in such a way that the boundary conditions are satisfied

Step 2: Extend $f(x)$ and $g(x)$ to the Real Line

We extend $f(x)$ and $g(x)$ as follows:

Boundary Conditions:

- Neumann at $x = 0 \Rightarrow$ Extend as an even function about $x = 0$
- Dirichlet at $x = 1 \Rightarrow$ Extend as an odd function about $x = 1$

This leads to a 2-periodic extension over $\mathbb{R}$. The steps:

1. Reflect $f(x)$, $g(x)$ evenly about $x = 0$
2. Reflect them oddly about $x = 1$
3. Periodically extend with period 4

We denote the extended functions as $\tilde{f}(x)$, $\tilde{g}(x)$, defined for all $x \in \mathbb{R}$ and this pattern repeats every 4 units:

$$\tilde{f}(x + 4) = \tilde{f}(x)$$

Step 3: Apply D'Alembert Formula

Using the extended functions, the D'Alembert solution becomes:

$$u(x,t) = \frac{1}{2}\left[\tilde{f}(x+t) + \tilde{f}(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{g}(s)\,ds$$

Now substitute $x = \frac{3}{8}$, $t = 3$:

$$u\left(\frac{3}{8}, 3\right) = \frac{1}{2}\left[\tilde{f}\left(\frac{3}{8}+3\right) + \tilde{f}\left(\frac{3}{8}-3\right)\right] + \frac{1}{2}\int_{\frac{3}{8}-3}^{\frac{3}{8}+3} \tilde{g}(s)\,ds$$

- Construction of $\tilde{f}(x)$ on $[0, 4)$:

Interval	Definition of $\tilde{f}(x)$	Reasoning
$0 \le x < 1$	$(1-x)^3$	Original definition
$1 \le x < 2$	$(1-x)^3$	Odd reflection about $x = 1$
$2 \le x < 3$	$(x-3)^3$	Even reflection about $x = 2$ of previous term
$3 \le x < 4$	$(x-3)^3$	Odd reflection about $x = 3$ of previous term

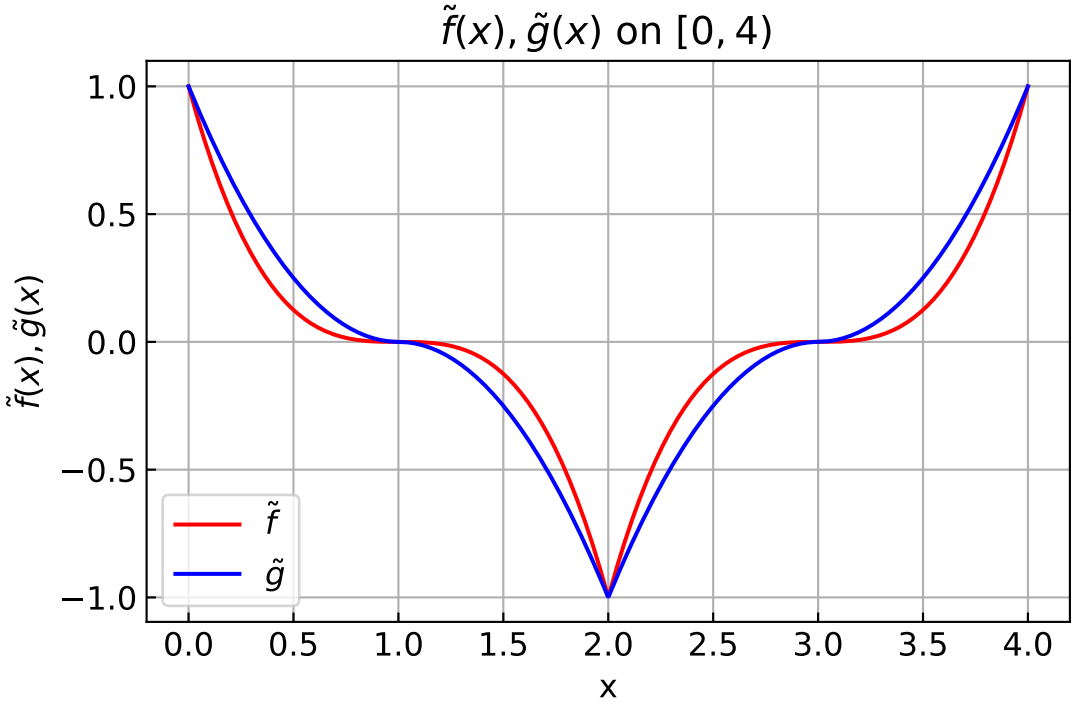
- Construction of $\tilde{g}(x)$ on $[0, 4)$:

Interval	Definition of $\tilde{g}(x)$	Reasoning

Interval	Definition of $\tilde{g}(x)$	Reasoning
$0 \leq x < 1$	$(1-x)^2$	Original definition
$1 \leq x < 2$	$-(1-x)^2$	Odd reflection about $x = 1$
$2 \leq x < 3$	$-(x-3)^2$	Even reflection about $x = 2$ of previous term
$3 \leq x < 4$	$(x-3)^2$	Odd reflection about $x = 3$ of previous term

```python
import numpy as np
import matplotlib.pyplot as plt

def tilde_fg(x):
    x = np.array(x)
    f = np.zeros_like(x)
    g = np.zeros_like(x)

    # (0, 1): original
    mask1 = (0 <= x) & (x < 1)
    f[mask1] = (1 -x[mask1])**3
    g[mask1] = (1 -x[mask1])**2

    # (1, 2): odd reflection about x = 1
    mask2 = (1 <= x) & (x < 2)
    f[mask2] = (1 -x[mask2])**3
    g[mask2] = -(1 -x[mask2])**2

    # (2, 3): even reflection about x = 2
    mask3 = (2 <= x) & (x < 3)
    f[mask3] = (x[mask3] -3)**3  # no sign change
    g[mask3] = -(x[mask3] -3)**2

    # (3, 4): odd reflection about x = 3
    mask4 = (3 <= x) & (x <= 4)
    f[mask4] = (x[mask4] -3)**3
    g[mask4] = (x[mask4] -3)**2

    return f, g

# Domain
x_vals = np.linspace(0, 4, 1000)
f_vals, g_vals = tilde_fg(x_vals)

# Plot
plt.figure(figsize=(6, 4))
plt.plot(x_vals, f_vals, color='red', label=r'$\tilde{f}$')
plt.plot(x_vals, g_vals, color='blue', label=r'$\tilde{g}$')

plt.title(r'$\tilde{f}(x), \tilde{g}(x)$ on $[0, 4)$')
```

```
plt.xlabel('x')
plt.ylabel(r'$\tilde{f}(x), \tilde{g}(x)$')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()
```



Step 4: Evaluate $\tilde{f}(x \pm t)$

$$x = \frac{3}{8} = 0.375, \quad t = 3 \Rightarrow x + t = 3.375, \quad x - t = -2.625$$

Using 4-periodicity:

- $3.375 \mod 4 = 3.375$
- $-2.625 \mod 4 = 1.375$

Thus:

- $\tilde{f}(3.375) = (3.375 - 3)^3 = (0.375)^3 = \frac{27}{512}$
- $\tilde{f}(1.375) = (1 - 1.375)^3 = -(0.375)^3 = -\frac{27}{512}$

So the average is:

$$\frac{1}{2}\left[\tilde{f}(3.375) + \tilde{f}(-2.625)\right] = \frac{1}{2}\left(\frac{27}{512} - \frac{27}{512}\right) = 0$$

Step 5: Evaluate the Integral Term

We now compute:

$$\int_{-2.625}^{3.375} \tilde{g}(s)\,ds = \underbrace{\int_{-2.625}^{-2} \tilde{g}(s)\,ds}_{A} + \underbrace{\int_{-2}^{2} \tilde{g}(s)\,ds}_{0} + \underbrace{\int_{2}^{3.375} \tilde{g}(s)\,ds}_{B}$$

We computed the tails:

- $A = \displaystyle\int_{-2.625}^{-2} \tilde{g}(s)\,ds = -\int_{1.375}^{2} (s-1)^2 ds = \boxed{-\frac{485}{1536}}$
- $B = \displaystyle\int_{2}^{3.375} \tilde{g}(s)\,ds$

We split this into two parts, because the definition of $\tilde{g}(s)$ changes at $s = 3$:

- On $[2, 3)$: $\tilde{g}(s) = (3-s)^2$

$$\int_2^3 (3-s)^2 ds = \int_1^0 u^2(-du) = \int_0^1 u^2 du = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3}$$

- On $[3, 3.375]$: $\tilde{g}(s) = -(s-3)^2$

$$\int_3^{3.375} -(s-3)^2 ds = -\int_0^{3/8} v^2 dv = -\left[\frac{v^3}{3}\right]_0^{3/8} = -\frac{(3/8)^3}{3} = -\frac{27}{1536}$$

So:

$$\int_2^{3.375} \tilde{g}(s)\,ds = \frac{1}{3} - \frac{27}{1536} = \frac{512-27}{1536} = \boxed{\frac{485}{1536}}$$

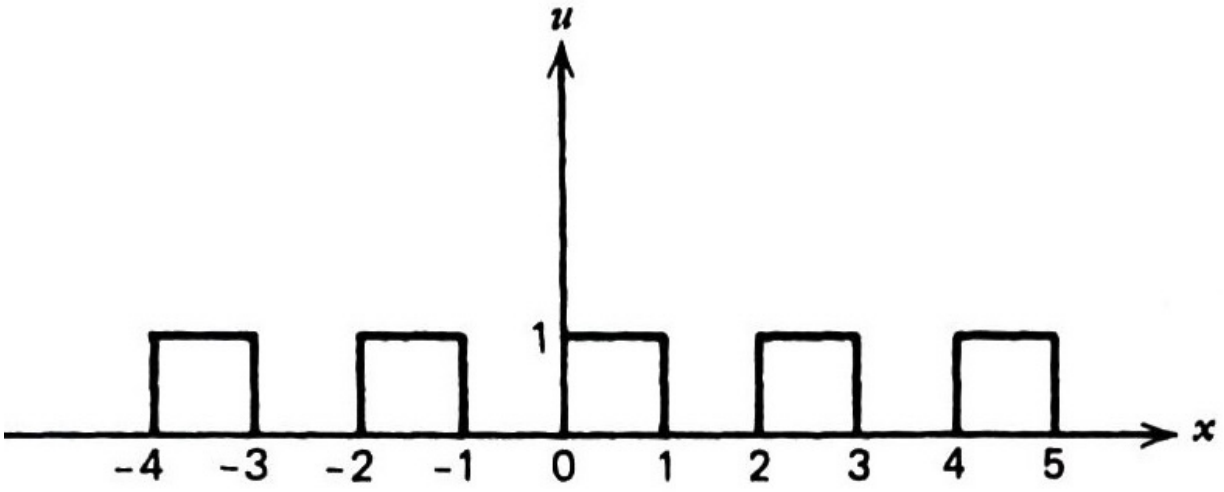
Final Answer:

Recall that from the previous step:

$$\frac{1}{2}[\tilde{f}(x+t) + \tilde{f}(x-t)] = 0 \quad \text{and} \quad (\text{integral term}) = 0$$

$$\boxed{u\left(\frac{3}{8}, 3\right) = 0}$$

7. Suppose the vibration of a string is described by $u_{tt} = u_{xx}$ and has an initial displacement as given by the following diagram:



Assuming the initial velocity $u_t(x,0) = 0$, describe the solution of this problem in the xt-plane

8. Consider the one-dimensional wave equation on the interval $0 < x < L$:

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t), & 0 < x < L, \\ u(0,t) = 0,\ u(L,t) &= 0, & t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) &= v_0(x), & 0 \le x \le L \end{aligned}$$

(a) Write down the eigenfunctions and eigenvalues for this boundary-value problem

(b) Using separation of variables, provide explicit expressions for the solution in terms of u_0 and v_0

Solution

(a) Eigenfunctions and Eigenvalues

Solve $X'' + \lambda X = 0$ with $X(0) = 0$ and $X(L) = 0$. Nontrivial solutions occur when

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, ...$$

and the eigenfunctions are

$$\phi_n(x) = \sin\frac{n\pi x}{L}$$

(b)-1 Series representation

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\frac{cn\pi t}{L} + B_n \sin\frac{cn\pi t}{L}\right) \sin\frac{n\pi x}{L}$$

This follows by separation of variables: $u(x,t) = X(x)T(t)$ leads to the spatial eigenfunctions above, and $T(t)$ solves $T'' + c^2\lambda_n T = 0$

(b)-2 Coefficients A_n and B_n

Using orthogonality:

$$A_n = \frac{2}{L}\int_0^L u_0(x) \sin\frac{n\pi x}{L}\, dx$$

$$B_n = \frac{2}{cn\pi}\int_0^L v_0(x)\sin\frac{n\pi x}{L}\,dx$$

Answers

Answer 1 (Section 13.3)

$u(x,t) = \frac{1}{2}\,\chi_{(1+ct,\ 2+ct)}(x) + \frac{1}{2}\,\chi_{(1-ct,\ 2-ct)}(x) - \frac{1}{2}\,\chi_{(ct-2,\ ct-1)}(x), \quad x>0$

($\chi_{(a,b)} = 1$ on (a,b) and 0 elsewhere): the left-moving pulse is reflected upside down at the fixed end

Answer 2 (Section 13.3)

$$u(x,t) = \frac{\sqrt{\pi}}{4c\sqrt{a}}\int_0^t e^{-b\tau}\Big[\operatorname{erf}(\sqrt{a}\,(x+c(t-\tau))) - \operatorname{erf}(\sqrt{a}\,(x-c(t-\tau)))\Big]d\tau$$

Answer 3 (Section 13.3)

$$u(r,t) = \begin{cases} t, & ct \le 1-r \\ \dfrac{1-(r-ct)^2}{4cr}, & |r-ct| < 1 < r+ct \\ 0, & \text{otherwise} \end{cases}$$ a spherical shell of thickness 2 moving outward; behind it $u=0$ again (Huygens' principle)

Answer 4 (Section 13.3)

$$u(r,t) = \frac{1}{2\pi c}\int_0^{ct}\frac{\Theta(\rho;r)\,\rho}{\sqrt{c^2t^2-\rho^2}}\,d\rho, \qquad u(0,t) = \begin{cases} t, & ct \le 1 \\ \dfrac{ct-\sqrt{c^2t^2-1}}{c}, & ct>1 \end{cases}$$

Unlike 3D, the wave leaves a tail that decays slowly instead of vanishing

Answer 5 (Section 13.6)

$$u(x,t) = \cos\frac{\pi ct}{L}\,\sin\frac{\pi x}{L} + 0.5\cos\frac{3\pi ct}{L}\,\sin\frac{3\pi x}{L}$$

Answer 6 (Section 13.6)

$$u(x,t) = \frac{L}{3\pi c}\,\sin\frac{3\pi ct}{L}\,\sin\frac{3\pi x}{L}$$

Answer 7 (Section 13.6)

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin n\pi x, \qquad b_n = 2\int_0^1 f(x)\sin n\pi x\,dx, \qquad \omega_n = n\pi c$$

$T_n = b_n e^{-\beta t/2}\left[\cos\nu_n t + \dfrac{\beta}{2\nu_n}\sin\nu_n t\right]$, $\nu_n = \sqrt{\omega_n^2 - \beta^2/4}$ if $\omega_n > \beta/2$; $T_n = b_n e^{-\beta t/2}\left(1+\frac{\beta}{2}t\right)$ if $\omega_n = \beta/2$

