

Process Dynamics and Control

1. Mathematical Preliminaries and Dynamic Response

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What are we up to?

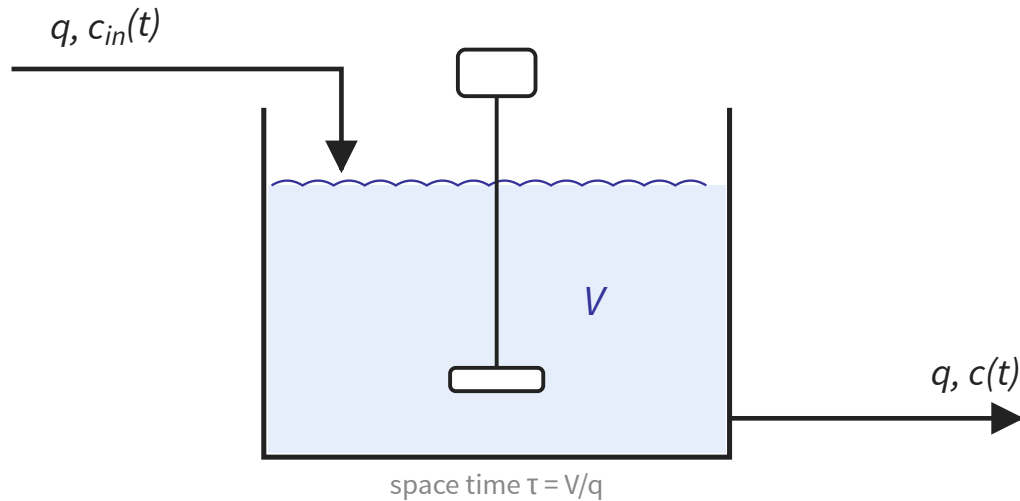
Before we design controllers we need a language for **how processes respond in time**: models in deviation variables, the Laplace transform, and transfer functions whose **poles** tell us the shape of the response without solving for it.

Section	Topic	Key result
1.1	process models, deviation variables, linearization	linear ODEs with zero initial conditions
1.2	Laplace transform	table look-up, initial- and final-value theorems
1.3	partial-fraction expansion	each pole gives one term of $f(t)$
1.4	transfer functions, poles and zeros	poles fix the time dependence, zeros the coefficients
1.5–1.6	first- and second-order systems	τ, K, ζ ; overshoot, decay ratio, settling time
1.7–1.9	dead time, higher order, zeros	Padé, FOPDT approximation, inverse response

Any linear model breaks into first-order terms by partial fractions, and complex-conjugate pairs combine into second-order terms, so first- and second-order systems are all we need to study in detail.

1.1 Process Models and Deviation Variables

A simple differential-equation model



Mass balance of a well-mixed tank:

$$\tau \frac{dc}{dt} = c_{in} - c, \quad c(0) = c_0, \quad \tau = \frac{V}{q}$$

Define **deviation variables** from the initial steady state:

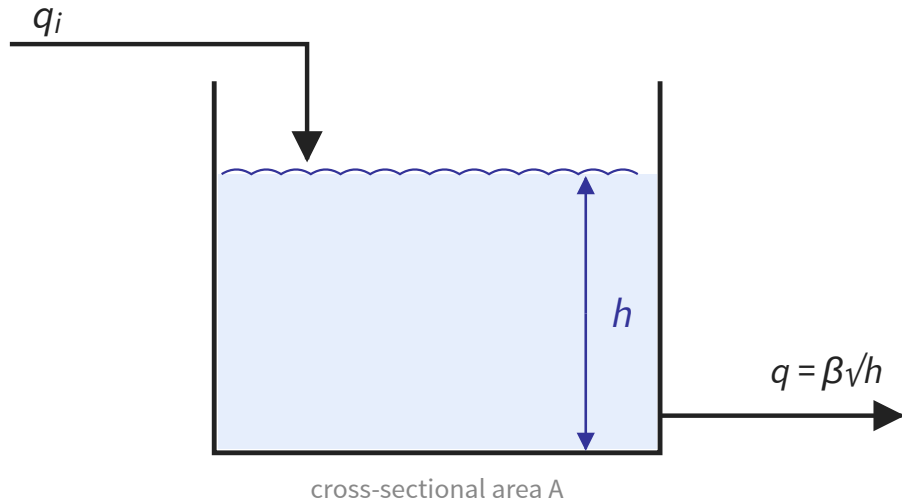
$$\bar{c} = c - c_0, \quad \bar{c}_{in} = c_{in} - c_0.$$

$$\tau \frac{d\bar{c}}{dt} + \bar{c} = \bar{c}_{in}, \quad \bar{c}(0) = 0$$

Why deviation variables? The initial condition becomes **zero**, so the Laplace transform of the model contains no initial-condition terms and gives a clean input-output ratio. For reference, the solution

is the convolution
$$\bar{c}(t) = \frac{1}{\tau} \int_0^t \bar{c}_{in}(\alpha) e^{-(t-\alpha)/\tau} d\alpha.$$

Linearization of nonlinear models



$$A \frac{dh}{dt} = q_i - \beta\sqrt{h}, \quad h(0) = h_s$$

Steady state: $0 = q_{i,s} - \beta\sqrt{h_s}$. Taylor-expand $\sqrt{h}$ about h_s :

$$A \frac{dh}{dt} \simeq q_i - \beta \left[\sqrt{h_s} + \frac{1}{2\sqrt{h_s}}(h - h_s) \right]$$

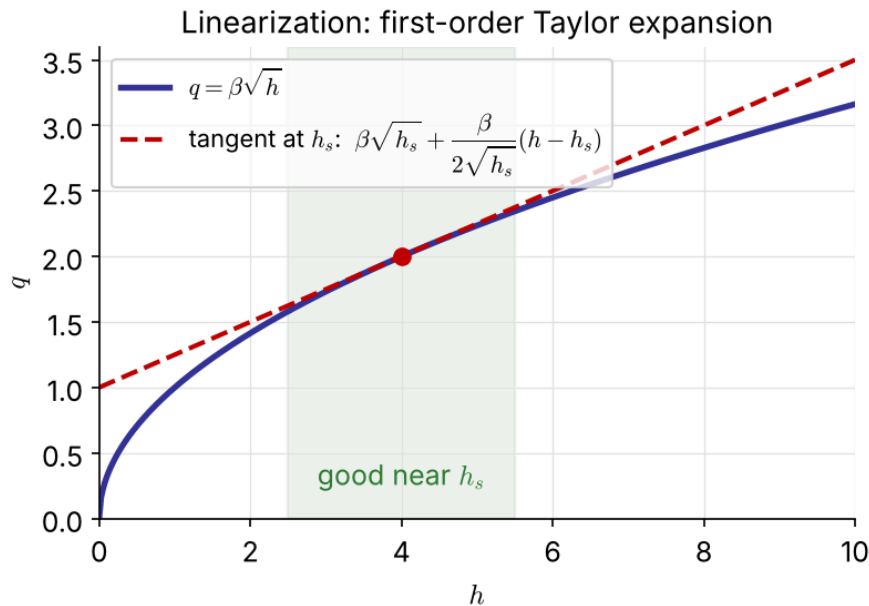
Subtract the steady state and use $\bar{q}_i = q_i - q_{i,s}$, $\bar{h} = h - h_s$:

$$A \frac{d\bar{h}}{dt} + \frac{\beta}{2\sqrt{h_s}} \bar{h} = \bar{q}_i, \quad \bar{h}(0) = 0$$

General case (one state y , inputs $\mathbf{u}$): with $y' = y - y_s$, $\mathbf{u}' = \mathbf{u} - \mathbf{u}_s$,

$$\frac{dy'}{dt} = \left. \frac{\partial f}{\partial y} \right|_s y' + \nabla_{\mathbf{u}}^T f|_s \mathbf{u}'$$

The model is accurate only **near the operating point**.



