

그 밖의 연산 성질

변환의 미분, 합성곱, 적분방정식, 주기함수

공학수학 I · 강의노트 §4.4 Example 풀이

1 $\mathcal{L}\{t \sin \omega t\}$

$$\mathcal{L}\{t \sin \omega t\} = -\frac{d}{ds} \frac{\omega}{s^2 + \omega^2} = \frac{2\omega s}{(s^2 + \omega^2)^2}.$$

2 $x'' + 16x = \cos 4t, x(0) = 1, x'(0) = 1$

$$(s^2 + 16)X - s - 1 = \frac{s}{s^2 + 16} \Rightarrow X = \frac{s}{s^2 + 16} + \frac{1}{s^2 + 16} + \frac{s}{(s^2 + 16)^2}.$$

앞 절에서 $\omega = 4$ 이면 $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+16)^2}\right\} = \frac{1}{8}t \sin 4t$.

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$$x = \cos 4t + \frac{1}{4} \sin 4t + \frac{1}{8} t \sin 4t$$

3 $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+\omega^2)^2}\right\}$

$\frac{1}{(s^2+\omega^2)^2} = \frac{1}{\omega^2} \cdot \frac{\omega}{s^2+\omega^2} \cdot \frac{\omega}{s^2+\omega^2}$ 이므로 합성곱 정리에서

$$\mathcal{L}^{-1} = \frac{1}{\omega^2} \int_0^t \sin \omega \tau \sin \omega(t - \tau) d\tau = \frac{1}{2\omega^2} \int_0^t [\cos \omega(2\tau - t) - \cos \omega t] d\tau.$$

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$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + \omega^2)^2}\right\} = \frac{\sin \omega t - \omega t \cos \omega t}{2\omega^3}$$

4 $\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\}$

$$\int_0^t f(\tau) d\tau = f * 1 \text{ 이므로 } \mathcal{L} = F(s) \cdot \frac{1}{s} = \frac{F(s)}{s}.$$

5 적분방정식: $f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau) e^{t-\tau} d\tau$

적분은 $f * e^t$ 이므로

$$F = \frac{6}{s^3} - \frac{1}{s+1} - \frac{F}{s-1} \Rightarrow F \frac{s}{s-1} = \frac{6}{s^3} - \frac{1}{s+1} \Rightarrow F = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}.$$

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$$f(t) = 3t^2 - t^3 + 1 - 2e^{-t}$$

6 여러 함수의 변환

$$\mathcal{L}\{te^{-10t}\} = \frac{1}{(s+10)^2}, \quad \mathcal{L}\{t \cos 2t\} = -\frac{d}{ds} \frac{s}{s^2+4} = \frac{s^2-4}{(s^2+4)^2}, \quad \mathcal{L}\{te^{2t} \sin 6t\} = \frac{12(s-2)}{[(s-2)^2+36]^2} \text{ (먼저 } \mathcal{L}\{t \sin 6t\} = \frac{12s}{(s^2+36)^2} \text{, 그다음 } s \rightarrow s-2).$$

7 초기값 문제

(1) $y' + y = t \sin t, y(0) = 0. (s+1)Y = \frac{2s}{(s^2+1)^2}, \text{ 즉 } Y = \frac{2s}{(s+1)(s^2+1)^2}. \text{ 부분분수}$

$$Y = -\frac{1/2}{s+1} + \frac{1}{2} \frac{s-1}{s^2+1} + \frac{s+1}{(s^2+1)^2},$$

$$\mathcal{L}^{-1} \left\{ \frac{s}{(s^2+1)^2} \right\} = \frac{t}{2} \sin t, \quad \mathcal{L}^{-1} \left\{ \frac{1}{(s^2+1)^2} \right\} = \frac{\sin t - t \cos t}{2}.$$

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$$y = \frac{1}{2} (t \sin t - t \cos t + \cos t - e^{-t})$$

(부분분수 계수는 $\frac{2s}{(s+1)(s^2+1)^2} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{(s^2+1)^2}$ 에서 $A = -\frac{1}{2}, B = \frac{1}{2}, C = -\frac{1}{2}, D = 1, E = 1$ 로 정해진다.)