$T_n = b_n\dfrac{r_2 e^{r_1 t} - r_1 e^{r_2 t}}{r_2 - r_1}$, $r_{1,2} = -\frac{\beta}{2} \pm \sqrt{\frac{\beta^2}{4} - \omega_n^2}$ if $\omega_n < \beta/2$

Answer 8 (Section 13.8)

$$u(r,t) = \sum_{m=1}^{\infty}\frac{8}{\alpha_{0m}^3\,J_1(\alpha_{0m})}\,J_0(\alpha_{0m}r)\,\cos(\alpha_{0m}t), \qquad J_0(\alpha_{0m}) = 0$$

Answer 9 (Section 13.9)

$\xi = \dfrac{x}{v} \implies u_t + u_\xi = 0$

Answer 10 (Section 13.10)

$u(x,y) = F(y) + G(y-4x)$, with arbitrary twice-differentiable F, G

Answer 11 (Section 13.11)

$$u(x,y,t) = \exp\left[-\left(x - \frac{a}{c}t\right)^2 - \left(y - \frac{b}{c}t\right)^2\right] e^{-dt/c}$$

Answer 12 (Section 13.12)

For $t \ge \frac{1}{3}$: $u(x,t) = \begin{cases} \dfrac{3(1+x)}{1+3t}, & -1 \le x < x_s(t) \\ 0, & \text{otherwise} \end{cases}$, shock at $x_s(t) = \sqrt{2(1+3t)} - 1$

Answer 13 (Section 13.12)

$u = \phi[x - g(u)t]$ satisfies both the PDE and the initial condition as long as $1 + t\,g'(u)\,\phi'(\xi) \neq 0$

Answer 14 (Section 13.13)

$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \phi(x-3t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \psi(x+t)\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, with arbitrary differentiable ϕ, ψ

14 Elliptic Partial Differential Equations

Learning objectives After this chapter you should be able to

- interpret the Laplacian, and set up Dirichlet, Neumann and Robin problems, including the compatibility condition for the Neumann problem
- solve Laplace's equation in a disk, an annulus and an exterior region by separation of variables, and derive the Poisson integral formula
- solve Laplace's equation in spherical coordinates with Legendre functions and spherical harmonics
- solve a nonhomogeneous Dirichlet problem with a Green's function
- apply these methods to the hydrogen atom

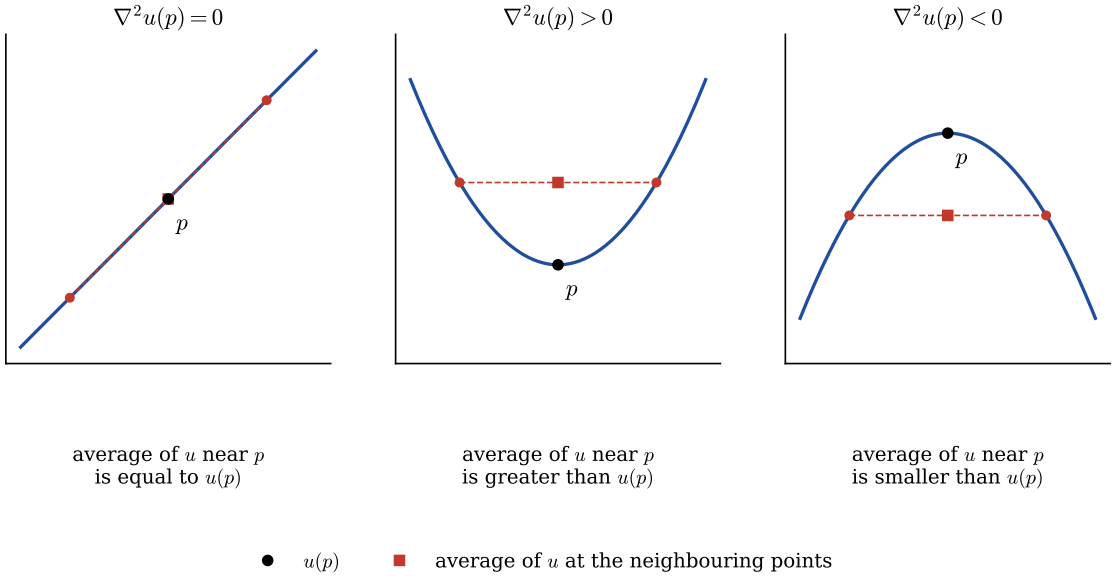
14.1 The Laplacian (an Intuitive Description)

- The Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

  is probably the most important operator in mathematical physics. The question is, what does it mean and why should the sum of three second derivatives have anything to do with the laws of nature?

- The answer to this lies in the fact that the Laplacian of a function allows us to compare the function at a point with the function at neighboring points. It does what the second derivative did in one dimension and might be thought of as a second derivative generalized to higher dimensions

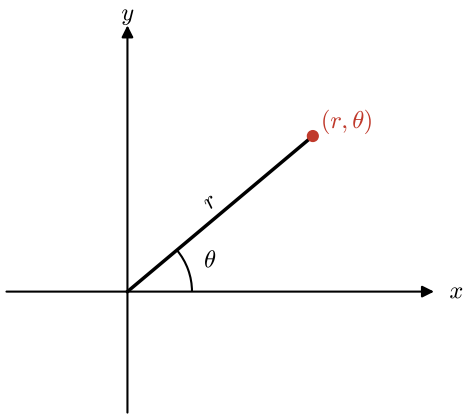


- The heat equation $u_t = \alpha \nabla^2 u$ measures temperature u, and the equation can be interpreted to mean that the change in temperature with respect to time u_t is proportional to $\nabla^2 u$. That is, the temperature at a point is increasing if the temperature at that point is less than the average of the temperature on a circle around the point
- The wave equation $u_{tt} = c^2 \nabla^2 u$ measures the displacement of a drumhead and can be interpreted to mean that the acceleration (or force) of a point on the drumhead u_{tt} is proportional to $\nabla^2 u$. That is, the drumhead at a point is accelerating upward if the drumhead at that point is less than the average of its neighbors
- Laplace equation $\nabla^2 u = 0$ says that the solution u is always equal to the average of its neighbors. For example, a steady-state, stretched rubber membrane satisfies Laplace equation, hence, the height of the membrane at any point is equal to the average height of the membrane on a circle around the point
- Poisson equation $\nabla^2 u = -f$, where f is a function that depends only on the space variables
  - $\nabla^2 u = -\rho$ describes the potential of an electrostatic field where ρ represents a constant charge density
  - $\nabla^2 u = -g(x, y)$ describes the steady-state temperature $u(x, y)$ due to a heat source $g(x, y)$
- $\nabla^2 u + \lambda u = 0$ is known as the Helmholtz equation which describes the fundamental shapes of a stretched membrane

- **Changing Coordinates**

  Before we start, however, let's review briefly the three major coordinate systems in two or three dimensions except for Cartesian system:

  - Polar Coordinates

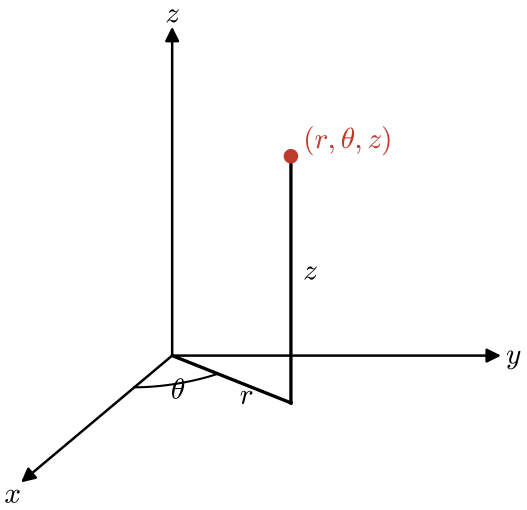


$$\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \quad \Rightarrow \quad \begin{array}{l} r^2 = x^2 + y^2 \\ \tan\theta = y/x \end{array}$$

As an illustration, we see how the two-dimensional Laplacian is transformed into polar coordinates:

$$\begin{aligned}
\nabla^2 u &= u_{xx} + u_{yy} \\
&\Updownarrow \\
u_x = u_r r_x + u_\theta \theta_x &= u_r \cos\theta - u_\theta \frac{\sin\theta}{r} \\
u_y = u_r r_y + u_\theta \theta_y &= u_r \sin\theta + u_\theta \frac{\cos\theta}{r} \\
&\Downarrow \\
u_{xx} = (u_x)_r r_x + (u_x)_\theta \theta_x &= \left(u_{rr}\cos\theta - u_{\theta r}\frac{\sin\theta}{r} + u_\theta \frac{\sin\theta}{r^2} \right)\cos\theta - \left(u_{r\theta}\cos\theta - u_r \sin\theta - u_{\theta\theta}\frac{\sin\theta}{r} - u \right. \\
u_{yy} = (u_y)_r r_y + (u_y)_\theta \theta_y &= \left(u_{rr}\sin\theta + u_{\theta r}\frac{\cos\theta}{r} - u_\theta \frac{\cos\theta}{r^2} \right)\sin\theta + \left(u_{r\theta}\sin\theta + u_r \cos\theta + u_{\theta\theta}\frac{\cos\theta}{r} - u \right. \\
&\Updownarrow \\
\nabla^2 u &= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}
\end{aligned}$$

– Cylindrical Coordinates

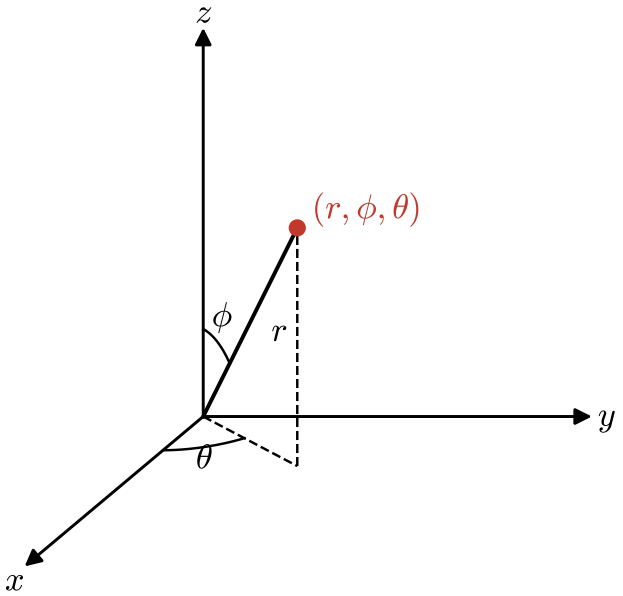


$$\begin{array}{lll} x = r\cos\theta & & r^2 = x^2 + y^2 \\ y = r\sin\theta & \Rightarrow & \tan\theta = y/x \\ z = z & & z = z \end{array}$$

Changing the Laplacian $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$ to cylindrical coordinates, we can show

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$$

– Spherical Coordinates



$$\begin{array}{ll} x = r\sin\phi\cos\theta & r^2 = x^2 + y^2 + z^2 \\ y = r\sin\phi\sin\theta & \Rightarrow \quad \cos\phi = z/r \\ z = r\cos\phi & \tan\theta = y/x \end{array}$$

Finally, if we write the Laplacian in spherical coordinates, we have

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$$

$$\Updownarrow$$

$$u_x = u_r r_x + u_\phi \phi_x + u_\theta \theta_x = u_r \sin\phi\cos\theta + u_\phi \frac{\cos\phi\cos\theta}{r} - u_\theta \frac{\sin\theta}{r\sin\phi}$$

$$u_y = u_r r_y + u_\phi \phi_y + u_\theta \theta_y = u_r \sin\phi\sin\theta + u_\phi \frac{\cos\phi\sin\theta}{r} + u_\theta \frac{\cos\theta}{r\sin\phi}$$

$$u_z = u_r r_z + u_\phi \phi_z + u_\theta \theta_z = u_r \cos\phi - u_\phi \frac{\sin\phi}{r}$$

$$\Downarrow$$

$$u_{xx} = (u_x)_r r_x + (u_x)_\phi \phi_x + (u_x)_\theta \theta_x = \Big(u_{rr}\sin\phi\cos\theta + u_{\phi r}\frac{\cos\phi\cos\theta}{r} - u_\phi \frac{\cos\phi\cos\theta}{r^2} - u_{\theta r}\frac{\sin\theta}{r\sin\phi} +$$
$$+ \big(u_{r\phi}\sin\phi\cos\theta + u_r\cos\phi\cos\theta + u_{\phi\phi}\frac{\cos\phi\cos\theta}{r} - u_\phi\frac{\sin\phi\cos\theta}{r} - u_{\theta\phi}\frac{\sin\theta}{r\sin\phi} + u_\theta$$
$$- \big(u_{r\theta}\sin\phi\cos\theta - u_r\sin\phi\sin\theta + u_{\phi\theta}\frac{\cos\phi\cos\theta}{r} - u_\phi\frac{\cos\phi\sin\theta}{r} - u_{\theta\theta}\frac{\sin\theta}{r\sin\phi} - u_\theta$$

$$u_{yy} = (u_y)_r r_y + (u_y)_\phi \phi_y + (u_y)_\theta \theta_y = \Big(u_{rr}\sin\phi\sin\theta + u_{\phi r}\frac{\cos\phi\sin\theta}{r} - u_\phi\frac{\cos\phi\sin\theta}{r^2} + u_{\theta r}\frac{\cos\theta}{r\sin\phi} - u_\theta$$
$$+ \big(u_{r\phi}\sin\phi\sin\theta + u_r\cos\phi\sin\theta + u_{\phi\phi}\frac{\cos\phi\sin\theta}{r} - u_\phi\frac{\sin\phi\sin\theta}{r} + u_{\theta\phi}\frac{\cos\theta}{r\sin\phi} - u_\theta$$
$$+ \big(u_{r\theta}\sin\phi\sin\theta + u_r\sin\phi\cos\theta + u_{\phi\theta}\frac{\cos\phi\sin\theta}{r} + u_\phi\frac{\cos\phi\cos\theta}{r} + u_{\theta\theta}\frac{\cos\theta}{r\sin\phi} - u_\theta$$
$$u_{zz} = (u_z)_r r_z + (u_z)_\phi \phi_z + (u_z)_\theta \theta_z = \Big(u_{rr}\cos\phi - u_{\phi r}\frac{\sin\phi}{r} + u_\phi\frac{\sin\phi}{r^2}\Big)\cos\phi + \big(-u_{r\phi}\cos\phi + u_r\sin\phi$$

$$\Downarrow$$

$$\nabla^2 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\phi\phi} + \frac{\cot\phi}{r^2}u_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta}$$
$$= \frac{1}{r^2}\left(r^2 u_r\right)_r + \frac{1}{r^2\sin\phi}\left(\sin\phi\, u_\phi\right)_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta}$$

NOTES

- The Laplacian in cartesian coordinates is the only one with constant coefficients. This is one reason why problems in other coordinate systems are harder to solve. It is still possible to use the separation of variables for these equations with variable coefficients; it's just that some of the resulting ordinary differential equations have variable coefficients
- We arrive at a lot of fairly complicated equations, such as Bessel's equation, Legendre equation, and other so-called classical equations of physics. To solve these equations, we must resort to infinite-series solutions and, in particular, the method of Frobenius