Example 1.1 Linearize a mixed vessel

$$V \frac{dc}{dt} = q c_i - q c \quad (q \text{ and } c_i \text{ both vary with time})$$

The product terms are nonlinear. Expand about $(q_s, c_{i,s}, c_s)$:

$$q c_i \simeq q_s c_{i,s} + c_{i,s} q' + q_s c'_i$$

$$q c \simeq q_s c_s + c_s q' + q_s c'$$

Subtract the steady state $0 = q_s(c_{i,s} - c_s)$:

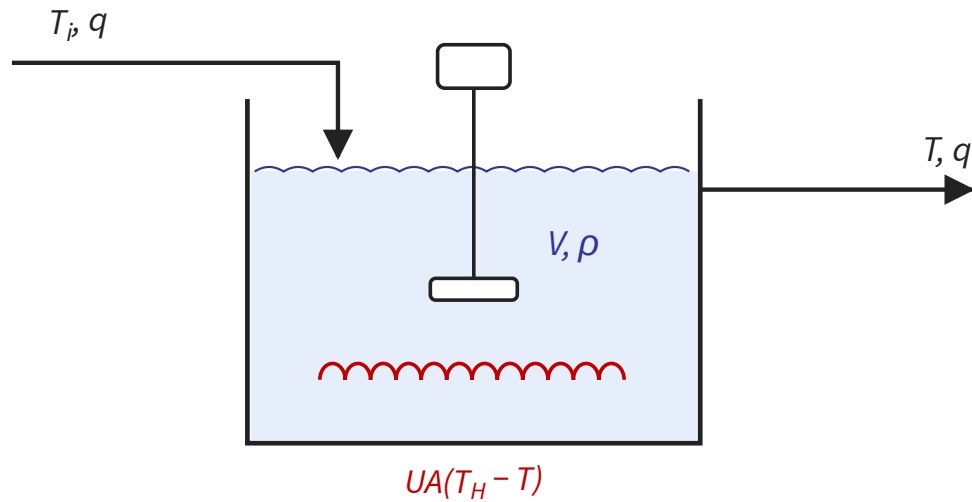
$$V \frac{dc'}{dt} = q_s(c'_i - c') + (c_{i,s} - c_s) q'$$

At steady state $c_s = c_{i,s}$, so the flow term vanishes:

$$\frac{V}{q_s} \frac{dc'}{dt} + c' = c'_i$$

To first order, a flow change does not move the concentration; only c_i does.

Example 1.2 Stirred-tank heater



$$\rho C_p V \frac{dT}{dt} = \rho C_p q (T_i - T) + UA(T_H - T)$$

With $\tau = V/q$ and $\kappa = UA/(\rho C_p q)$, in deviation variables:

$$\tau \frac{d\bar{T}}{dt} + (1 + \kappa)\bar{T} = \bar{T}_i + \kappa\bar{T}_H$$

Pole-zero form

$$T(s) = \frac{K_i}{s + a} T_i(s) + \frac{K_H}{s + a} T_H(s)$$

$$a = \frac{1 + \kappa}{\tau}, \quad K_i = \frac{1}{\tau}, \quad K_H = \frac{\kappa}{\tau}$$

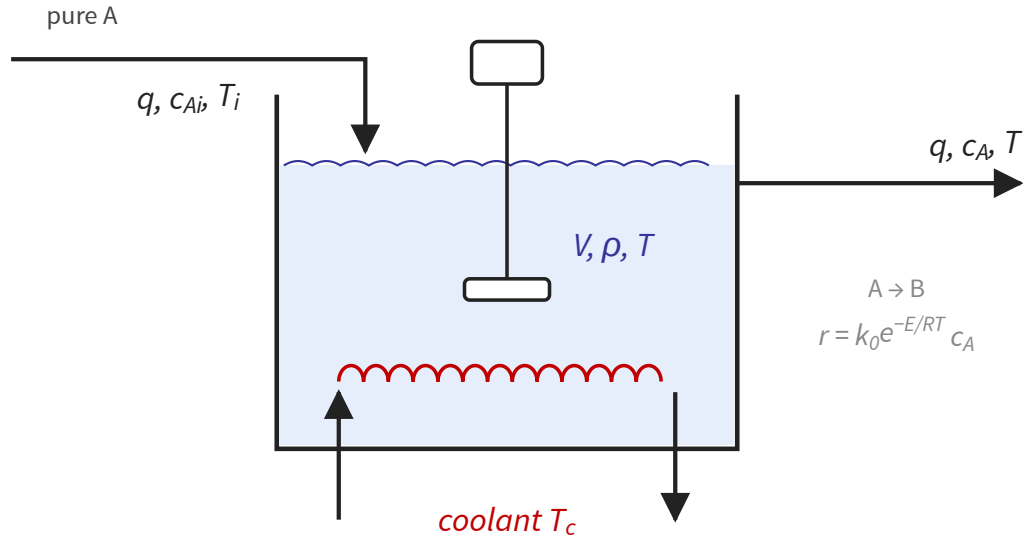
Time-constant form (used in control)

$$T(s) = \frac{K_d}{\tau_p s + 1} T_i(s) + \frac{K_p}{\tau_p s + 1} T_H(s)$$

$$\tau_p = \frac{\tau}{1 + \kappa}, \quad K_d = \frac{1}{1 + \kappa}, \quad K_p = \frac{\kappa}{1 + \kappa}$$

One output, two inputs: the heater temperature T_H will be our **manipulated** variable, the inlet temperature T_i a **disturbance**. Both see the same pole $-1/\tau_p$.

Example 1.3 Non-isothermal CSTR



$$V \frac{dc_A}{dt} = q(c_{Ai} - c_A) - V k(T) c_A$$

$$\rho V C_p \frac{dT}{dt} = \rho q C_p (T_i - T) + (-\Delta H_R) V k(T) c_A + U A (T_c - T)$$

$k(T)c_A$ is nonlinear in **both** states. With q, c_{Ai}, T_i constant, linearize ($k'_s = k_s E / RT_s^2$):

$$\frac{d}{dt} \begin{bmatrix} c'_A \\ T' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c'_A \\ T' \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} T'_c$$

$$a_{11} = -\frac{q}{V} - k_s, \quad a_{12} = -k'_s c_{As}, \quad a_{21} = \frac{(-\Delta H_R) k_s}{\rho C_p}$$

$$a_{22} = -\frac{q}{V} - \frac{U A}{\rho V C_p} + \frac{(-\Delta H_R) k'_s c_{As}}{\rho C_p}, \quad b_2 = \frac{U A}{\rho V C_p}$$

$$\frac{C'_A(s)}{T'_c(s)} = \frac{a_{12} b_2}{(s - a_{11})(s - a_{22}) - a_{12} a_{21}}$$

second order: the two balances are coupled.

Problem #1 (homework)

For the CSTR of Example 1.3, find all the approximate transfer functions relating the outputs c_A and T to the inputs $c_{A,i}$, T_i and T_c .

Show that all six transfer functions share **one** denominator, and explain why.

Hint: write the Laplace-transformed linear model as $(s\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{B}\mathbf{u}$ and solve with Cramer's rule; the denominator is $\det(s\mathbf{I} - \mathbf{A})$.

1.2 Laplace Transform

Definition and the look-up table approach

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}dt = F(s), \quad f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} F(s)e^{st}ds$$

- In control we **never** use the inverse integral: we break $F(s)$ into simple parts and read the answer from a **table**.
- $\mathcal{L}$ is a **linear operator**:

$$\mathcal{L}[af_1 + bf_2] = aF_1(s) + bF_2(s)$$

- A linear ODE becomes an **algebraic** equation in s

The real goal: predict the **time response without the inverse transform**. The answer lies in the roots of the characteristic equation (§1.4).