(2) $y'' + 9y = \cos 3t, y(0) = 2, y'(0) = 5. Y = \frac{2s+5}{s^2+9} + \frac{s}{(s^2+9)^2}.$

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$$y = 2 \cos 3t + \frac{5}{3} \sin 3t + \frac{1}{6} t \sin 3t$$

8 합성곱

(1) $4t * 3t^2 = \int_0^t 4\tau \cdot 3(t-\tau)^2 d\tau = 12 \cdot \frac{t^4}{12} = t^4, \quad \mathcal{L} = \frac{24}{s^5} \text{ (확인: } \frac{4}{s^2} \cdot \frac{6}{s^3}).$

(2) $e^{-t} * e^t = \int_0^t e^{-\tau} e^{t-\tau} d\tau = e^t \cdot \frac{1-e^{-2t}}{2} = \sinh t, \quad \mathcal{L} = \frac{1}{s^2-1} \text{ (확인: } \frac{1}{s+1} \cdot \frac{1}{s-1}).$

(3) $\mathcal{L}\{e^{-t} * e^t \cos t\} = \frac{1}{s+1} \cdot \frac{s-1}{(s-1)^2+1}.$

9 $\mathcal{L}^{-1} \left\{ \frac{1}{s^3(s-1)} \right\}$

$\frac{1}{s^3(s-1)} = \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^3}$ 이므로 $\mathcal{L}^{-1} = e^t - 1 - t - \frac{t^2}{2}. \text{ (합성곱으로 보면 } \frac{t^2}{2} * e^t = \int_0^t \frac{\tau^2}{2} e^{t-\tau} d\tau \text{ 와 같다.)}$

10 적분방정식과 적분미분방정식

(1) $f + 2f * \cos t = 4e^{-t} + \sin t: F \left(1 + \frac{2s}{s^2+1} \right) = \frac{4}{s+1} + \frac{1}{s^2+1}, \text{ 즉 } F \frac{(s+1)^2}{s^2+1} = \frac{4s^2+s+5}{(s+1)(s^2+1)} \text{ 이고}$

$$F = \frac{4s^2+s+5}{(s+1)^3} = \frac{4}{s+1} - \frac{7}{(s+1)^2} + \frac{8}{(s+1)^3}.$$

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$$f(t) = (4 - 7t + 4t^2)e^{-t}$$

(2) $y' = 10 - e^{-4t} * y, y(0) = 5: sY - 5 = \frac{10}{s} - \frac{Y}{s+4}$, 따라서

$$Y = \frac{5(s+2)(s+4)}{s(s^2+4s+1)} = \frac{40}{s} - \frac{35(s+2)+60}{(s+2)^2-3}.$$

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$$y = 40 - e^{-2t} (35 \cosh \sqrt{3}t + 20\sqrt{3} \sinh \sqrt{3}t)$$

11 $\mathcal{L}\{e^{-t^2}\}$ 를 구하지 않고: $y'' + 9y = 3e^{-t^2}, y(0) = y'(0) = 0$

$Y = \frac{3}{s^2+9} \mathcal{L}\{e^{-t^2}\}$ 이고 $\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \sin 3t$ 이므로 합성곱 정리에서

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$$y(t) = \sin 3t * e^{-t^2} = \int_0^t e^{-\tau^2} \sin 3(t-\tau) d\tau$$

12 $f(t) = e^t + e^t \int_0^t e^{-\tau} f(\tau) d\tau$

$e^t \int_0^t e^{-\tau} f(\tau) d\tau = \int_0^t e^{t-\tau} f(\tau) d\tau = e^t * f$ 이므로 $F = \frac{1}{s-1} + \frac{F}{s-1}, F = \frac{1}{s-2}$.

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$$f(t) = e^{2t}$$