Example What is the wave equation $u_{tt} = c^2\nabla^2 u$ in polar coordinates if you know that the solution u depends only on r and t

Answer 1 (end of chapter) · Solution

Example What is Laplace equation in spherical coordinates if the solution u depends only on r? Can you find the solutions of this equation? These are the spherically symmetric potentials in three dimensions

Answer 2 (end of chapter) · Solution

14.2 General Nature of Boundary-Value Problems

- Many important problems whose outcomes do not change with time are described by elliptic boundary-value problems. There are two common situations in physical problems that give rise to PDEs that don't involve time; they are:

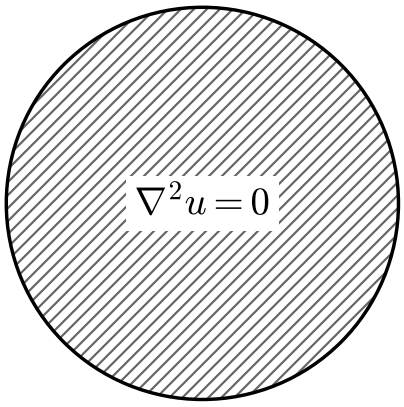
  1. Steady-state problems
  2. Problems where we factor out the time component in the solution

- When studying boundary-value problems (BVPs), there are three types of BCs that are most common; we discuss these three types now

- **Dirichlet Problems**

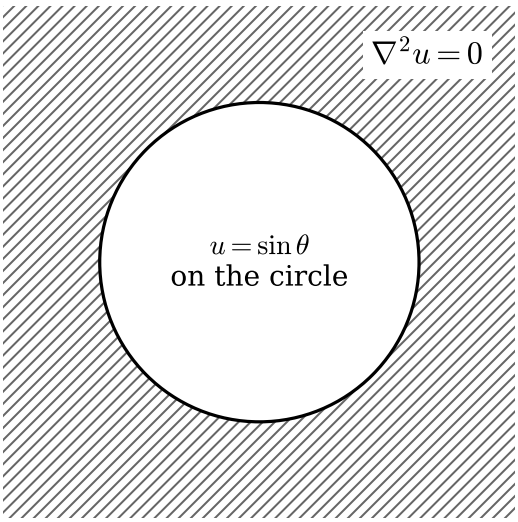
  Here, the PDE holds over a given region of space, and the solution is specified on the boundary of the region. An example would be to find the steady-state temperature inside a circle with the temperature given on the boundary

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$
$$u(1,\theta) = \sin\theta, \quad 0 < r < 1, \quad 0 \le \theta < 2\pi$$



$u = \sin\theta$ on the boundary

Another example would be an exterior Dirichlet problem in which we are looking for the solution of Laplace equation outside the unit circle, and the boundary condition is given on the circle



- **Neumann Problems**

  Here, the PDE holds in some region of space, but now the outward normal derivative

$$\frac{\partial u}{\partial n}$$

  (which is proportional to the inward flux) is specified on the boundary. For example, suppose the inward flow of heat varies around the circle according to

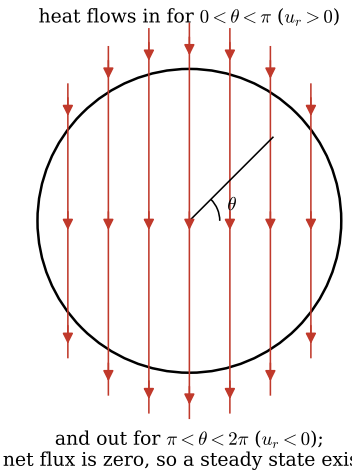
$$\frac{\partial u}{\partial r} = \sin\theta$$

  The steady-state temperature inside the circle would then be given by the solution of the BVP

$$\nabla^2 u = 0, \quad 0 < r < 1$$

$$\frac{\partial u}{\partial r} = \sin\theta, \quad r = 1, \ \ 0 \leq \theta < 2\pi$$

  Here, we can see that the flux of heat across the boundary is inward for $0 \leq \theta \leq \pi$ and outward for $\pi \leq \theta < 2\pi$



However, since the total flux

$$\int_0^{2\pi} \frac{\partial u}{\partial r}\, d\theta \;= \int_0^{2\pi} \sin\theta\, d\theta = 0$$

(a condition that must be true for Neumann problems), we can say that the temperature at each point inside circle does not change with respect to time. In other words, Neumann problems make sense only if the net gain in heat across the boundary is zero

The Neumann problem is somewhat different from other boundary conditions, in that solutions are not unique. In other words, the above Neumann problem has an infinite number of solutions $u(r, \theta)$. Once we have one solution, we can get the others just by adding a constant. For example, one solution to our Neumann problem is

$$u(r, \theta) = r \sin\theta$$

and it is obvious that if we add a constant to this solution, another one is obtained. For this reason, if we want to find one solution to the Neumann problem, we must have some additional information (like knowing the solution at one point)

- **Robin Problems**

  These problems correspond to the PDEs being given in some region of space, but now the condition on the boundary is a mixture of the first two kinds

$$\frac{\partial u}{\partial n} = -h(u - g)$$

  where h is a constant and g is a given function that can vary over the boundary. This BC says the inward flux across the boundary is proportional to the difference between the boundary value of u and specified environment value g

  In heat transfer, this, of course, is just Newton's law of cooling. The constant h is a physical parameter that measures the amount of flux across the boundary per difference between u and g. If h is large, the solution looks very much like the solution of the Dirichlet problem $u = g$. On the other hand, if $h = 0$, then the BC is reduced to the insulated BC

$$\frac{\partial u}{\partial r} = 0$$

Example Does the following Neumann problem have a solution inside the circle?

$$\nabla^2 u = 0, \quad 0 < r < 1$$

$$\frac{\partial u}{\partial r} = \sin^2\theta, \quad r = 1, \ \ 0 \le \theta < 2\pi$$

Answer 3 (end of chapter) · Solution

Example For different values of h, imagine the solution $u(r,\theta)$ to

$$\nabla^2 u = 0, \quad 0 < r < 1$$

$$\frac{\partial u}{\partial r} + h(u - \sin\theta) = 0, \quad r = 1, \ \ 0 \le \theta < 2\pi$$

Answer 4 (end of chapter) · Solution

14.3 Interior Dirichlet Problem for a Circle

- This section presents a number of new ideas to solve the interior Dirichlet problem for the circle

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < 1$$

$$u(1,\theta) = g(\theta), \quad 0 \le \theta < 2\pi$$

The method of separation of variables will be the usual procedure

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0 \\ u(1,\theta) &= g(\theta) \\ \Downarrow\ u(r,\theta) &= R(r)\Theta(\theta) \end{aligned}$$

$$-\frac{r^2R'' + rR'}{R} = \frac{\Theta''}{\Theta} = -\lambda \leq 0$$

$$\Downarrow$$

$$\Theta'' + \lambda\Theta = 0 \xrightarrow{\Theta(0)=\Theta(2\pi),\ \Theta'(0)=\Theta'(2\pi)} \Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, \quad \lambda_n = n^2, \quad n = 0, 1, 2, \cdots$$

$$\Downarrow$$

$$r^2R_0'' + rR_0' = 0 \xrightarrow{|R_0(0)|<\infty} R_0(r) = 1$$

$$r^2R_n'' + rR_n' - n^2R_n = 0, \quad n = 1, 2, \cdots \xrightarrow{|R_n(0)|<\infty} R_n(r) = r^n$$

$$\Downarrow$$

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta \, d\theta$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin n\theta \, d\theta$$

- The solution can be interpreted as expanding the boundary function $g(\theta)$ as a Fourier series

$$g(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

and solve the problem for $\cos n\theta$ and $\sin n\theta$ in the series. Since each of these terms will give rise to solutions $r^n \cos n\theta$ and $r^n \sin n\theta$, we can then say (by superposition) that

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

- Note that the constant term $\frac{a_0}{2}$ in the solution represents the average of $g(\theta)$

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\, d\theta$$

- If the radius of the circle was arbitrary (say R), then the solution would be

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

This completes our discussion of the separation of variables solution. We now go to the interesting Poisson integral formula

Example The solution of

$$\nabla^2 u = 0, \quad 0 < r < 1$$

$$u(1,\theta) = 1 + \sin\theta + \frac{1}{2}\sin 3\theta + \cos 4\theta$$

would be

$$u(r,\theta) = 1 + r\sin\theta + \frac{r^3}{2}\sin 3\theta + r^4\cos 4\theta$$

- **Poisson Integral Formula**

  We start with the separation of variables solution

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (a_n\cos n\theta + b_n\sin n\theta)$$

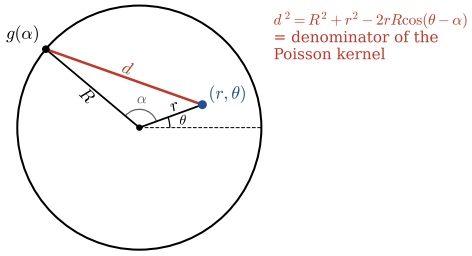
  and substitute the coefficients a_n and b_n. After a few manipulations, we have

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} g(\alpha)\,d\alpha + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \int_0^{2\pi} g(\alpha)\left(\cos n\alpha\cos n\theta + \sin n\alpha\sin n\theta\right)\,d\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos n(\theta-\alpha)\right\} g(\alpha)\,d\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left\{1 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \left[e^{in(\theta-\alpha)} + e^{-in(\theta-\alpha)}\right]\right\} g(\alpha)\,d\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left\{1 + \frac{re^{i(\theta-\alpha)}}{R - re^{i(\theta-\alpha)}} + \frac{re^{-i(\theta-\alpha)}}{R - re^{-i(\theta-\alpha)}}\right\} g(\alpha)\,d\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\theta-\alpha)}\right\} g(\alpha)\,d\alpha \end{aligned}$$

  This last equation is what we were looking for; it's the Poisson Integral Formula

  – We can interpret the Poisson integral solution as finding the potential u at (r,θ) as a weighted average of the boundary potentials $g(\theta)$ weighted by the Poisson kernel

$$\mathcal{P}(r,\theta) = \frac{1}{2\pi}\frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\theta-\alpha)}$$



  For boundary values $g(\alpha)$ close to (r,θ), the Poisson kernel gets larger, since the denominator of the Poisson kernel is the square of the distance from (r,θ) to (R,α)

– If we evaluate the potential at the center of the circle by the Poisson integral, we find

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,d\theta$$

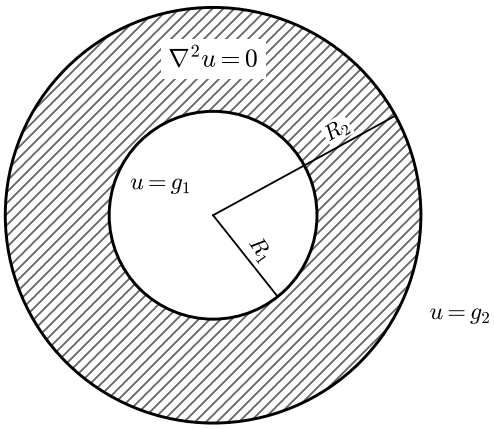
In other words, the potential at the center of the circle is the average of the boundary potential

14.4 The Dirichlet Problem in an Annulus and in an Exterior

- The Dirichlet problem between two circles (annulus) is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad R_1 < r < R_2$$

$$\begin{matrix} u(R_1,\theta) = g_1(\theta) \\ u(R_2,\theta) = g_2(\theta) \end{matrix}, \quad 0 \le \theta < 2\pi$$



- We begin by looking for solutions of the form

$$u(r,\theta) = R(r)\Theta(\theta)$$

Substituting this into Laplace equation, we get the two following ODEs in $R(r)$ and $\Theta(\theta)$:

$$\begin{aligned} &r^2R'' + rR' - \lambda R = 0 \quad \text{(Euler's equation)} \\ &\Theta'' + \lambda\Theta = 0 \end{aligned}$$

Note that in the two equations, we required the separation constant λ to be greater than, or equal to zero, or else the solution for $\Theta(\theta)$ would not be periodic

- Solving these two ODEs, we now have

$$\lambda = 0 \quad \begin{cases} \Theta(\theta) = \alpha + \beta\theta \\ R(r) = \gamma + \delta \ln r \end{cases}$$
$$\lambda > 0 \quad \begin{cases} \Theta(\theta) = a\cos\sqrt{\lambda}\theta + b\sin\sqrt{\lambda}\theta \\ R(r) = cr^{\sqrt{\lambda}} + dr^{-\sqrt{\lambda}} \end{cases}$$

  and using the requirement that $\Theta(\theta)$ must be periodic with period 2π, we have that $\beta = 0$ at $\lambda = 0$ and $\lambda(> 0)$ must be n^2, $n = 1, 2, 3, \cdots$

- Hence, we arrive at the following solutions to Laplace equation

$$\begin{array}{l} 1 \quad \text{(constants)} \\ \ln r \\ r^n \cos n\theta \\ r^n \sin n\theta \\ r^{-n} \cos n\theta \\ r^{-n} \sin n\theta \end{array}$$

- Since any sum of these solutions is also a solution, we arrive at our general solution

$$u(r,\theta) = \frac{a_0}{2} + \frac{\hat{a}_0}{2}\ln r + \sum_{n=1}^{\infty}\left[(a_n r^n + \hat{a}_n r^{-n})\cos n\theta + \left(b_n r^n + \hat{b}_n r^{-n}\right)\sin n\theta\right]$$

  The only task left is to determine the constants in the sum so that $u(r,\theta)$ satisfies the BCs

$$\begin{array}{l} u(R_1,\theta) = g_1(\theta) \\ u(R_2,\theta) = g_2(\theta) \end{array}, \quad 0 \le \theta < 2\pi$$

- Substituting the general solution into these BCs and integrating gives the following equations

$$\begin{cases} a_0 + \hat{a}_0 \ln R_1 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} g_1(\alpha)\, d\alpha \\ a_0 + \hat{a}_0 \ln R_2 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} g_2(\alpha)\, d\alpha \end{cases} \quad \text{Solve for } a_0, \hat{a}_0$$

$$\begin{cases} a_n R_1^n + \hat{a}_n R_1^{-n} = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} g_1(\alpha)\cos n\alpha\, d\alpha \\ a_n R_2^n + \hat{a}_n R_2^{-n} = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} g_2(\alpha)\cos n\alpha\, d\alpha \end{cases} \quad \text{Solve for } a_n, \hat{a}_n$$

$$\begin{cases} b_n R_1^n + \hat{b}_n R_1^{-n} = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} g_1(\alpha)\sin n\alpha\, d\alpha \\ b_n R_2^n + \hat{b}_n R_2^{-n} = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} g_2(\alpha)\sin n\alpha\, d\alpha \end{cases} \quad \text{Solve for } b_n, \hat{b}_n$$

Example Suppose the potential on the inside circle is zero, while the outside potential is $\sin\theta$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < 2$$

$$\begin{matrix} u(1,\theta) = 0 \\ u(2,\theta) = \sin\theta \end{matrix}, \quad 0 \le \theta < 2\pi$$