Transforms of common functions

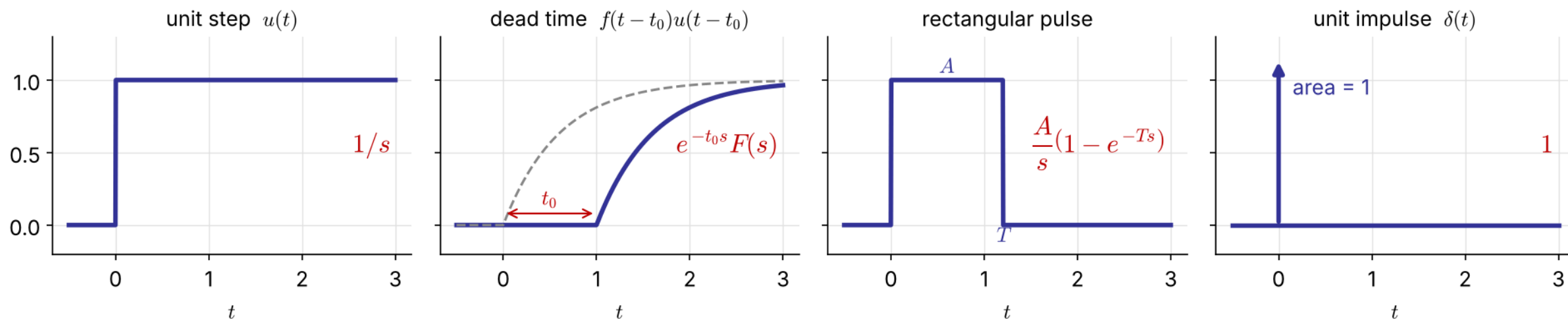
$f(t)$	$F(s)$
$\delta(t)$ (unit impulse)	1
$u(t)$ (unit step)	$\frac{1}{s}$
t (ramp)	$\frac{1}{s^2}$
$t^{n-1}/(n-1)!$	$\frac{1}{s^n}$
e^{-at}	$\frac{1}{s+a}$
$t^{n-1}e^{-at}/(n-1)!$	$\frac{1}{(s+a)^n}$
$\sin \omega t, \cos \omega t$	$\frac{\omega}{s^2 + \omega^2}, \frac{s}{s^2 + \omega^2}$
$e^{-at} \sin \omega t, e^{-at} \cos \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}, \frac{s+a}{(s+a)^2 + \omega^2}$

Operation	Transform
$\frac{df}{dt}$	$sF(s) - f(0)$
$\frac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0) - f'(0)$
$\int_0^t f dt$	$\frac{F(s)}{s}$
$f(t-t_0)u(t-t_0)$	$e^{-st_0}F(s)$
$e^{-at}f(t)$	$F(s+a)$

Derivation by integration by parts, e.g. $\mathcal{L}[df/dt] = fe^{-st}|_0^\infty + s \int_0^\infty fe^{-st}dt = sF(s) - f(0)$. With deviation variables all $f^{(k)}(0) = 0$, so $d/dt \rightarrow s$ and $\int dt \rightarrow 1/s$.

Try: derive $\mathcal{L}[\cos \omega t]$ from $\cos \omega t = (e^{j\omega t} + e^{-j\omega t})/2$.

Inputs common to control problems



- Pulse: $A[u(t) - u(t - T)] \Rightarrow \frac{A}{s}(1 - e^{-sT})$

- Impulse: $\delta(t) = \lim_{T \rightarrow 0} \frac{u(t) - u(t - T)}{T}$, so

$$\mathcal{L}[\delta] = \lim_{T \rightarrow 0} \frac{1 - e^{-sT}}{Ts} = 1$$

- $\int_{-\infty}^{\infty} \delta dt = 1, \quad \int_{-\infty}^{\infty} f(t)\delta(t - t_0) dt = f(t_0)$

- **Dead time** (time shift) multiplies by e^{-st_0} : an exponential, not a polynomial, in s (§1.7).

Initial- and final-value theorems

Values of $f(t)$ at the two extremes **without** the inverse transform:

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s),$$

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

Proof of the final value:

$$\lim_{s \rightarrow 0} \int_0^{\infty} \frac{df}{dt} e^{-st} dt = \int_0^{\infty} df = f(\infty) - f(0), \text{ and the left side is } \lim_{s \rightarrow 0} [sF(s) - f(0)].$$

The final-value theorem holds **only if** $f(\infty)$ exists: all poles of $sF(s)$ must lie in the left half-plane.

Example 1.4 What is $f(\infty)$?

a. $F = \frac{6(s-2)(s+2)}{s(s+1)(s+3)(s+4)}$: $sF(0) = \frac{6(-2)(2)}{(3)(4)} = -2$

b. $F = \frac{1}{s-2}$, i.e. $f = e^{2t}$: $sF(0) = 0$, but $f \rightarrow \infty$: theorem **not applicable** (pole at $+2$).

c. $F = \frac{6(s+3)}{s^3 + s^2 - 4s - 4} = \frac{6(s+3)}{(s+1)(s+2)(s-2)}$: again a pole at $+2$; $f(\infty)$ does not exist.

1.3 Partial-Fraction Expansion

Breaking $F(s)$ into table entries

$$F(s) = \frac{q(s)}{p(s)} = \frac{\alpha_1}{s + a_1} + \frac{\alpha_2}{s + a_2} + \cdots + \frac{\alpha_n}{s + a_n} \Rightarrow f(t) = \sum_i \alpha_i e^{-a_i t}$$

$-a_i$ are the roots of $p(s) = 0$ (order $n >$ order of q). Heaviside's cover-up rule:

$$\alpha_i = (s + a_i) \frac{q(s)}{p(s)} \Big|_{s=-a_i}$$

1. The **time dependence** of $f(t)$ comes entirely from the roots of the **denominator**.
2. The **numerator** affects only the coefficients α_i .

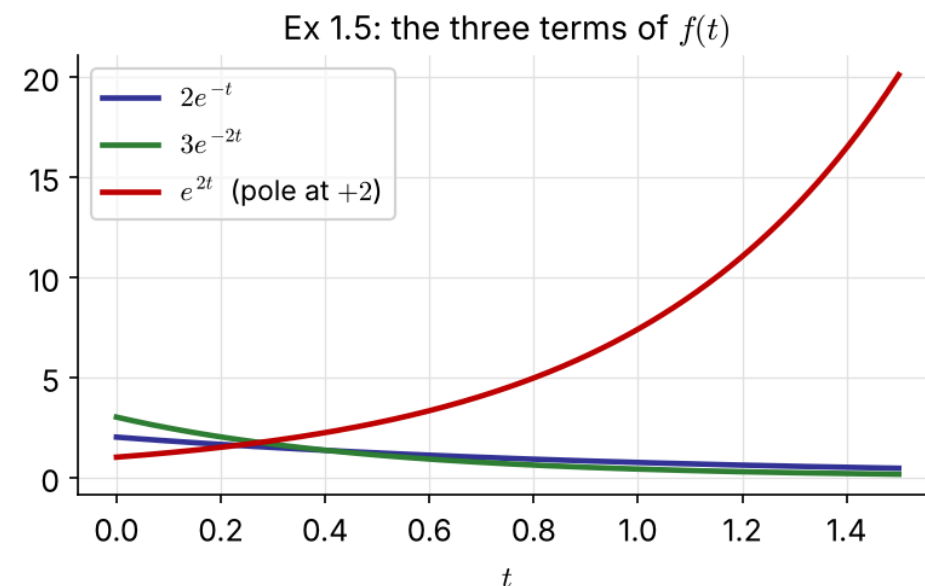
3. Poles **closer to the origin** give terms that decay more slowly: they are **dominant**.
4. Three cases: distinct real, complex, and repeated roots.