Solving the necessary equations for $a_0, \hat{a}_0$, $a_n, \hat{a}_n$, b_n, and $\hat{b}_n$ yields

$$u(r,\theta) = \frac{2}{3}\left(r - \frac{1}{r}\right)\sin\theta$$

Example Consider the problem with constant potentials on the boundaries

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < 2$$

$$\begin{matrix} u(1,\theta) = 3 \\ u(2,\theta) = 5 \end{matrix}, \quad 0 \le \theta < 2\pi$$

In this case, since it's obvious that the solution is independent of θ, we know our solution must be of the form $a_0 + \hat{a}_0 \ln r$. Using our two equations for a_0 and $\hat{a}_0$, we obtain

$$u(r,\theta) = 3 + \frac{2}{\ln 2}\ln r$$

The only solutions of the two dimensional Laplace equation that depend only on r are constant and $\ln r$. The potential $\ln r$ is very important and is called the logarithmic potential

Example Another interesting problem is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < 2$$

$$\begin{matrix} u(1,\theta) = \sin\theta \\ u(2,\theta) = \sin\theta \end{matrix}, \quad 0 \le \theta < 2\pi$$

A quick check of the coefficients reveals that they are all zero except for b_1 and $\hat{b}_1$. Solving for b_1 and $\hat{b}_1$ gives the solution

$$u(r,\theta) = \left(\frac{r}{3} + \frac{2}{3r}\right)\sin\theta$$

- **Exterior Dirichlet Problem**

  The exterior Dirichlet problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad R_1 < r < \infty$$

$$u(R_1, \theta) = g(\theta), \quad 0 \le \theta < 2\pi$$

  is solved exactly like the interior Dirichlet problem except that now we throw out the solutions that are unbounded as r goes to infinity

  Hence, we are left with the solution

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{R_1}\right)^{-n} \left(\hat{a}_n \cos n\theta + \hat{b}_n \sin n\theta\right)$$

  where a_0, $\hat{a}_n$ and $\hat{b}_n$ are exactly as Fourier series. In other words, we merely expand $u(R_1, \theta) = g(\theta)$ as a Fourier series

$$g(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\hat{a}_n \cos n\theta + \hat{b}_n \sin n\theta\right)$$

  and then insert the factor $\left(\frac{r}{R_1}\right)^{-n}$ in each term to get the solution

Example The exterior problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < \infty$$

$$u(1, \theta) = 1 + \sin\theta + \cos 3\theta, \quad 0 \le \theta < 2\pi$$

has the solution

$$u(r, \theta) = 1 + \frac{1}{r}\sin\theta + \frac{1}{r^3}\cos 3\theta, \quad 0 \le \theta < 2\pi$$

Example The exterior Neumann problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < \infty$$

$$\frac{\partial u}{\partial r}(1, \theta) = g(\theta), \quad 0 \le \theta < 2\pi$$

has a solution that is the same form as the Dirichlet problem

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}\left(\hat{a}_n \cos n\theta + \hat{b}_n \sin n\theta\right)$$

but now the coefficients a_0, $\hat{a}_n$ and $\hat{b}_n$ must satisfy the new BC. Of course, once you have this solution, any constant plus this solution is also a solution

14.5 Laplace Equation in Spherical Coordinates (Spherical Harmonics)

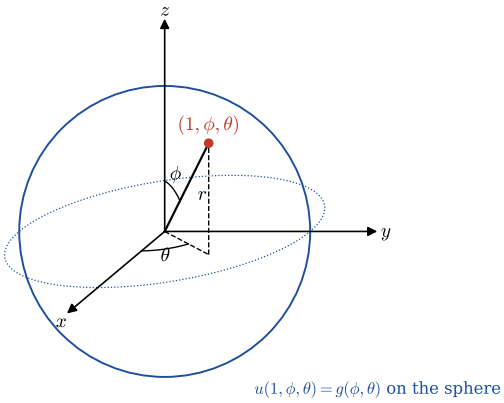
- An important problem in physics is to find the potential inside or outside a sphere when the potential is given on the boundary
- For the interior problem, we must find the solution $u(r,\phi,\theta)$ that satisfies

$$\left(r^2 u_r\right)_r + \frac{1}{\sin\phi}\left(\sin\phi\, u_\phi\right)_\phi + \frac{1}{\sin^2\phi} u_{\theta\theta} = 0$$

$$u(1,\phi,\theta) = g(\phi,\theta),\ 0 \le \theta < 2\pi,\ 0 \le \phi \le \pi$$

  Note that this spherical Laplacian is written in a different form than those we've seen before. This form is slightly more compact and easier to use

- Quite often $g(\phi,\theta)$ has a specific form, so that it isn't necessary to solve the problem in its most general form. Two important special cases are considered in this section. One is the case when $g(\phi,\theta)$ is constant, and the other is when it depends only on the angle ϕ (the angle from the north pole)



- **Special Case 1** – $g(\phi,\theta) =$ constant

  In this case, it is clear that the solution is independent of ϕ and θ, and so Laplace equation reduces to the ODE

$$\left(r^2 u_r\right)_r = 0$$

  This is a simple ODE that the student can easily solve; the general solution is

$$u(r) = \frac{a}{r} + b$$

In other words, constant and $\frac{a}{r}$ are the only potentials that depend only on the radial distance from the origin. The potential $\frac{1}{r}$ is very important in physics and is called the Newtonian potential

Example (Potential interior to a sphere)

$$\nabla^2 u = 0, \quad 0 < r < 1$$

$$u(1, \phi, \theta) = 3$$

Here solution must be $u(r, \phi, \theta) = 3$ in order to be bounded

Example (Potential between two spheres each at constant potential)

Suppose we want to find the steady-state temperature between two spheres held at different temperatures

$$\nabla^2 u = 0, \quad R_1 < r < R_2$$

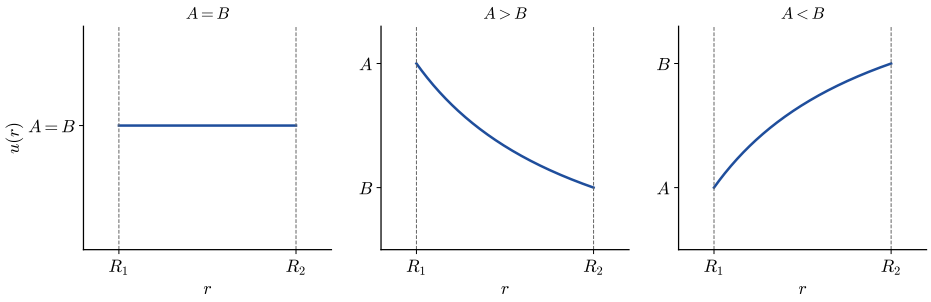
$$u(R_1, \phi, \theta) = A$$
$$u(R_2, \phi, \theta) = B$$

Since we know the potential has the general solution

$$u(r) = \frac{a}{r} + b$$

we substitute it in the BCs and solve for a and b; doing this gives

$$u(r) = (A - B)\frac{R_1 R_2}{R_2 - R_1}\frac{1}{r} + \frac{R_2 B - R_1 A}{R_2 - R_1}$$



- **Special Case 2** – $g(\phi, \theta)$ depends only on ϕ

  Here, the Dirichlet problem takes the form

$$r^2 u_{rr} + 2r u_r + \frac{1}{\sin\phi}\left(\sin\phi\, u_\phi\right)_\phi = 0, \quad 0 < r < 1$$

$$u(1, \phi) = g(\phi), \quad 0 \le \phi \le \pi$$

  Using separation of variables, we look for solutions of the form

$$u(r, \phi) = R(r)\Phi(\phi)$$

  and arrive at the two ODEs

$$\begin{aligned} &(\sin\phi\,\Phi')' + \lambda \sin\phi\Phi = 0 \quad \text{Legendre's equation} \\ &r^2 R'' + 2rR' - \lambda R = 0 \quad \text{Euler's equation} \end{aligned}$$

  Legendre's equation isn't easy; the general strategy in solving this equation is to make the substitution

$$x = \cos\phi$$

  Making this change of variable gives rise to the new Legendre's equation

$$\begin{aligned} &(\sin\phi\,\Phi')' + \lambda\sin\phi\Phi = 0 \\ &\qquad \Downarrow \quad x = \cos\phi, \quad \Phi' = \frac{d\Phi}{dx}\frac{dx}{d\phi} = -\sin\phi\frac{d\Phi}{dx} \\ &(\sin\phi\,\Phi')' = -\frac{d}{dx}\left(\sin^2\phi\frac{d\Phi}{dx}\right)\frac{dx}{d\phi} = \left(\sin^2\phi\frac{d^2\Phi}{dx^2} + 2\sin\phi\cos\phi\frac{d\phi}{dx}\frac{d\Phi}{dx}\right)\sin\phi \\ &\qquad = \left[(1-x^2)\frac{d^2\Phi}{dx^2} - 2x\frac{d\Phi}{dx}\right]\sin\phi \\ &\qquad \Downarrow \\ &(1-x^2)\frac{d^2\Phi}{dx^2} - 2x\frac{d\Phi}{dx} + \lambda\Phi = 0, \quad -1 \le x \le 1 \\ &\qquad \Downarrow \\ &\frac{d}{dx}\left[(1-x^2)\frac{d\Phi}{dx}\right] + \lambda\Phi = 0 \end{aligned}$$

  – One of the difficulties in this equation is that

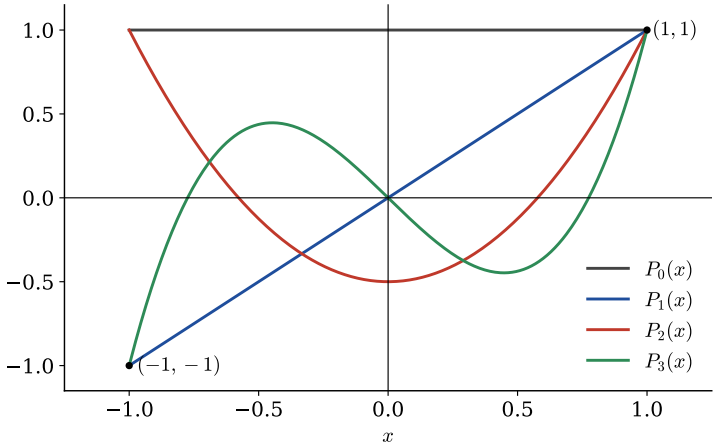
    the coefficient $(1-x^2)$ of $\dfrac{d^2\Phi}{dx^2}$ is zero at the ends of the domain $-1 \le x \le 1$

    Equations like this are called singular differential equations. We arrive at a very interesting conclusion

– The only bounded solutions of Legendre's equation occur when $\lambda = n(n+1)$, $n = 0, 1, 2, \cdots$ and these solutions are Legendre polynomials $P_n(x)$

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ &\vdots \\ P_n(x) &= \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right] \end{aligned}$$

The graphs of a few Legendre polynomials are shown in



– We now solve Euler's equation by substituting $\lambda = n(n+1)$ and $R(r) = r^\alpha$ in the equation and solving for α. Doing this, we get two values

$$\alpha = \begin{cases} n \\ -(n+1) \end{cases}$$

and, hence, Euler's equation has the general solution

$$R_n(r) = a_n r^n + b_n r^{-(n+1)} \xrightarrow{\text{bounded solution}} a_n r^n$$

– The final step is to form the sum

$$u(r, \phi) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\phi)$$

in such a way that it agrees with the BC $u(1, \phi) = g(\phi)$. Substituting the above solution into the BC gives

$$g(\phi) = \sum_{n=0}^{\infty} a_n P_n(\cos\phi)$$

If we multiply each side of this equation by $P_m(\cos\phi)\sin\phi$ and integrate ϕ from 0 to π, we get

$$\begin{aligned}\int_0^\pi g(\phi)P_m(\cos\phi)\,\sin\phi\,d\phi &= \sum_{n=0}^{\infty} a_n \int_0^\pi P_n(\cos\phi)P_m(\cos\phi)\,\sin\phi\,d\phi \\ &= \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x)P_m(x)\,dx \\ &= \begin{cases} 0 & n \neq m \\ \dfrac{2}{2m+1}\,a_m & n = m \end{cases} \\ &\Downarrow \\ a_n &= \frac{2n+1}{2}\int_0^\pi g(\phi)P_n(\cos\phi)\,\sin\phi\,d\phi\end{aligned}$$

Example Suppose the temperature on the surface of the sphere is given by

$$g(\phi) = 1 - \cos 2\phi, \quad 0 \le \phi \le \pi$$

and suppose we would like to find the temperature inside the sphere. In this problem, the temperature is constant on circles of constant latitude. To find u, we must solve

$$(r^2 u_r)_r + \frac{1}{\sin\phi}\left(\sin\phi\, u_\phi\right)_\phi = 0, \quad 0 < r < 1$$

$$u(1,\phi) = 1 - \cos 2\phi, \quad 0 \le \phi \le \pi$$

Our goal now is to expand $g(\phi)$ as a series of Legendre polynomials;

$$\begin{aligned}1 - \cos 2\phi &= 1 - (2\cos^2\phi - 1) \\ &= \frac{4}{3} - \frac{4}{3}\left[\frac{1}{2}(3\cos^2\phi - 1)\right] \\ &= \frac{4}{3}P_0(\cos\phi) - \frac{4}{3}P_2(\cos\phi) \\ &\Downarrow \\ a_0 = \frac{4}{3}, \;\; a_2 = -\frac{4}{3}, \;\; a_1 = a_3 = a_4 = a_5 = \cdots = 0\end{aligned}$$

Hence, the solution to the problem is

$$u(r,\phi) = \frac{4}{3}P_0(\cos\phi) - \frac{4}{3}r^2 P_2(\cos\phi) = \frac{4}{3} - \frac{2}{3}r^2(3\cos^2\phi - 1)$$

NOTE The solution of the exterior Dirichlet problem

$$\begin{gathered}(r^2 u_r)_r + \frac{1}{\sin\phi}\left(\sin\phi\, u_\phi\right)_\phi = 0, \;\; 1 < r < \infty \\ u(1,\phi) = g(\phi), \quad 0 \le \phi \le \pi\end{gathered}$$

is

$$u(r,\phi)=\sum_{n=0}^{\infty}\frac{b_n}{r^{n+1}}P_n(\cos\phi)$$

where $b_n=\dfrac{2n+1}{2}\displaystyle\int_0^{\pi}g(\phi)P_n(\cos\phi)\,\sin\phi\,d\phi$

For example, the BC $g(\phi)=3$ would yield the solution $u(r,\phi)=3/r$. Note that in this problem, the solution goes to zero, while in two dimensions, the exterior solution with constant BC was itself a constant

- **General Case** – $g(\phi,\theta)$

  We consider the boundary value problem

$$r^2u_{rr}+2ru_r+\frac{1}{\sin\phi}\left(\sin\phi u_\phi\right)_\phi+\frac{1}{\sin^2\phi}u_{\theta\theta}=0,\quad r<R$$
$$u(R,\phi,\theta)=g(\phi,\theta)$$

  in a sphere of radius R

  1. Solve this problem by separation of variables,
  2. Derive $K(r,\phi,\theta;R,\varphi,\vartheta)$ in the equivalent integral formula:

$$u(r,\phi,\theta)=\int_0^{2\pi}\int_0^{\pi}K(r,\phi,\theta;R,\varphi,\vartheta)\,g(\varphi,\vartheta)\sin\varphi\,d\varphi\,d\vartheta$$

  1. Applying separation of variables, we find that the equation has the solution of the form $R(r)\,\Phi(\phi)\,\Theta(\theta)$, provided

$$-\sin^2\phi\,\frac{r^2R''+2rR'}{R}-\sin\phi\frac{(\sin\phi\,\Phi')'}{\Phi}=\frac{\Theta''}{\Theta}=-\mu\le 0$$

  Since Θ must be periodic of period 2π, we have $\Theta=\cos m\theta$ or $\sin m\theta$, where $\mu=m^2$, $m=0,1,2,\cdots$. Then

$$-\frac{r^2R''+2rR'}{R}=\frac{(\sin\phi\,\Phi')'}{\sin\phi\,\Phi}-\frac{m^2}{\sin^2\phi}=-\lambda\le 0$$
$$\Downarrow$$
$$(\sin\phi\,\Phi')'+\left(\lambda\sin\phi-\frac{m^2}{\sin\phi}\right)\Phi=0$$
$$r^2R''+2rR'-\lambda R=0$$

  – The equation for $\Phi(\phi)$ is singular at its two endpoints $\phi=0$ and $\phi=\pi$. In lieu of boundary conditions, we impose the condition that Φ and Φ' remain bounded at both ends. This gives an eigenvalue problem with two singular endpoints. We introduce the new variable $x=\cos\phi$ and let $\Phi(\phi)=P(x)$. Then the equation for $\Phi(\phi)$ becomes

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[\lambda-\frac{m^2}{1-x^2}\right]P=0 \qquad \text{(AL)}$$

– In the case of $m=0$:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\lambda P=0 \qquad \text{(LG)}$$

we obtain a function which is bounded for $-1\le x\le 1$ if and only if

$$\lambda=n(n+1),\quad n=0,1,2,\cdots$$

These, then, are the eigenvalues. Setting $P_n(1)=1$, we obtain the eigenfunction $P_n(x)$ corresponding to the eigenvalue $\lambda_n=n(n+1)$:

$$P_n(x)=\sum_{k=0}^{n}\frac{(n+k)!}{2^k(k!)^2(n-k)!}(x-1)^k$$

It is called a **Legendre polynomial**

* Since $P_n(x)$ is of exactly degree n in x, any polynomial of degree k can be expressed as a linear combination of $P_n(x)$ with $n=0,1,\cdots,k$. Since $P_n(x)$ are orthogonal, each $P_n(x)$ must therefore be orthogonal to all powers of x less than n :

$$\int_{-1}^{1}x^kP_n(x)\,dx=0,\quad k=0,1,\cdots,n-1$$

These n linear conditions, together with the fact that $P_n(x)$ is of degree n, determine $P_n(x)$ uniquely

* We can verify the identity

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$$

which is called the **Rodrigues formula**. It is clear from this formula that $P_n(x)$ is even in x if n is even, and odd if n is odd

* The Legendre polynomials can also be defined as the coefficients in a formal expansion in powers of t of the **generating function**

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n$$

– We now wish to consider the equation (AL) with a positive m. For this purpose, we first differentiate the equation (LG) m times with respect to x. This gives

$$(1-x^2)\frac{d^{m+2}}{dx^{m+2}}P - 2(m+1)x\frac{d^{m+1}}{dx^{m+1}}P + [\lambda - m(m+1)]\frac{d^m}{dx^m}P = 0$$

We now introduce the new dependent variable

$$Q(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P(x)$$

– The above equation becomes

$$\frac{d}{dx}\left[(1-x^2)\frac{dQ}{dx}\right] + \left[\lambda - \frac{m^2}{1-x^2}\right]Q = 0$$

Thus any solution $P(x)$ of (LG) leads to a solution $Q(x)$ of (AL), unless $P(x)$ is a polynomial of degree less than m

– Since $P_n(x)$ is a polynomial of degree n,

$$\frac{d^m}{dx^m}P_n = 0 \text{ for } n < m,$$

and we get no eigenfunction. Therefore, the eigenvalues are precisely $n(n+1)$ with $n = m, m+1, \cdots$. The eigenfunctions

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) = \frac{1}{2^n n!}(1-x^2)^{m/2}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n$$

are called the **associated Legendre functions**. $P_n^m(x)$ corresponds to the eigenvalue $n(n+1)$, and $n \geq m$

– In Fourier expressions, it is useful to know the integrals of $[P_n^m(x)]^2$. It follows from the Rodrigues formula and n integrations by parts that

$$\begin{aligned}
\int_0^\pi [P_n(\cos\phi)]^2 \sin\phi\, d\phi &= \int_{-1}^1 [P_n(x)]^2\, dx \\
&= \frac{1}{2^{2n}n!^2}\int_{-1}^1 \left[\frac{d^n}{dx^n}(1-x^2)^n\right]^2 dx \\
&= \frac{1}{2^{2n}n!^2}\left[\underbrace{\frac{d^{n-1}}{dx^{n-1}}(1-x^2)^n\frac{d^n}{dx^n}(1-x^2)^n\Big|_{-1}^1}_{=0} - \int_{-1}^1 \frac{d^{n-1}}{dx^{n-1}}(1-x^2)^n\frac{d^{n+1}}{dx^{n+1}}(\right. \\
&\vdots \\
&= \frac{(-1)^n}{2^{2n}n!^2}\int_{-1}^1 (1-x^2)^n \underbrace{\frac{d^{2n}}{dx^{2n}}(1-x^2)^n}_{(-1)^n(2n)!}\, dx \\
&= \frac{(2n)!}{2^{2n}n!^2}\int_{-1}^1 (1-x^2)^n\, dx = \frac{(2n)!}{2^{2n}n!^2}2^{2n+1}\frac{n!^2}{(2n+1)!} = \frac{2}{2n+1}
\end{aligned}$$

– Similarly,

$$\int_0^\pi [P_n^m(\cos\phi)]^2 \sin\phi\, d\phi = \int_{-1}^1 [P_n^m(x)]^2\, dx = \int_{-1}^1 \left[(1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)\right]^2 dx$$

$$= \underbrace{\frac{d^{m-1}}{dx^{m-1}} P_n(x) \left\{(1-x^2)^m \frac{d^m}{dx^m} P_n(x)\right\}\Big|_{-1}^{1}}_{=0} - \int_{-1}^1 \frac{d^{m-1}}{dx^{m-1}} P_n(x) \frac{d}{dx} \left\{(1-x^2)\right.$$

$$\Downarrow \quad (1-x^2)^m = (-1)^m x^{2m} + \cdots, \quad \frac{d^m}{dx^m} P_n(x) = c_n \frac{n!}{(n-m)!} x^{n-m} + \cdots$$

$$= (-1)^m \int_{-1}^1 P_n(x) \underbrace{\frac{d^m}{dx^m} \left\{(1-x^2)^m \frac{d^m}{dx^m} P_n(x)\right\}}_{(-1)^m c_n \frac{n!}{(n-m)!} \frac{(n+m)!}{n!} x^n + \cdots} dx$$

$$= \frac{(n+m)!}{(n-m)!} \int_{-1}^1 P_n(x) c_n x^n\, dx$$

$$= \frac{(n+m)!}{(n-m)!} \int_{-1}^1 [P_n(x)]^2\, dx$$

$$= \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}$$

in which c_n is the coefficient of x^n in $P_n(x)$

– Putting $\lambda = n(n+1)$ in the equation for $R(r)$, we obtain the two solutions r^n and $r^{-(n+1)}$. Only the first of these is bounded at $r = 0$

– The method of separation of variables thus gives the harmonic functions

$$r^n P_n^m(\cos\phi) \cos m\theta$$
$$r^n P_n^m(\cos\phi) \sin m\theta$$

* We note that $P_n^m(\cos\phi)$ is $\sin^m\phi$ times a polynomial of degree $(n-m)$ in $\cos\phi$ which is even or odd according as $(n-m)$ is even or odd
* Since $r\cos\phi = z$, it follows that r^{n-m} times this polynomial is a polynomial in r^2 and z, and hence in x, y, and z
* Also, $r\sin\phi = \sqrt{x^2+y^2}$. Thus $r^m \sin^m\phi \cos m\theta$ and $r^m \sin^m\phi \sin m\theta$ are the polynomial solutions in x and y obtained by separating the two-dimensional Laplace equation in polar coordinates
* We see then that the functions $r^n P_n^m(\cos\phi)\cos m\theta$ and $r^n P_n^m(\cos\phi)\sin m\theta$ are homogeneous polynomials of degree n in x, y, and z. They are of degree $(n-m)$ in z. These polynomials are called **spherical harmonics**

The formal solution of the problem is then

$$u(r,\phi,\theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n \left[\frac{1}{2} a_{n0} P_n(\cos\phi) + \sum_{m=1}^{n} (a_{nm}\cos m\theta + b_{nm}\sin m\theta) P_n^m(\cos\phi)\right]$$

where

$$a_{nm} = \frac{(2n+1)}{2\pi}\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_0^{\pi} g(\phi,\theta)\, P_n^m(\cos\phi)\, \cos m\theta\, \sin\phi\, d\phi\, d\theta$$
$$b_{nm} = \frac{(2n+1)}{2\pi}\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_0^{\pi} g(\phi,\theta)\, P_n^m(\cos\phi)\, \sin m\theta\, \sin\phi\, d\phi\, d\theta$$

are the n-th Fourier coefficients of $g(\phi,\theta) = u(R,\phi,\theta)$

2. We can write this solution as

$$u(r,\phi,\theta) = \int_0^{2\pi}\int_0^{\pi} K(r,\phi,\theta;R,\varphi,\vartheta)\, g(\varphi,\vartheta)\sin\varphi\, d\varphi\, d\vartheta$$

where $K(r,\phi,\theta;R,\varphi,\vartheta)$ is

$$\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n \underbrace{\left[\frac{2n+1}{4\pi}P_n(\cos\phi)P_n(\cos\varphi) + \sum_{m=1}^{n}\frac{(2n+1)}{2\pi}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\phi)P_n^m(\cos\varphi)\cos m(\theta-\vartheta)\right]}_{K_n(\phi,\theta;\varphi,\vartheta)} \tag{KL}$$

– For a fixed n,

$$\int_0^{2\pi}\int_0^{\pi} K_n(\phi,\theta;\varphi,\vartheta)\, r^l\, P_l^m(\cos\varphi)\,(a\cos m\vartheta + b\sin m\vartheta)\,\sin\varphi\, d\varphi\, d\vartheta$$
$$= \begin{cases} r^n P_n^m(\cos\phi)(a\cos m\theta + b\sin m\theta) & \text{for } l = n \\ 0 & \text{for } l \neq n \end{cases}$$

– This means that any homogeneous polynomial of degree l which is a solution of Laplace equation can be written as a linear combination of the $2l+1$ spherical harmonics $r^l\, P_l^m(\cos\phi)\cos m\theta$ and $r^l\, P_l^m(\cos\phi)\sin m\theta$. The kernel $K_n(\phi,\theta;\varphi,\vartheta)$ has the property that it reproduces all homogeneous harmonic polynomials of degree n, and reduces those of other degrees to zero

– Since rotation of the coordinate axes takes a homogeneous harmonic polynomial of degree l into another such polynomial in the new variable, the kernel K_n must be unchanged by such a rotation. That is, if the new polar coordinates are (r',ϕ',θ'), we have $K_n(\phi,\theta;\varphi,\vartheta) = K_n(\phi',\theta';\varphi',\vartheta')$. In particular, we can rotate our coordinates so that the new z-axis passes through (r,ϕ,θ); that is, so that $\phi' = 0$

– By definition, $P_n(1) = 1$. On the other hand, $P_n^m(x)$ with $m \geq 1$ has a factor $(1-x^2)^{m/2}$, so that $P_n^m(1) = 0$, for $m \geq 1$. Therefore

$$K_n(\phi,\theta;\varphi,\vartheta) = K_n(0,\theta';\varphi',\vartheta') = \frac{2n+1}{4\pi}P_n(\cos\varphi') \tag{KS}$$

where φ' is the angle between the direction (ϕ,θ) and (φ,ϑ) :

$$\cos\varphi' = \cos\phi\cos\varphi + \sin\phi\sin\varphi\cos(\theta-\vartheta)$$

– Substituting the identity (KS) in (KL), we find

$$K(r,\phi,\theta;R,\varphi,\vartheta) = \frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)\left(\frac{r}{R}\right)^n P_n(\cos\varphi') = \frac{1}{4\pi}\left(2r\frac{\partial}{\partial r}+1\right)\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\varphi')$$

– Evaluating the generating function at $x=\cos\varphi'$ and $t=\frac{r}{R}$, we have the identity

$$\frac{R}{\sqrt{r^2+R^2-2rR\cos\varphi'}} = \sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\varphi')$$

Since

$$\frac{1}{4\pi}\left(2r\frac{\partial}{\partial r}+1\right)\frac{R}{\sqrt{r^2+R^2-2rR\cos\varphi'}} = \frac{1}{4\pi}\frac{R\left(R^2-r^2\right)}{\left(r^2+R^2-2rR\cos\varphi'\right)^{3/2}} = \frac{1}{4\pi}\frac{1-\left(\frac{r}{R}\right)^2}{\left(1+\left(\frac{r}{R}\right)^2-2\frac{r}{R}\cos\varphi'\right)^{3/2}} = \mathcal{P}(r,R,\varphi')$$

where $\left(r^2+R^2-2rR\cos\varphi'\right)^{1/2}$ is the distance from the point (r,ϕ,θ) to the point (R,φ,ϑ), thus, the solution formula is equivalent to the integral formula

$$u(r,\phi,\theta) = \int_0^{2\pi}\int_0^{\pi}\mathcal{P}(r,R,\varphi')\,g(\varphi,\vartheta)\sin\varphi\,d\varphi\,d\vartheta$$

This is **Poisson integral formula** in three dimensions

Poisson Integral Formula in 2D and 3D — Summary Table

Dimension	Domain	Poisson Kernel $\mathcal{P}$	Poisson Integral Formula
2D	Unit disk $D=\{\mathbf{x}\in\mathbb{R}^2:\lvert\mathbf{x}\rvert<1\}$	$\mathcal{P}(\mathbf{x},\xi)=\frac{1}{2\pi}\frac{1-\lvert\mathbf{x}\rvert^2}{\lvert\mathbf{x}-\xi\rvert^2}$	$u(\mathbf{x})=\int_{\partial D}\mathcal{P}(\mathbf{x},\xi)\,g(\xi)\,d\xi$
3D	Unit ball $B=\{\mathbf{x}\in\mathbb{R}^3:\lvert\mathbf{x}\rvert<1\}$	$\mathcal{P}(\mathbf{x},\xi)=\frac{1}{4\pi}\frac{1-\lvert\mathbf{x}\rvert^2}{\lvert\mathbf{x}-\xi\rvert^3}$	$u(\mathbf{x})=\int_{\partial B}\mathcal{P}(\mathbf{x},\xi)\,g(\xi)\,d\xi$

Try it in Python: Spherical harmonics Y_l^m evaluated with `scipy.special.sph_harm_y` and visualized in 3D: Hydrogen Atomic Orbitals

14.6 A Nonhomogeneous Dirichlet Problem (Green's Function)

- A common problem in applied mathematics is to find the potential in some region of space in response to a forcing term f. A typical example would be to find the potential inside a circle in two dimensions that satisfies