Example 1.5 Case 1: distinct real roots

$s^3 + s^2 - 4s - 4 = (s + 1)(s + 2)(s - 2)$. Two numerators, same denominator:

	$\frac{6s^2 - 12}{(s + 1)(s + 2)(s - 2)}$	$\frac{6s}{(s + 1)(s + 2)(s - 2)}$
$\alpha_1 = \frac{q}{(s + 2)(s - 2)} \Big _{-1}$	$\frac{-6}{-3} = 2$	$\frac{-6}{-3} = 2$
$\alpha_2 = \frac{q}{(s + 1)(s - 2)} \Big _{-2}$	$\frac{12}{4} = 3$	$\frac{-12}{4} = -3$
$\alpha_3 = \frac{q}{(s + 1)(s + 2)} \Big _2$	$\frac{12}{12} = 1$	$\frac{12}{12} = 1$
$f(t)$	$2e^{-t} + 3e^{-2t} + e^{2t}$	$2e^{-t} - 3e^{-2t} + e^{2t}$

Same three exponentials, different coefficients: the numerator changes only the weights.



The same procedure gives

$$\frac{6}{(s + 1)(s + 2)(s + 3)} = \frac{3}{s + 1} - \frac{6}{s + 2} + \frac{3}{s + 3}.$$

Example 1.6 Case 2: complex roots

$$F(s) = \frac{s+5}{s^2+4s+13}, \quad s^2+4s+13=0 \Rightarrow s = -2 \pm 3j$$

Method 1: cover-up with complex numbers

$$\alpha = \frac{s+5}{s-(-2-3j)} \Big|_{s=-2+3j} = \frac{3+3j}{6j} = \frac{1-j}{2}, \quad \alpha^* = \frac{1+j}{2} \quad \frac{s+5}{(s+2)^2+3^2} = \frac{s+2}{(s+2)^2+3^2} + \frac{3}{(s+2)^2+3^2}$$

$$f(t) = \frac{1-j}{2} e^{(-2+3j)t} + \frac{1+j}{2} e^{(-2-3j)t} \quad f(t) = e^{-2t}(\cos 3t + \sin 3t)$$

Method 2: complete the square (faster)

Euler's formula collects the terms into real functions.

Phase-lag form (the one we use): $\cos 3t + \sin 3t = \sqrt{2} \sin(3t + \phi)$, $\phi = \tan^{-1} 1 = \pi/4$, so $f(t) = \sqrt{2} e^{-2t} \sin(3t + \pi/4)$. Roots $-a \pm bj$ always give e^{-at} times an oscillation of frequency b

Example 1.7 Case 3: repeated roots

$$\frac{2}{(s+1)^3(s+2)} = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{(s+1)^2} + \frac{\alpha_3}{(s+1)^3} + \frac{\alpha_4}{s+2}$$

$$\alpha_4 = \frac{2}{(s+1)^3} \Big|_{-2} = -2. \text{ Multiply by } (s+1)^3:$$

$$\frac{2}{s+2} = \alpha_1(s+1)^2 + \alpha_2(s+1) + \alpha_3 + \alpha_4 \frac{(s+1)^3}{s+2}$$

Set $s = -1$, then differentiate and set $s = -1$ again:

$$\alpha_3 = 2, \quad \alpha_2 = -\frac{2}{(s+2)^2} \Big|_{-1} = -2, \quad \alpha_1 = \frac{1}{2} \frac{4}{(s+2)^3} \Big|_{-1}$$

$$f(t) = 2 \left[\left(1 - t + \frac{t^2}{2} \right) e^{-t} - e^{-2t} \right]$$

In general

$$\mathcal{L}^{-1} \left[\sum_{k=1}^n \frac{\alpha_k}{(s+a)^k} \right] = \left[\alpha_1 + \alpha_2 t + \cdots + \frac{\alpha_n t^{n-1}}{(n-1)!} \right] e^{-at}$$

Still based on the root $-a$, but the polynomial in t makes it decay more slowly.

Partial fractions by computer

Python

```

1 import sympy as sp
2 s, t = sp.symbols('s t'); tp = sp.Symbol('t', positive=True)
3
4 for F in [(6*s**2 - 12)/(s**3 + s**2 - 4*s - 4),      # Ex 1.5
            (s + 5)/(s**2 + 4*s + 13),                # Ex 1.6
            2/((s + 1)**3*(s + 2))]:                  # Ex 1.7
5     print(sp.apart(F, s), ' -> ', sp.inverse_laplace_transform(F, s, tp))
6
7 import numpy as np, scipy.signal as sig
8
9 r, p, k = sig.residue([6, 0, -12], [1, 1, -4, -4])    # residues, poles, direct term
10 print(np.round(r, 4), np.round(p, 4))               # [3, 2, 1] at [-2, -1, 2]
11

```

MATLAB

```

1 [r, p, k] = residue([6 0 -12], [1 1 -4 -4])          % Ex 1.5: residues and poles
2 roots([1 1 -4 -4])                                  % 2, -2, -1
3
4 syms s t
5 ilaplace((s + 5)/(s^2 + 4*s + 13))                  % Ex 1.6
6 ilaplace(2/((s + 1)^3*(s + 2)))                     % Ex 1.7
7 partfrac(2/((s + 1)^3*(s + 2)), s)

```

Problem #2 (homework)

a. Find $f(t)$ for

$$F(s) = \frac{s + \beta}{(s + a)^2 + \omega^2}$$

b. Find $f(t)$ for

$$F(s) = \frac{s \sin \phi + \omega \cos \phi}{s^2 + \omega^2}$$

Check your answers with `sympy.inverse_laplace_transform` or MATLAB `ilaplace`.

1.4 Transfer Functions, Poles and Zeros

The transfer function

For a linear ODE with zero initial conditions ($n \geq m$)

$$a_n y^{(n)} + \dots + a_1 y' + a_0 y = b_m x^{(m)} + \dots + b_1 x' + b_0 x$$

the Laplace transform gives the **transfer function**, the ratio of output to input:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0} = \frac{Q(s)}{P(s)}$$

Pole-zero form

$$G(s) = \frac{b_m}{a_n} \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

$P(s)$ is the **characteristic polynomial**; its roots are the **poles**, the roots of $Q(s)$ the **zeros**.

Time-constant form (real roots)