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = -f(r,\theta), \quad 0 < r < 1$$

$$u(1,\theta) = 0, \quad 0 \le \theta < 2\pi$$

- Note that we have chosen the boundary values to be zero. If we wanted to solve the general case, where both the equation and BC were nonhomogeneous, we could add the Poisson integral formula from Section 14.3 to the solution from this section
- In order to gain a little intuition about nonhomogeneous differential equations, let's consider graphing the solution to the following Poisson equation:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = -q, \quad 0 < r < 1$$

$$u(1,\theta) = 0, \quad 0 \le \theta < 2\pi$$

  Here, the potential is fixed at zero on the boundary, and the Laplacian of u is always equal to $-q$ inside the circle. Since $\nabla^2 u(p)$ measures the difference between $u(p)$ and the average of its neighbors, it will look like a thin membrane fixed at the boundary that was continuously being pushed up by a stream of air from below

- **Potentials from Point Sources and Sinks**
  - In solving a nonhomogeneous linear equation, it is sufficient to solve the equation with a point source since we can find the solution to the general problem by summing the responses to point sources. Our goal here is to find the potential in some region of space due to a point source (or sink)
  - Suppose we have a single point source of magnitude q located at the origin. It is clear that the heat will flow outward along radial lines, and, hence, if we compute the total outward flux across a circle of radius r, we have

$$\text{Total outward flux across the circle } = -\int_0^{2\pi} \underbrace{u_r(r)}_{<0}\, r\, d\theta = -2\pi r\, u_r(r)$$

  - But the outward flux must be equal to the heat generated within the circle (conservation of energy), so we have

$$-2\pi r u_r(r) = q$$

    Solving this simple differential equation for $u_r(r)$, we get

$$u(r) = \frac{q}{2\pi} \ln \frac{1}{r}$$

A sink, on the other hand, is represented by a negative source, and so a sink with magnitude $-q$ would give rise to a potential field

$$u(r) = -\frac{q}{2\pi} \ln \frac{1}{r}$$

- **Poisson Equation Inside a Circle**
  - We will now solve the important problem

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = -f(r, \theta), \quad 0 < r < 1$$

$$u(1, \theta) = 0, \quad 0 \leq \theta < 2\pi$$

  - The Green function technique (impulse-response method) consists of two steps:
    * Finding the potential $G(r, \theta, \rho, \phi)$ at (r, θ), which we force to be zero on the boundary and which is due to a single charge (magnitude 1) at (ρ, ϕ)
    * Summing the individual responses $G(r, \theta, \rho, \phi)$ weighted by $f(\rho, \phi)$ over all (ρ, ϕ) in the circle to get the solution

$$u(r, \theta) = \int_0^{2\pi} \int_0^1 G(r, \theta, \rho, \phi)\, f(\rho, \phi)\, \rho \, d\rho d\phi$$

STEP 1 Since the function

$$\frac{1}{2\pi} \ln \frac{1}{R}$$

is the potential at (r, θ) due to a single unit charge at (ρ, ϕ), where R is the distance from (r, θ) to (ρ, ϕ), the only thing left to do is to modify the function so that it is zero on the boundary

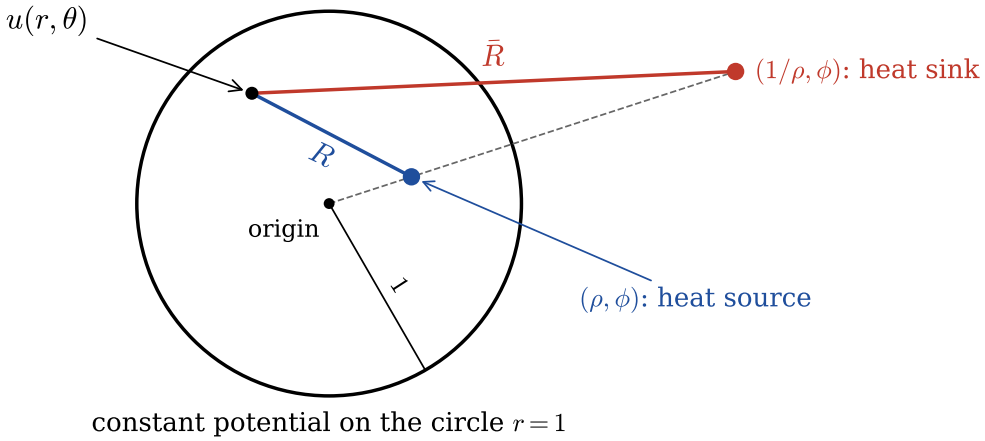
STEP 2 The strategy is to place a sink outside the circle at such a point that the potential due to both is constant on the circle at $r = 1$. We can then subtract this constant value to obtain a zero potential on the boundary. The big question is, of course, where do we place the sink outside circle, so that the potential on the boundary is constant?

Without going into the details, we can show rather easily that if the sink is placed at $(1/\rho, \phi)$, then the potential due to the source and the sink will be constant on the circle $r = 1$

$$\begin{aligned} u^*(r, \theta) &= \frac{1}{2\pi} \ln \frac{1}{R} - \frac{1}{2\pi} \ln \frac{1}{\bar{R}} = \frac{1}{2\pi} \ln \frac{\bar{R}}{R} \\ &= \frac{1}{2\pi} \ln \frac{\sqrt{r^2 + 1/\rho^2 - 2r/\rho \cos(\theta - \phi)}}{\sqrt{r^2 + \rho^2 - 2r\rho \cos(\theta - \phi)}} \\ &\Downarrow \quad r = 1 \\ u^*(1, \theta) &= -\frac{1}{2\pi} \ln \rho \end{aligned}$$

With these steps in mind, we construct Green's function

$$G(r,\theta,\rho,\phi) = \frac{1}{2\pi}\ln\frac{1}{R} - \frac{1}{2\pi}\ln\frac{1}{\bar{R}} + \frac{1}{2\pi}\ln\rho$$



constant potential on the circle $r=1$

To find the solution to our original problem, we merely superimpose the impulse functions; this brings us to the final solution

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \ln\left(\rho\frac{\bar{R}}{R}\right) f(\rho,\phi)\,\rho\,d\rho\,d\phi \qquad \text{(CG)}$$

This is Green's function solution of Poisson equation inside a circle

NOTES

1. It is also possible to solve

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < 1$$

$$u(1,\theta) = g(\theta), \quad 0 \le \theta < 2\pi$$

by means of the Green's function approach

- Choose any point (r_0,θ_0) inside a circle. Let $v(r,\theta)$ be any twice continuously differentiable function in a circle such that $v = g$ on the boundary, and $v = 0$ near (r_0,θ_0). Let $w = u - v$. Then w satisfies

$$\nabla^2 w = -\nabla^2 v, \quad 0 < r < 1$$

$$w(1,\theta) = 0, \quad 0 \le \theta < 2\pi$$

Therefore

$$w(r_0,\theta_0) = \int_0^{2\pi}\int_0^1 G(r_0,\theta_0,\rho,\phi)\,\nabla^2 v\,\rho\,d\rho d\phi$$

- Since $v = 0$ near the singularity (r_0, θ_0) of G, we can apply the divergence theorem to the identity

$$G\nabla^2 v = G\nabla^2 v - v\nabla^2 G = \nabla \cdot (G\nabla v - v\nabla G)$$

to find that

$$\begin{aligned} w(r_0,\theta_0) &= \int_0^{2\pi}\int_0^1 G(r_0,\theta_0,\rho,\phi)\,\nabla^2 v\,\rho\,d\rho d\phi \\ &\Downarrow \quad \text{Divergence Theorem} \\ &= \int_0^{2\pi}\left[G(r_0,\theta_0,1,\phi)\frac{\partial v}{\partial \rho}(1,\phi) - v(1,\phi)\frac{\partial G}{\partial \rho}(r_0,\theta_0,1,\phi)\right]\,d\phi \\ &\Downarrow \quad \text{On the boundary, } G=0,\ v=g \\ &= -\int_0^{2\pi}\frac{\partial G}{\partial \rho}(r_0,\theta_0,1,\phi)\,g(\phi)\,d\phi \end{aligned}$$

- But $v(r_0,\theta_0) = 0$, so that $w(r_0,\theta_0) = u(r_0,\theta_0)$. Since (r_0,θ_0) was any point in a circle, the solution is

$$u(r,\theta) = -\int_0^{2\pi}\frac{\partial G}{\partial \rho}(r,\theta,1,\phi)\,g(\phi)\,d\phi$$

which, if we compute $\dfrac{\partial G}{\partial \rho}$, gives

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}\right]g(\phi)\,d\phi \tag{PI}$$

which is the Poisson integral formula

2. The solution to the general Dirichlet problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = -f(r,\theta), \quad 0 < r < 1$$

$$u(1,\theta) = g(\theta), \quad 0 \le \theta < 2\pi$$

would be the sum of solutions (CG) and (PI)

- **Poisson Equation Inside a Sphere**
  - We consider the problem

$$\begin{aligned} &u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\phi\phi} + \frac{\cot\phi}{r^2}u_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta} = -F(r,\phi,\theta), \quad r < R \\ &u(R,\phi,\theta) = 0 \end{aligned} \tag{SK}$$

  - The solution procedure consists of two steps:

* Solve by taking the finite Fourier Transforms

$$\mathcal{F}_{c,nm}\left[f(r,\phi,\theta)\right] = \frac{2n+1}{2\pi}\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_0^{\pi} f(r,\phi,\theta)\,P_n^m(\cos\phi)\,\cos m\theta\,\sin\phi\,d\phi\,d\theta$$
$$\mathcal{F}_{s,nm}\left[f(r,\phi,\theta)\right] = \frac{2n+1}{2\pi}\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_0^{\pi} f(r,\phi,\theta)\,P_n^m(\cos\phi)\,\sin m\theta\,\sin\phi\,d\phi\,d\theta$$

* Derive $G(r,\phi,\theta;\rho,\varphi,\vartheta)$ in the equivalent integral formula

$$u(r,\phi,\theta) = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} G(r,\phi,\theta;\rho,\varphi,\vartheta)\,F(\rho,\varphi,\vartheta)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\vartheta$$

STEP 1 Using the finite Fourier transforms of u and F:

$$\begin{aligned} a_{nm}(r) &= \mathcal{F}_{c,nm}\left[u(r,\phi,\theta)\right] \\ b_{nm}(r) &= \mathcal{F}_{s,nm}\left[u(r,\phi,\theta)\right] \\ A_{nm}(r) &= \mathcal{F}_{c,nm}\left[F(r,\phi,\theta)\right] \\ B_{nm}(r) &= \mathcal{F}_{s,nm}\left[F(r,\phi,\theta)\right] \end{aligned}$$

we obtain the finite Fourier transforms of the differential equation and integrating by parts leads to the system

$$\begin{aligned} r^2a_{nm}'' + 2ra_{nm}' - n(n+1)a_{nm} &= -r^2A_{nm},\ a_{nm}(R) = 0,\ |a_{nm}| < \infty \\ r^2b_{nm}'' + 2rb_{nm}' - n(n+1)b_{nm} &= -r^2B_{nm},\ b_{nm}(R) = 0,\ |b_{nm}| < \infty \end{aligned}$$

These problems have the solutions

$$\begin{aligned} a_{nm}(r) &= \int_0^R G_n(r,\rho)\,A_{nm}(\rho)\,\rho^2\,d\rho \\ b_{nm}(r) &= \int_0^R G_n(r,\rho)\,B_{nm}(\rho)\,\rho^2\,d\rho \end{aligned} \qquad \text{(SG)}$$

where (See Section 3.10)

$$G_n(r,\rho) = \begin{cases} \dfrac{1}{(2n+1)R}\left(\dfrac{r}{R}\right)^n\left[\left(\dfrac{\rho}{R}\right)^{-(n+1)} - \left(\dfrac{\rho}{R}\right)^n\right] & \text{for } r \le \rho \\ \dfrac{1}{(2n+1)R}\left(\dfrac{\rho}{R}\right)^n\left[\left(\dfrac{r}{R}\right)^{-(n+1)} - \left(\dfrac{r}{R}\right)^n\right] & \text{for } r \ge \rho \end{cases}$$

Thus the problem (SK) has a solution

$$u(r,\phi,\theta) = \sum_{n=0}^{\infty}\left[\frac{1}{2}a_{n0}(r)P_n(\cos\phi) + \sum_{m=1}^{n}\left(a_{nm}(r)\cos m\theta + b_{nm}(r)\sin m\theta\right)P_n^m(\cos\phi)\right] \qquad \text{(SS)}$$

STEP 2 We substitute the definitions of $A_{nm}(r)$ and $B_{nm}(r)$ in (SG), and formally interchange integration and summation in (SS). This process gives the formal solution

$$u(r,\phi,\theta) = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} G(r,\phi,\theta;\rho,\varphi,\vartheta)\, F(\rho,\varphi,\vartheta)\, \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\vartheta$$

where for $r > \rho$

$$\begin{aligned} G(r,\phi,\theta;\rho,\varphi,\vartheta) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)R} \left(\frac{\rho}{R}\right)^n \left[\left(\frac{r}{R}\right)^{-(n+1)} - \left(\frac{r}{R}\right)^n\right] K_n(\phi,\theta;\varphi,\vartheta) \\ &= \frac{1}{4\pi R} \sum_{n=0}^{\infty} \left(\frac{\rho}{R}\right)^n \left[\left(\frac{r}{R}\right)^{-(n+1)} - \left(\frac{r}{R}\right)^n\right] P_n(\cos\varphi') \end{aligned}$$

For $r < \rho$, we must only interchange r and ρ in this formula

To evaluate the series in the above equation, we use the generating function:

$$G(r,\phi,\theta;\rho,\varphi,\vartheta) = \frac{1}{4\pi} \frac{1}{\sqrt{r^2+\rho^2-2r\rho\cos\varphi'}} - \frac{1}{4\pi} \frac{1}{\sqrt{R^2 + \frac{r^2\rho^2}{R^2} - 2r\rho\cos\varphi'}}$$

This, then, is the Green's function for $r > \rho$. It is already symmetric in r and ρ, so that the same formula holds for $r < \rho$, and even for $r = \rho$. We note that $G = 0$ for $r = R$

Green Function in 2D and 3D - Summary Table

Dimension - Domain	Object	Formula	Purpose
2D - Unit disk $D = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| < 1\}$	**Green Function**	$G(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\left(\ln\frac{1}{\|\mathbf{x}-\mathbf{y}\|} - \ln\frac{1}{\|\mathbf{y}\|\,\|\mathbf{x}-\mathbf{y}^*\|}\right)$, $\mathbf{y}^* = \frac{\mathbf{y}}{\|\mathbf{y}\|^2}$	Solve **Poisson equation** $\Delta u = f$: $u(\mathbf{x}) = \int_D G(\mathbf{x},\mathbf{y}) f(\mathbf{y}) d\mathbf{y}$
3D - Unit ball $B = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| < 1\}$	**Green Function**	$G(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|} - \frac{1}{\|\mathbf{y}\|\,\|\mathbf{x}-\mathbf{y}^*\|}\right)$, $\mathbf{y}^* = \frac{\mathbf{y}}{\|\mathbf{y}\|^2}$	Solve **Poisson equation** $\Delta u = f$: $u(\mathbf{x}) = \int_B G(\mathbf{x},\mathbf{y}) f(\mathbf{y}) d\mathbf{y}$