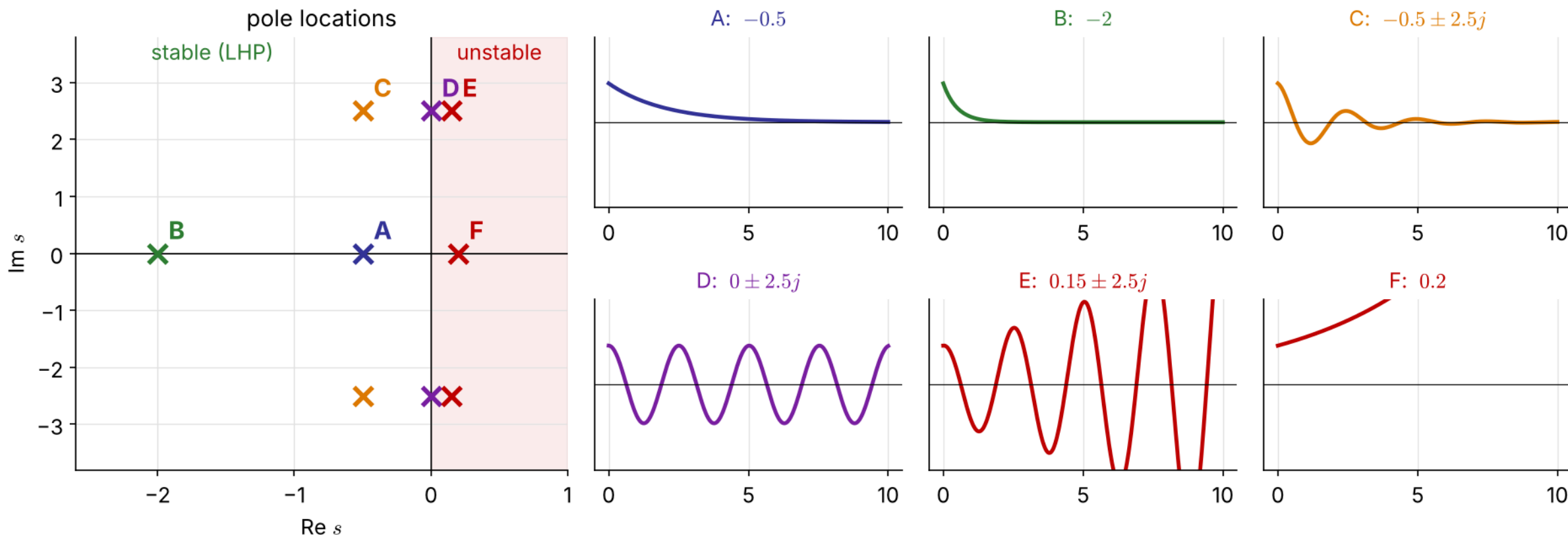
$$G(s) = K \frac{(\tau_a s + 1) \cdots (\tau_m s + 1)}{(\tau_1 s + 1) \cdots (\tau_n s + 1)}$$

Steady-state gain: final change of y per unit step in x ,

$$K = \lim_{s \rightarrow 0} sG(s) \frac{1}{s} = G(0) = \frac{b_0}{a_0}$$

Poles tell the shape of the response

Each pole contributes a term e^{pt} : the real part sets decay or growth, the imaginary part the oscillation



- real pole $p \rightarrow e^{pt}$; repeated m times $\rightarrow (\alpha_1 + \dots + \alpha_m t^{m-1})e^{pt}$
- complex pair $-\zeta/\tau \pm j\sqrt{1-\zeta^2}/\tau \rightarrow e^{-\zeta t/\tau} \sin(\sqrt{1-\zeta^2} t/\tau + \phi)$

- the discussion of poles is **independent of the input**; the actual $y(t)$ also depends on $x(t)$
- zeros change only the coefficients
- **Stability:** all poles must have **negative real parts** (Ch 4).

Back to the first-order tank: step and impulse

From §1.1, $\tau \frac{d\bar{c}}{dt} + \bar{c} = \bar{c}_{in} \Rightarrow (\tau s + 1)C(s) = C_{in}(s) \Rightarrow G(s) = \frac{C(s)}{C_{in}(s)} = \frac{1}{\tau s + 1}$, one pole at $-1/\tau$.

Unit step $C_{in} = 1/s$

$$C = \frac{1}{(\tau s + 1)s} = \frac{1}{s} - \frac{1}{s + 1/\tau}$$

$$\bar{c}(t) = 1 - e^{-t/\tau}$$

Unit impulse $C_{in} = 1$

$$C = \frac{1}{\tau s + 1} = \frac{1}{\tau} \frac{1}{s + 1/\tau}$$

$$\bar{c}(t) = \frac{1}{\tau} e^{-t/\tau}$$

Without specifying the input we already know $\bar{c}(t)$ must contain $e^{-t/\tau}$: that is what the pole tells us.

1.5 First-Order Systems

First-order model

$$a_1 \frac{dy}{dt} + a_0 y = b x(t) \quad \Rightarrow \quad \tau \frac{dy}{dt} + y = K x(t), \quad G(s) = \frac{Y(s)}{X(s)} = \frac{K}{\tau s + 1}$$

$\tau = a_1/a_0$: time constant (시간상수), $K = b/a_0$: steady-state gain (정상상태 이득), one real pole at $-1/\tau$.

Step $x = M u(t)$:

$$Y = \frac{KM}{s(\tau s + 1)} \Rightarrow y(t) = KM \left(1 - e^{-t/\tau}\right)$$

Impulse $x = M \delta(t)$:

$$y(t) = \frac{KM}{\tau} e^{-t/\tau}$$

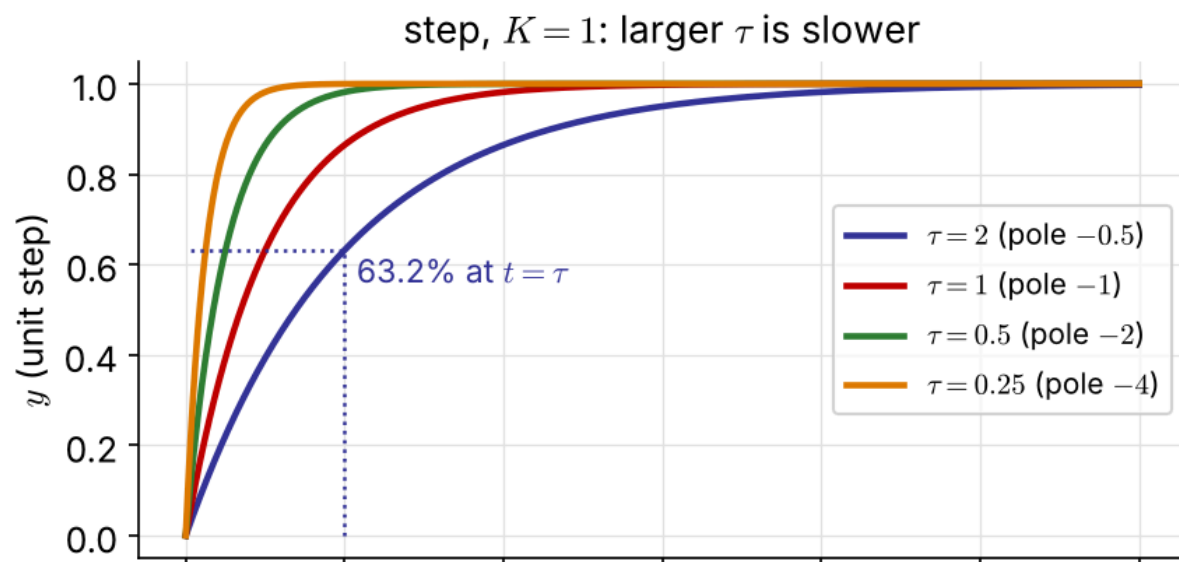
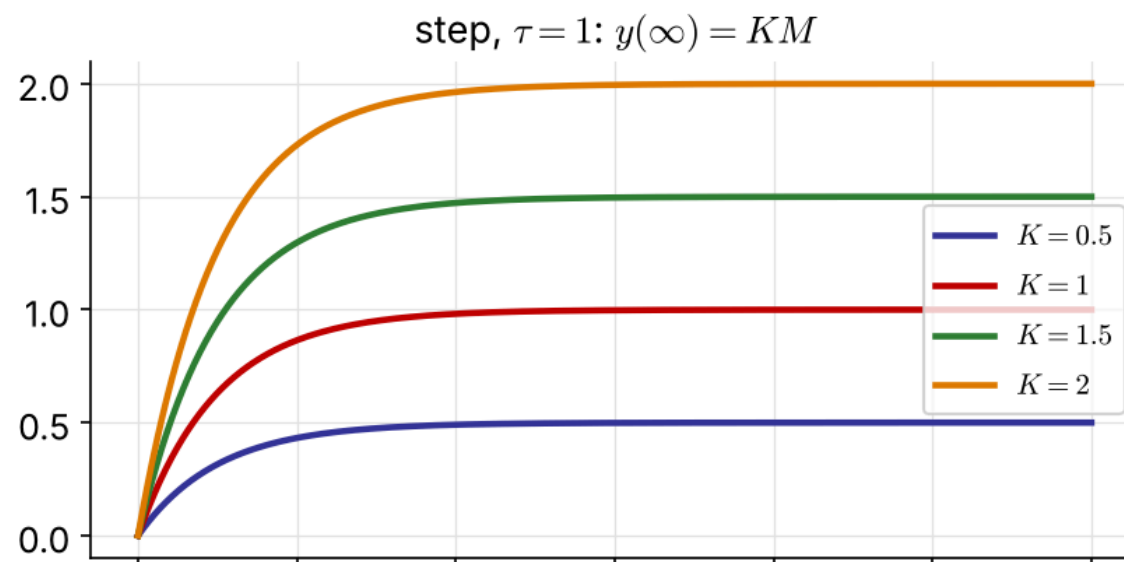
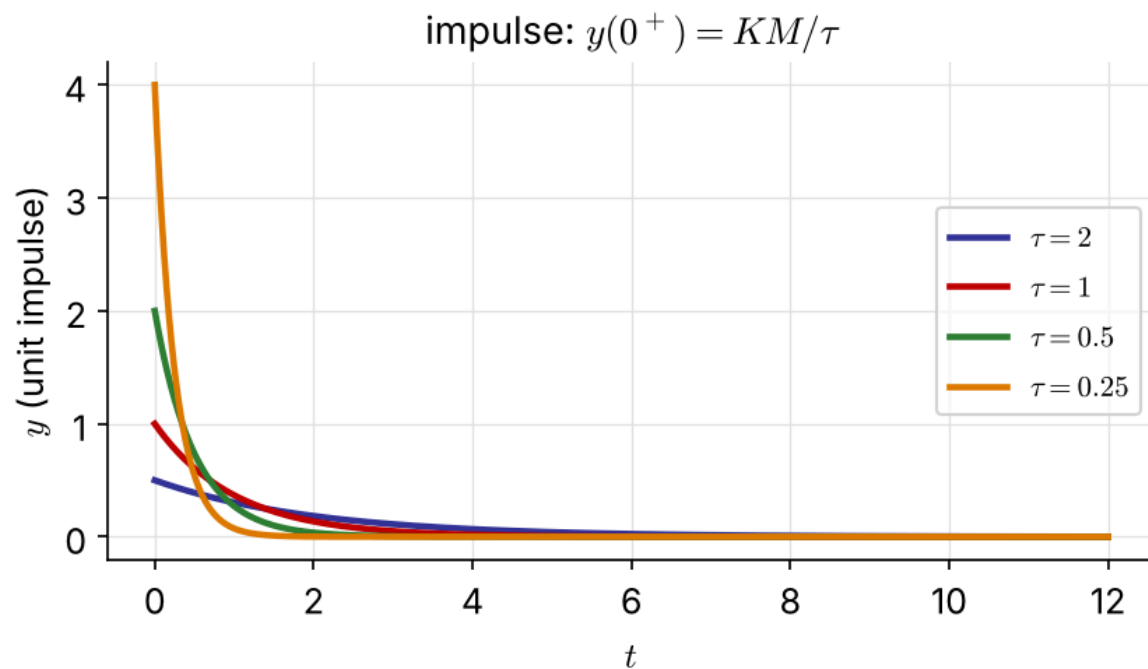
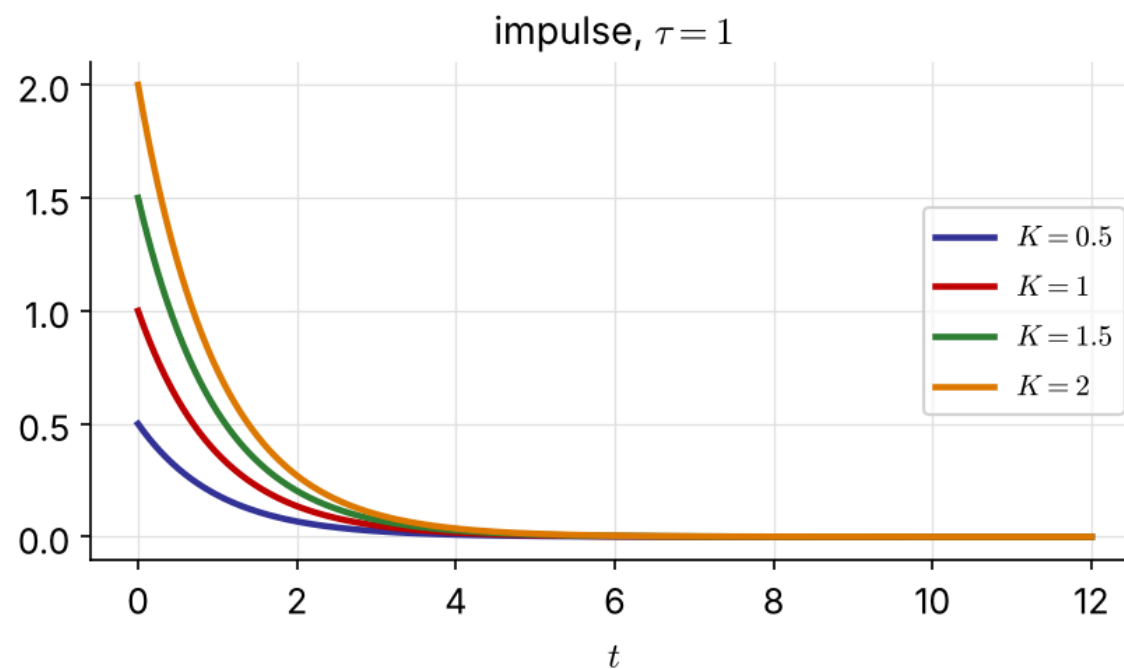
Final value check: $\lim_{s \rightarrow 0} s \frac{KM}{s(\tau s + 1)} = KM.$

t	$y/KM = 1 - e^{-t/\tau}$
τ	0.632
2τ	0.865
3τ	0.950
4τ	0.982
5τ	0.993

After 3τ within 5%, after 5τ within 1%: “settled”.

Effect of τ and K

The larger the time constant, the slower the response. The larger the gain, the more sensitive the output to the input.



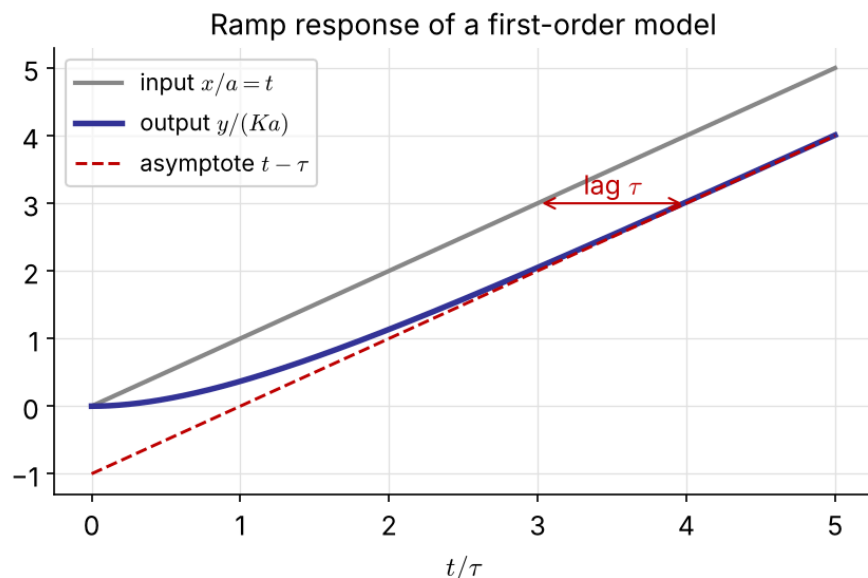
Ramp input and the integrating process

Ramp $x = at$, $X = a/s^2$:

$$Y = \frac{Ka}{s^2(\tau s + 1)} = \frac{Ka\tau^2}{\tau s + 1} - \frac{Ka\tau}{s} + \frac{Ka}{s^2}$$

$$y(t) = Ka\tau \left(e^{-t/\tau} - 1 \right) + Kat$$

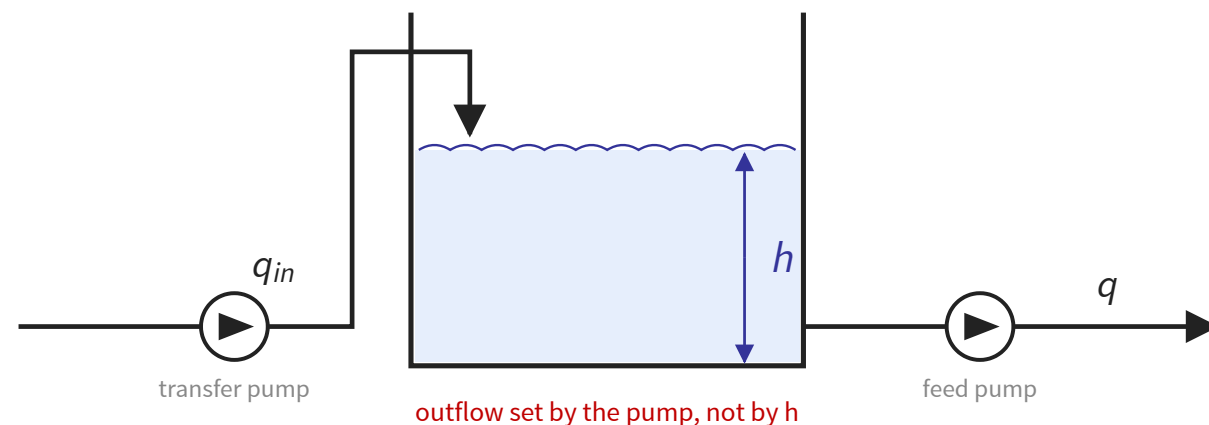
→ $Ka(t - \tau)$ for $t \gg \tau$: the output lags the input by τ .



Integrating process ($a_0 = 0$):

$$\frac{dy}{dt} = K_I x(t) \Rightarrow G(s) = \frac{K_I}{s}$$

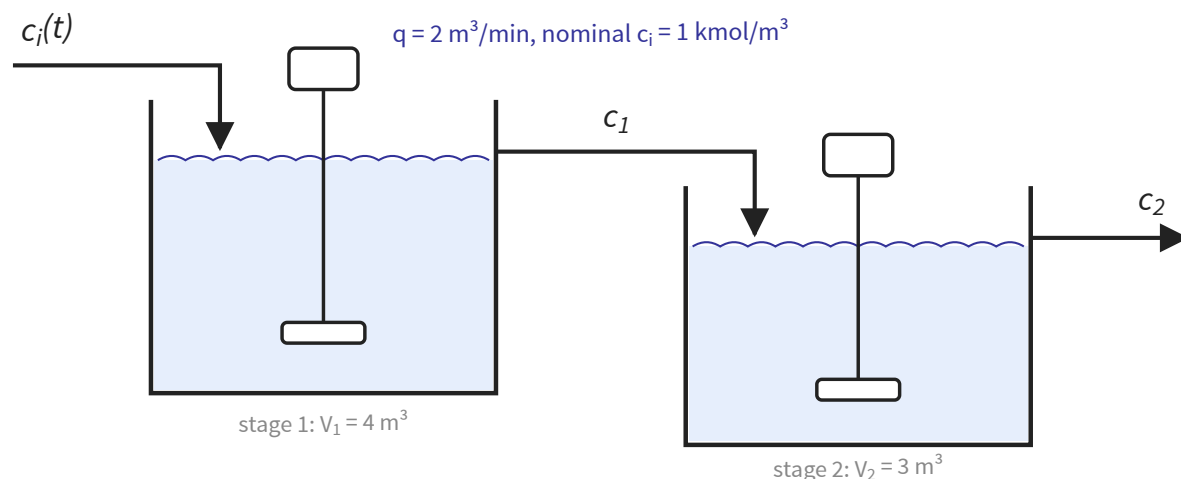
a pole at the origin.



$$A \frac{d\bar{h}}{dt} = \bar{q}_{in} - \bar{q} \Rightarrow H(s) = \frac{1}{A} \frac{Q_{in}(s) - Q(s)}{s}$$

A step input gives a **ramp**; an impulse is **not** forgotten, the level keeps whatever was added. Not self-regulating: levels need control.

Example 1.8 Two stirred tanks in series



Feed: $c_i = 6$ for $0 < t < 0.25$ min, otherwise 1, so $\bar{c}_i$ is a pulse of height 5:

$$C_i(s) = \frac{5}{s} (1 - e^{-s/4}) \quad \text{or, as an impulse,} \quad C_i(s) = \frac{5}{4}$$

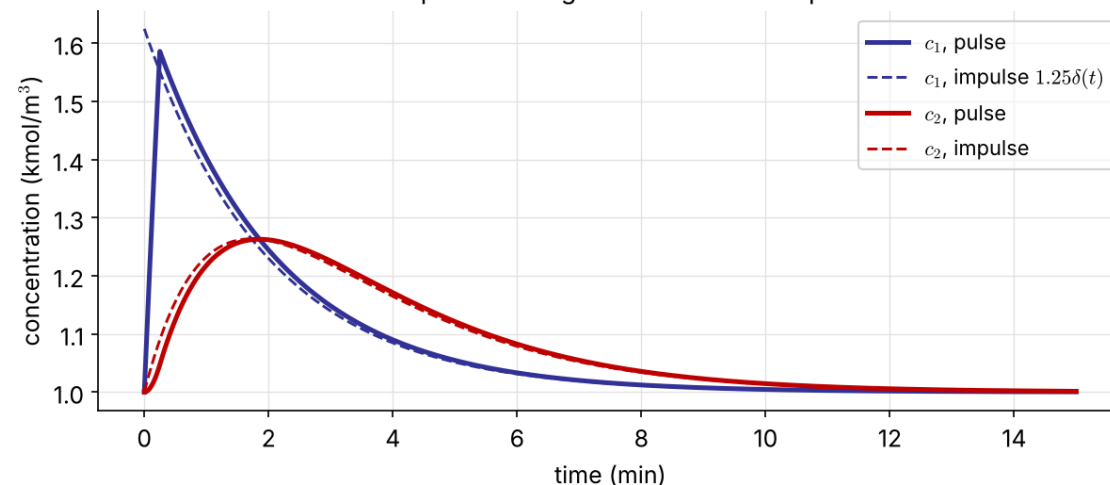
$$C_2(s) = \frac{5(1 - e^{-s/4})}{s(2s + 1)(1.5s + 1)} \quad \text{or} \quad \frac{5/4}{(2s + 1)(1.5s + 1)}$$

$$\tau_1 \frac{d\bar{c}_1}{dt} + \bar{c}_1 = \bar{c}_i, \quad \tau_2 \frac{d\bar{c}_2}{dt} + \bar{c}_2 = \bar{c}_1$$

$$\frac{C_2(s)}{C_i(s)} = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

$$\tau_1 = V_1/q = 2 \text{ min}, \quad \tau_2 = V_2/q = 1.5 \text{ min}$$

Ex 1.8: a 0.25-min pulse of height 5 is almost an impulse of area 1.25



The original MATLAB script used `pulseLength = 1.25`, a pulse 5× too long.