14.7 Hydrogen Atom

- As the electron circles around the nucleus, the system (the proton and the electron) rotates around the center of gravity. For the rotating system, we can write the reduced mass, μ, as

$$\frac{1}{\mu} = \frac{1}{m_e} + \frac{1}{m_p}$$

or

$$\mu = \frac{m_e m_p}{m_e + m_p}$$

Since the mass of the electron is so much less than that of the proton, $m_e + m_p \approx m_p$ and $\mu \approx m_e$. Assuming that the nucleus is stationary and that the electron does all of the moving (the infinite-nuclear-mass approximation) leads to the same result

- The Hamiltonian can be written as the sum of the kinetic and potential energies

$$H = T + V$$

in which $T = p^2/2\mu$ and the potential energy V for the interaction of the electron with the proton is $-e^2/r$ (Gaussian units; $e^2 \to e^2/4\pi\varepsilon_0$ in SI units). Therefore, the Hamiltonian operator H is

$$H = -\frac{\hbar^2}{2\mu}\nabla^2 - \frac{e^2}{r}$$

- Therefore, the time-independent wave function ψ gives

$$\begin{gathered} H\psi = E\psi \\ \Downarrow \\ \nabla^2\psi + \frac{2\mu}{\hbar^2}\left(\frac{e^2}{r} + E\right)\psi = 0 \\ \Downarrow \\ \frac{1}{r^2}\left(r^2\psi_r\right)_r + \frac{1}{r^2\sin\phi}\left(\sin\phi\,\psi_\phi\right)_\phi + \frac{1}{r^2\sin^2\phi}\psi_{\theta\theta} + \frac{2\mu}{\hbar^2}\left(\frac{e^2}{r} + E\right)\psi = 0 \end{gathered}$$

- We assume that $\psi(r,\phi,\theta) = R(r)\,\Phi(\phi)\,\Theta(\theta)$ and make the substitution. For simplicity, we will write the partial solutions as R, Φ, and Θ without showing the functionality. We now divide both sides by $R\,\Phi\,\Theta$ and multiply by $r^2\sin^2\phi$:

$$\frac{\sin^2\phi}{R}\left(r^2R_r\right)_r + \frac{\sin\phi}{\Phi}\left(\sin\phi\,\Phi_\phi\right)_\phi + \frac{1}{\Theta}\Theta_{\theta\theta} + \frac{2\mu r^2\sin^2\phi}{\hbar^2}\left(\frac{e^2}{r} + E\right) = 0$$

- Inspection shows that, of the four terms on the left side of the equation, there is no functional dependence on θ except in the third term. Therefore, with respect to the other variables, the third term can be treated as a constant:

$$\begin{gathered} -\frac{\sin^2\phi}{R}\left(r^2R_r\right)_r - \frac{\sin\phi}{\Phi}\left(\sin\phi\,\Phi_\phi\right)_\phi - \frac{2\mu r^2\sin^2\phi}{\hbar^2}\left(\frac{e^2}{r} + E\right) = \frac{1}{\Theta}\Theta_{\theta\theta} = -\lambda \le 0 \\ \Downarrow \\ \Theta_{\theta\theta} + \lambda\Theta = 0, \quad \Theta(0) = \Theta(2\pi) \\ \Downarrow \quad e^{\pm i2\pi\sqrt{\lambda}} = 1 \\ \lambda_m = m^2,\ m = 0, \pm1, \pm2, \cdots \\ \Theta_m(\theta) = \frac{1}{\sqrt{2\pi}}e^{im\theta} \end{gathered}$$

Actually, there are two solutions: $e^{im\theta}$ and $e^{-im\theta}$, but we cover the latter by allowing m to run negative

- We can now write the wave equation as

$$\frac{\sin^2\phi}{R}\left(r^2R_r\right)_r+\frac{\sin\phi}{\Phi}\left(\sin\phi\,\Phi_\phi\right)_\phi-m^2+\frac{2\mu r^2\sin^2\phi}{\hbar^2}\left(\frac{e^2}{r}+E\right)=0$$

  If we divide the equation by $\sin^2\phi$ and rearrange, we obtain

$$\frac{1}{R}\left(r^2R_r\right)_r+\frac{2\mu r^2}{\hbar^2}\left(\frac{e^2}{r}+E\right)+\frac{1}{\Phi\sin\phi}\left(\sin\phi\,\Phi_\phi\right)_\phi-\frac{m^2}{\sin^2\phi}=0$$

$$\Downarrow$$

$$\frac{1}{\Phi\sin\phi}\left(\sin\phi\,\Phi_\phi\right)_\phi-\frac{m^2}{\sin^2\phi}=-\beta$$

$$\frac{1}{R}\left(r^2R_r\right)_r+\frac{2\mu r^2}{\hbar^2}\left(\frac{e^2}{r}+E\right)=\beta$$

$$\Downarrow\quad\frac{1}{\sin\phi}\left(\sin\phi\,\Phi_\phi\right)_\phi+\left(\beta-\frac{m^2}{\sin^2\phi}\right)\Phi=0$$

$$\frac{1}{r^2}\left(r^2R_r\right)_r+\frac{2\mu}{\hbar^2}\left(\frac{e^2}{r}+E\right)R-\frac{\beta}{r^2}R=0$$

- We will now turn our attention to the equation involving ϕ. The standard method for solving this equation is to make the transformation $x=\cos\phi$

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\frac{\sin^2\phi}{\sin\phi}\frac{d\Phi}{d\phi}\right)+\left(\beta-\frac{m^2}{\sin^2\phi}\right)\Phi=0$$

$$\Downarrow\quad x=\cos\phi,\ dx=-\sin\phi\,d\phi$$

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{d\Phi}{dx}\right]+\left(\beta-\frac{m^2}{1-x^2}\right)\Phi=0$$

$$\Downarrow$$

$$\left(1-x^2\right)\frac{d^2\Phi}{dx^2}-2x\frac{d\Phi}{dx}+\left(\beta-\frac{m^2}{1-x^2}\right)\Phi=0$$

- This is the associated Legendre equation. If you start with the Legendre equation and differentiate it $|m|$ times, you end up with a new differential equation

$$\left(1-x^2\right)P_l''(x)-2xP_l'(x)+l(l+1)P_l(x)=0,\quad l=0,1,2,\cdots$$

$$\Downarrow\quad\text{differentiate it }|m|\text{ times, }y=\left(\tfrac{d}{dx}\right)^{|m|}P_l(x),\ |m|\le l$$

$$\left(1-x^2\right)y''-2|m|xy'-|m|(|m|-1)y-2xy'-2|m|y+l(l+1)y=0$$

$$\Downarrow$$

$$\left(1-x^2\right)y''-2x(|m|+1)y'+\left[l(l+1)-|m|(|m|+1)\right]y=0$$

- And the substitution

$$y=\left(1-x^2\right)^{-|m|/2}\Phi$$

$$y'=\left(1-x^2\right)^{-|m|/2}\left[\frac{d\Phi}{dx}+\frac{|m|x}{1-x^2}\Phi\right]$$

$$y''=\left(1-x^2\right)^{-|m|/2}\left[\frac{d^2\Phi}{dx^2}+\frac{2|m|x}{1-x^2}\frac{d\Phi}{dx}+\frac{|m|+|m|x^2+|m|^2x^2}{(1-x^2)^2}\Phi\right]$$

puts the equation into standard form with respect to Φ. Then the standard solutions of the associated Legendre equation were found to be

$$\Phi(x) = P_l^{|m|}(x) = (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|} P_l(x) = (1-x^2)^{|m|/2}\,\frac{1}{2^l l!}\left(\frac{d}{dx}\right)^{|m|+l}(x^2-1)^l$$

and $P_l^{|m|}(x)$ are orthogonal with the weight function 1

$$\int_{-1}^{1} P_l^{|m|}(x)\,P_{l'}^{|m|}(x)\,dx = \frac{2}{2l+1}\frac{(l+|m|)!}{(l-|m|)!}\,\delta_{ll'}$$

- Notice that l must be a nonnegative integer. For any given l, then, there are $(2l+1)$ possible values of m:

$$l = 0, 1, 2, \cdots; \quad m = -l, -l+1, \cdots, -1, 0, 1, \cdots, l-1, l$$

For historical reasons, l is called the azimuthal quantum number, and m the magnetic quantum number

- The normalized angular wave functions are called spherical harmonics:

$$Y_l^m(\phi,\theta) = \epsilon\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}P_l^{|m|}(\cos\phi)e^{im\theta}$$

where $\epsilon = (-1)^m$ for $m \geq 0$ and $\epsilon = 1$ for $m < 0$

- Notice that the angular part of the wave function, $Y(\phi,\theta)$, is the same for all spherically symmetric potentials; the actual shape of the potential $V(r)$ affects only the radial part of the wave function, $R(r)$, which is determined by

$$\frac{1}{r^2}\left(r^2 R_r\right)_r + \frac{2\mu}{\hbar^2}\left(\frac{e^2}{r} + E\right)R - \frac{l(l+1)}{r^2}R = 0$$

This equation simplifies if we change variables:

$$\begin{gathered}
\text{Let } u(r) = rR(r) \\
\Downarrow \text{ so that} \\
R = \frac{u}{r}, \quad \frac{dR}{dr} = \frac{1}{r^2}\left(r\frac{du}{dr} - u\right), \quad \frac{d}{dr}\left[r^2\frac{dR}{dr}\right] = r\frac{d^2u}{dr^2} \\
\Downarrow \quad \kappa = \frac{\sqrt{-2\mu E}}{\hbar} \quad \text{and} \quad \rho_0 = \frac{2\mu e^2}{\hbar^2\kappa} \\
\frac{d^2u}{dr^2} = \left[-\frac{\rho_0\kappa}{r} + \kappa^2 + \frac{l(l+1)}{r^2}\right]u \quad \overset{\rho=\kappa r}{\rightarrow} \quad \frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u
\end{gathered}$$

- The next step is to peel off the asymptotic behavior, introducing the new function $v(\rho)$

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

in the hope that $v(\rho)$ will turn out to be simpler than $u(\rho)$. In terms of $v(\rho)$, then, the radial equation reads

$$\rho \frac{d^2 v}{d\rho^2} + 2(l + 1 - \rho) \frac{dv}{d\rho} + [\rho_0 - 2(l + 1)]\, v = 0$$

- Finally, we assume the solution can be expressed as a power series in ρ:

$$v(\rho) = \sum_{k=0}^{\infty} c_k \rho^k$$

Inserting the power series into the equation and equating the coefficients of like powers yields

$$c_{k+1} = \frac{2(k + l + 1) - \rho_0}{(k + 1)(k + 2l + 2)} c_k$$

- Now let's see what the coefficients look like for large k. In this regime the recursion formula says

$$c_{k+1} \approx \frac{2}{k + 1} c_k$$

Suppose for a moment that this were exact. Then

$$c_k = \frac{2^k}{k!} c_0$$

- so

$$v(\rho) = c_0 \sum_{k=0}^{\infty} \frac{2^k}{k!} \rho^k = c_0 e^{2\rho} \quad \Rightarrow \quad u(\rho) = c_0 \rho^{l+1} e^{\rho}$$

which blows up at large ρ. The positive exponential is precisely the asymptotic behavior we didn't want

- There is only one way out of this dilemma: *The series must terminate.* There must occur some maximal integer, k_{max}, such that

$$c_{k_{max}+1} = 0$$

and beyond which all coefficients vanish automatically. Evidently

$$2(k_{max} + l + 1) - \rho_0 = 0$$

- Defining the so-called principal quantum number

$$n = k_{max} + l + 1$$

we have

$$\rho_0 = 2n$$

- But ρ_0 determines E

$$E = -\frac{\hbar^2\kappa^2}{2\mu} = -\frac{2\mu e^4}{\hbar^2\rho_0^2}$$

so the allowed energies are

$$E_n = -\left[\frac{\mu e^4}{2\hbar^2}\right]\frac{1}{n^2}, \quad n = 1, 2, 3, \cdots$$

This is the famous **Bohr formula**. The ground state (that is, the state of lowest energy) is the case $n = 1$; putting in the accepted values for the physical constants, we get

$$E_1 = -\left[\frac{\mu e^4}{2\hbar^2}\right] = -13.6\, eV$$

and we find that

$$\kappa = \frac{\mu e^2}{\hbar^2}\frac{1}{n} = \frac{1}{a_0 n}, \quad \rho = \frac{r}{a_0 n}$$

where

$$a_0 = \frac{\hbar^2}{\mu e^2} = 0.529 \times 10^{-10} m$$

is the so-called **Bohr radius**

- Anyway the radial equation gives

$$R_{nl}(r) = \frac{1}{r}\rho^{l+1}e^{-\rho}v(\rho)$$

in which $v(\rho)$ is a polynomial of degree $k_{max} = n - l - 1$ in ρ, whose coefficients are determined by the recursion formula

$$c_{k+1} = \frac{2(k + l + 1 - n)}{(k + 1)(k + 2l + 2)}c_k$$

For arbitrary n, the possible values of l are

$$l = 0, 1, 2, \cdots, n - 1$$

and for each l, there are $(2l+1)$ possible values of m, so the total degeneracy of the energy level E_n is

$$d(n) = \sum_{l=0}^{n-1} (2l+1) = n^2$$

- The polynomial $v(\rho)$ is a function well known to applied mathematicians; it can be written as

$$v(\rho) = L_{n-l-1}^{2l+1}(2\rho)$$

where

$$L_{q-p}^{p}(x) = (-1)^p \left(\frac{d}{dx}\right)^p L_q(x) = \sum_{k=0}^{q-p} \frac{(-1)^k}{k!} \binom{q}{q-p-k} x^k$$

is a generalized Laguerre polynomial, and

$$L_q(x) = \frac{e^x}{q!} \left(\frac{d}{dx}\right)^q (e^{-x} x^q) = \sum_{k=0}^{q} \frac{(-1)^k}{k!} \binom{q}{k} x^k$$

is the q-th Laguerre polynomial

- The radial wave functions are given by

$$R_{nl}(r) = \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-l-1)!}{2n(n+l)!}} e^{-r/na_0} \left(\frac{2r}{na_0}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na_0}\right)$$

The combinatorial factor in front assures that $R_{nl}(r)$ are normalized with respect to an integration over r:

$$\int_0^\infty R_{nl}^2(r)\, r^2 dr = 1$$

Note that the volume element here is $r^2 dr$, which is the r part of the spherical coordinate volume element $r^2 \sin\phi \, dr \, d\phi \, d\theta$

- The complete hydrogen atomic wave functions are

$$\psi_{nlm}(r, \phi, \theta) = R_{nl}(r)\, Y_l^m(\phi, \theta)$$

- The normalization condition is

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} |\psi_{nlm}(r, \phi, \theta)|^2 \, r^2 \sin\phi \, dr \, d\phi \, d\theta = 1$$

- The functions ψ_{nlm} must also be orthogonal. This orthonormality relationship is given by

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \psi^*_{n'l'm'}(r,\phi,\theta)\,\psi_{nlm}(r,\phi,\theta)\,r^2\sin\phi\,dr\,d\phi\,d\theta = \delta_{nn'}\delta_{ll'}\delta_{mm'}$$

See the Hydrogen Atomic Orbitals

Try it in Python: Interactive 3D plots of hydrogen orbitals ψ_{nlm}: Hydrogen Atomic Orbitals

Summary

Topic	Key result
Laplacian (cylindrical)	$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$
Neumann compatibility	$\nabla^2 u = 0$ in D, $\partial u/\partial n = g$ on the boundary needs $\oint g\,ds = 0$
Disk of radius R	$u = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (a_n\cos n\theta + b_n \sin n\theta)$
Poisson integral formula	$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\theta-\alpha)}\,g(\alpha)\,d\alpha$
Sphere (axisymmetric)	$u(r,\phi) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\phi)$ inside the unit sphere
Green's function	$u(r,\theta) = \int_0^{2\pi}\int_0^1 G(r,\theta,\rho,\phi)\,f(\rho,\phi)\,\rho\,d\rho\,d\phi$
Hydrogen atom	$E_n = -\left[\frac{\mu e^4}{2\hbar^2}\right]\frac{1}{n^2}$, $n = 1, 2, \cdots$; wave functions $R_{nl}(r)\,Y_l^m$

Worked Exercises

1. Solve the exterior Dirichlet problem in a circle:

$$\nabla^2 u = 0, \quad 1 < r < \infty$$
$$u(1,\theta) = \sin\theta + \cos 3\theta$$

Solution

Step 1: General Solution in Polar Coordinates

The general solution to Laplace's equation in polar coordinates for $r > 1$ (the exterior region) is:

$$u(r,\theta) = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n}) \cos n\theta + \sum_{n=1}^{\infty} (c_n r^n + d_n r^{-n}) \sin n\theta$$

Step 2: Apply the decay condition $u(r,\theta) \to 0$ as $r \to \infty$

This requires:

- $a_0 = 0$
- $b_0 = 0$
- $a_n = 0$, $c_n = 0$ for all $n \geq 1$

Hence, the solution becomes:

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n r^{-n} \cos n\theta + \sum_{n=1}^{\infty} d_n r^{-n} \sin n\theta$$

Step 3: Use Boundary Condition $u(1,\theta) = \sin\theta + \cos 3\theta$

At $r = 1$, we get:

$$u(1,\theta) = \sum_{n=1}^{\infty} b_n \cos n\theta + \sum_{n=1}^{\infty} d_n \sin n\theta = \sin\theta + \cos 3\theta$$

So:

- $b_3 = 1$, $d_1 = 1$
- All other b_n, d_n are 0

Final Solution

$$u(r,\theta) = \frac{\cos 3\theta}{r^3} + \frac{\sin\theta}{r}$$

This satisfies Laplace's equation in $r > 1$, the boundary condition at $r = 1$, and decays at infinity

2. Solve the interior Dirichlet problem in a sphere

$$\nabla^2 u = 0,\ r < 1$$
$$u(1,\phi) = 1 + \cos\phi$$

Solution

We are solving the interior Dirichlet problem for Laplace's equation inside the unit sphere. The boundary condition is given only in terms of ϕ, which suggests axial symmetry (i.e., the function does not depend on the azimuthal angle θ). Hence, we can solve the axisymmetric Laplace equation in spherical coordinates

Step 1: Axisymmetric Laplace Equation

In spherical coordinates with axial symmetry (no dependence), Laplace's equation becomes:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial u}{\partial \phi}\right) = 0$$

Step 2: Separation of Variables

Assume $u(r, \phi) = R(r)P(\phi)$. Substituting and separating variables gives:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{P\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dP}{d\phi}\right) = 0$$

Set both parts equal to $\ell(\ell+1)$. Then:

- $R(r) = A_\ell r^\ell$ (we discard $r^{-(\ell+1)}$ since u must be finite at $r = 0$),
- $P(\phi)$ satisfies Legendre's equation:

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dP}{d\phi}\right) + \ell(\ell+1)P = 0 \Rightarrow P(\phi) = P_\ell(\cos\phi)$$

Step 3: General Solution

The general solution is:

$$u(r, \phi) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\phi)$$

Step 4: Apply Boundary Condition

At $r = 1$, the boundary condition is:

$$u(1, \phi) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(\cos\phi) = 1 + \cos\phi$$

We expand $1 + \cos\phi$ in terms of Legendre polynomials

- $P_0(\cos\phi) = 1$
- $P_1(\cos\phi) = \cos\phi$

So:

$$1 + \cos\phi = P_0(\cos\phi) + P_1(\cos\phi) \Rightarrow A_0 = 1,\ A_1 = 1,\ A_\ell = 0 \text{ for } \ell \geq 2$$

Final Solution

$$u(r, \phi) = P_0(\cos\phi) + rP_1(\cos\phi) = 1 + r\cos\phi$$

This satisfies Laplace's equation inside the unit ball and matches the boundary condition

3. Solve the exterior Dirichlet problem in a sphere

$$\nabla^2 u = 0,\ 1 < r$$
$$u(1, \phi) = 1 + \cos 2\phi$$

Solution

We are solving the exterior Dirichlet problem for Laplace's equation in a spherically symmetric setting. The boundary condition depends only on the polar angle ϕ, so we assume axial symmetry (i.e., no θ-dependence)

Step 1: General Axisymmetric Solution to Laplace's Equation (Exterior)

The axisymmetric solution to Laplace's equation in spherical coordinates (no θ dependence) is:

$$u(r,\phi) = \sum_{\ell=0}^{\infty} \left(A_\ell r^{-\ell-1} + B_\ell r^\ell\right) P_\ell(\cos\phi)$$

To ensure the solution decays at infinity:

- Set $B_\ell = 0$

So the solution becomes:

$$u(r,\phi) = \sum_{\ell=0}^{\infty} A_\ell r^{-\ell-1} P_\ell(\cos\phi)$$

Step 2: Apply Boundary Condition

At $r = 1$, the boundary condition is:

$$u(1,\phi) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(\cos\phi) = 1 + \cos 2\phi$$

Recall:

- $P_0(\cos\phi) = 1$
- $P_2(\cos\phi) = \frac{1}{2}(3\cos^2\phi - 1) = \cos 2\phi$ (identity on the sphere)

So this corresponds to:

$$A_0 = 1, \quad A_2 = 1, \quad A_\ell = 0 \text{ for } \ell \neq 0, 2$$

Final Solution

$$u(r,\phi) = \frac{1}{r}P_0(\cos\phi) + \frac{1}{r^3}P_2(\cos\phi) = \frac{1}{r} + \frac{1}{r^3}\cos 2\phi$$

This is harmonic for $r > 1$, satisfies the boundary condition at $r = 1$, and decays as $r \to \infty$

4. Solve the Dirichlet problem in a circle:

$$\begin{aligned} \nabla^2 u &= 0, \quad 1 < r < 2 \\ u(1,\theta) &= \frac{1}{2} + \cos\theta \\ u(2,\theta) &= 1 + \sin\theta \end{aligned}$$

Solution

We solve this using separation of variables in polar coordinates (r,θ), where the Laplacian is:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

General Solution to Laplace's Equation in Annulus

We use the standard separation-of-variables solution:

$$u(r,\theta) = a_0 + b_0 \log r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos(n\theta) + (C_n r^n + D_n r^{-n}) \sin(n\theta)$$

Since $r \in (1,2)$, both r^n and r^{-n} are bounded, so we retain all terms

Step 1: Fourier Series of Boundary Data

We write the boundary functions as Fourier series

$$u(r,\theta) = a_0 + b_0 \log r + (A_1 r + B_1 r^{-1}) \cos\theta + (C_1 r + D_1 r^{-1}) \sin\theta$$

We'll determine the constants a_0, b_0, A_1, B_1, C_1, D_1 by applying the boundary conditions at $r = 1$ and $r = 2$

At $r = 1$:

$$u(1,\theta) = a_0 + (A_1 + B_1)\cos\theta + (C_1 + D_1)\sin\theta = \frac{1}{2} + \cos\theta$$

this gives:

- $a_0 = \frac{1}{2}$
- $A_1 + B_1 = 1$
- $C_1 + D_1 = 0$

At $r = 2$:

$$u(2,\theta) = a_0 + b_0 \log 2 + (2A_1 + \frac{1}{2}B_1)\cos\theta + (2C_1 + \frac{1}{2}D_1)\sin\theta = 1 + \sin\theta$$

Match terms:

- $a_0 + b_0 \log 2 = 1$
- $2A_1 + \frac{1}{2}B_1 = 0$
- $2C_1 + \frac{1}{2}D_1 = 1$

Now solve the system

Step 2: Solve the System

We already have:

- $a_0 = \frac{1}{2}$
- $a_0 + b_0 \log 2 = 1 \Rightarrow \frac{1}{2} + b_0 \log 2 = 1 \Rightarrow b_0 = \frac{1}{2\log 2}$

Solve for A_1, B_1:

- $A_1 + B_1 = 1$
- $2A_1 + \frac{1}{2}B_1 = 0$

Multiply first by 2 and subtract by second:

$$(2A_1 + 2B_1) - (2A_1 + \tfrac{1}{2}B_1) = 2 - 0 \Rightarrow \tfrac{3}{2}B_1 = 2 \Rightarrow B_1 = \frac{4}{3},\ A_1 = 1 - \frac{4}{3} = -\frac{1}{3}$$

Solve for C_1, D_1:

- $C_1 + D_1 = 0$
- $2C_1 + \frac{1}{2}D_1 = 1$

Substitute $D_1 = -C_1$ into second:

$$2C_1 - \frac{1}{2}C_1 = 1 \Rightarrow \frac{3}{2}C_1 = 1 \Rightarrow C_1 = \frac{2}{3},\ D_1 = -\frac{2}{3}$$

Final Solution

$$\boxed{\begin{aligned} u(r,\theta) = \frac{1}{2} + \frac{1}{2\log 2}\log r + \left(-\frac{1}{3}r + \frac{4}{3}r^{-1}\right)\cos\theta \\ + \left(\frac{2}{3}r - \frac{2}{3}r^{-1}\right)\sin\theta \end{aligned}}$$

This is the harmonic function in the annulus $1 < r < 2$ satisfying the given Dirichlet boundary conditions

5. Consider the Laplace equation on the upper half-plane:

$$\nabla^2 u(x,y) = 0, \quad y > 0,$$
$$u(x,0) = \phi(x), \quad x \in \mathbb{R}$$

where $\phi \in L^\infty(\mathbb{R})$

(a) Starting from the Fourier transform method, derive the Poisson kernel $\mathcal{P}$

(b) Show that the solution can be expressed as the Poisson integral

$$u(x,y) = \int_{\mathbb{R}} \mathcal{P}(x-\xi, y)\,\phi(\xi)\,d\xi.$$

(c) Let

$$\phi(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| \geq 1 \end{cases}$$

Compute $u(0,y)$ explicitly in terms of the arctangent function

Hint:

$$\int \frac{1}{\xi^2 + y^2}\,d\xi = \frac{1}{y}\arctan\left(\frac{\xi}{y}\right) + C$$

Solution

(a) Derivation of the Poisson kernel

Take the Fourier transform in x of the harmonic function $u(\cdot,\, y)$ for each fixed $y > 0$:

$$\widehat{u}(\omega, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega x} u(x, y)\, dx$$

Applying the Laplace equation $u_{xx} + u_{yy} = 0$ and transforming in x gives the ODE in y:

$$\frac{\partial^2}{\partial y^2} \widehat{u}(k, y) - \omega^2 \widehat{u}(\omega, y) = 0$$

The general solution (for fixed ω) is

$$\widehat{u}(\omega, y) = A(\omega) e^{|\omega| y} + B(\omega) e^{-|\omega| y}$$

For $u(\cdot,\, y)$ to remain bounded as $y \to +\infty$, we must set $A(\omega) = 0$. The boundary condition $u(x, 0) = \phi(x)$ implies

$$\widehat{u}(\omega, 0) = \widehat{\phi}(\omega) = B(\omega)$$

Hence

$$\widehat{u}(\omega, y) = \widehat{\phi}(\omega) e^{-|\omega| y}$$

Invert the Fourier transform:

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\phi}(\omega) e^{-|\omega| y}\, e^{i\omega x}\, d\omega$$

Write $\widehat{\phi}(\omega) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\mathbb{R}} e^{-i\omega\xi} \phi(\xi)\, d\xi$ and interchange integrals to obtain

$$u(x, y) = \int_{\mathbb{R}} \left(\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega(x-\xi)} e^{-|\omega| y}\, d\omega \right) \phi(\xi)\, d\xi$$

The inner integral is the inverse Fourier transform of $e^{-|\omega| y}$ to $z = x - \xi$ coordinate. A standard computation yields

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega z} e^{-|\omega| y}\, d\omega = \frac{1}{\pi} \frac{y}{z^2 + y^2} = \mathcal{P}(z, y)$$

so that the Poisson kernel for the upper half-plane is

$$\boxed{\mathcal{P}(z, y) = \frac{1}{\pi} \frac{y}{z^2 + y^2}, \quad y > 0}$$

One way to compute the inverse transform is to split the integral into $\omega > 0$ and $\omega < 0$ and evaluate each as a complex exponential integral

(b) Poisson integral representation

Plugging the kernel found above into the expression for $u(x, y)$ gives

$$\boxed{u(x,y) = \int_{\mathbb{R}} P(x-\xi, y)\, \phi(\xi)\, d\xi}$$

which is harmonic in $y > 0$, continuous in x for each fixed $y > 0$, and reproduces boundary values in the sense of non-tangential limits under mild regularity assumptions on ϕ

(c) Explicit value at $x = 0$ for indicator data

With $\phi = \mathbf{1}_{(-1,1)}$ we have

$$u(0,y) = \int_{\mathbb{R}} P(-\xi, y)\, \mathbf{1}_{(-1,1)}\, d\xi = \int_{-1}^{1} \frac{1}{\pi} \frac{y}{\xi^2 + y^2}\, d\xi$$

Directly using the antiderivative of $1/(\xi^2 + y^2)$:

$$\int \frac{1}{\xi^2 + y^2}\, d\xi = \frac{1}{y} \arctan\left(\frac{\xi}{y}\right) + C$$

Hence

$$u(0,y) = \frac{y}{\pi} \left[\frac{1}{y} \arctan\left(\frac{\xi}{y}\right)\right]_{\xi=-1}^{\xi=1} = \frac{1}{\pi} \left[\arctan\left(\frac{1}{y}\right) - \arctan\left(-\frac{1}{y}\right)\right]$$

Using the oddness of arctan, this simplifies to

$$u(0,y) = \frac{2}{\pi} \arctan\left(\frac{1}{y}\right), \quad y > 0$$

Boundary limit. As $y \to 0$ we have $\arctan(1/y) \to \pi/2$, so

$$\lim_{y \to 0} u(0,y) = \frac{2}{\pi} \cdot \frac{\pi}{2} = 1$$

which equals the boundary value $\phi(0) = 1$. This is consistent with the Poisson integral reproducing the continuous boundary values. If ϕ had discontinuities, the Poisson integral would converge to the average of the lateral limits at the discontinuity

Answers

Answer 1 (Section 14.1)

$u_{tt} = c^2 \left(u_{rr} + \frac{1}{r}\, u_r\right)$

Answer 2 (Section 14.1)

$u_{rr} + \frac{2}{r}\, u_r = 0, \quad u(r) = a + \frac{b}{r}$

Answer 3 (Section 14.2)

No: the total boundary flux $\int_0^{2\pi} \sin^2\theta\, d\theta = \pi \neq 0$ violates the compatibility condition

Answer 4 (Section 14.2)

$$u(r,\theta) = \frac{h}{1+h}\, r \sin\theta$$