Problem #3 (homework)

For Example 1.8, carry out the inverse transforms:

- the impulse response $\bar{c}_2(t)$ for $C_i = 5/4$;
- the pulse response $\bar{c}_2(t)$, using the dead-time property for the factor $e^{-s/4}$.

Plot both together with $\bar{c}_1(t)$ and compare with the figure on the previous slide.

1.6 Second-Order Systems

Second-order model

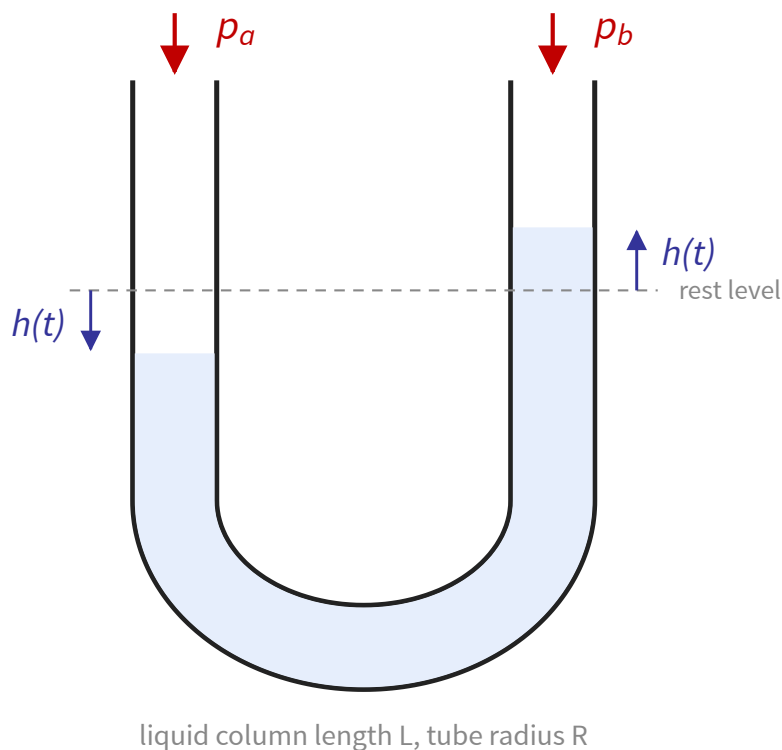
$$\tau^2 \frac{d^2 y}{dt^2} + 2\zeta\tau \frac{dy}{dt} + y = Kx(t) \quad \Rightarrow \quad G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

τ : natural period, $\omega_n = 1/\tau$: natural frequency, ζ : **damping ratio** (감쇠계수), K : steady-state gain.
Poles:

$$p_{1,2} = -\frac{\zeta}{\tau} \pm \frac{\sqrt{\zeta^2 - 1}}{\tau} \quad (\text{stable only if } \zeta > 0, \text{ since } \tau > 0)$$

ζ	poles	name	response
$\zeta > 1$	two distinct real, $-1/\tau_1, -1/\tau_2$	overdamped	sluggish, $G = K/[(\tau_1 s + 1)(\tau_2 s + 1)]$
$\zeta = 1$	repeated real, $-1/\tau$	critically damped	fastest without oscillation
$0 < \zeta < 1$	complex pair $-\zeta/\tau \pm j\sqrt{1 - \zeta^2}/\tau$	underdamped	overshoot, decaying oscillation; τ is not a time constant
$\zeta = 0$	purely imaginary $\pm j/\tau$	undamped	sustained oscillation at ω_n

Example 1.9 Oscillations of a damped manometer



A pressure step $p_a - p_b$ sets the liquid in motion (laminar, isothermal). A momentum balance on the liquid column gives

$$\frac{4}{3}\pi R^2 L \rho \frac{d^2 h}{dt^2} = -8\pi L \mu \frac{dh}{dt} - 2\pi R^2 \rho g h + \pi R^2 (p_a - p_b)$$

$$\frac{d^2 h}{dt^2} + \frac{6\mu}{R^2 \rho} \frac{dh}{dt} + \frac{3g}{2L} h = \frac{3}{4} \frac{p_a - p_b}{\rho L}$$

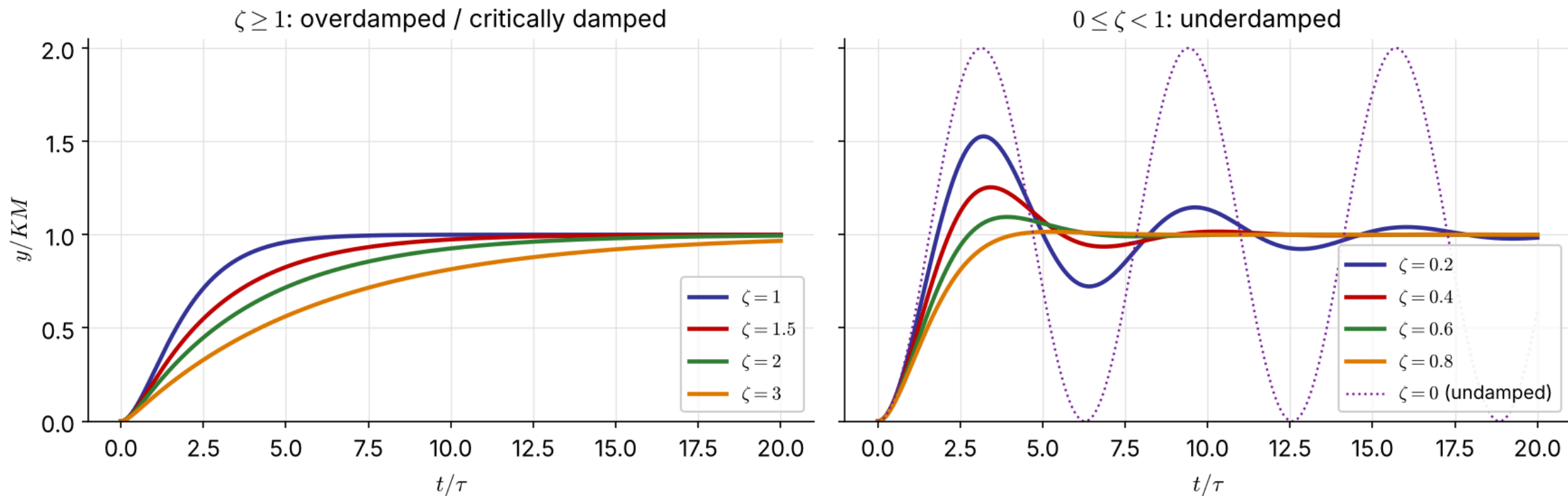
Compare with $s^2 + 2\zeta\omega_n s + \omega_n^2$:

$$\omega_n = \sqrt{\frac{3g}{2L}}, \quad \zeta = \frac{3\mu}{R^2 \rho \omega_n}$$

Critical damping $\zeta = 1$ at $R_{cr} = \left(\frac{6\mu^2 L}{\rho^2 g} \right)^{1/4}$. Wider tubes oscillate, narrower tubes creep.

Bird, Stewart & Lightfoot, *Transport Phenomena*, 2nd ed., pp. 219–221.

Step response of a second-order model



$\zeta > 1$:

$$y = KM \left[1 - e^{-\zeta t/\tau} \left(\cosh \frac{\sqrt{\zeta^2 - 1}}{\tau} t + \frac{\zeta}{\sqrt{\zeta^2 - 1}} \sinh \frac{\sqrt{\zeta^2 - 1}}{\tau} t \right) \right]$$

(simpler with $e^{-t/\tau_1}, e^{-t/\tau_2}$)

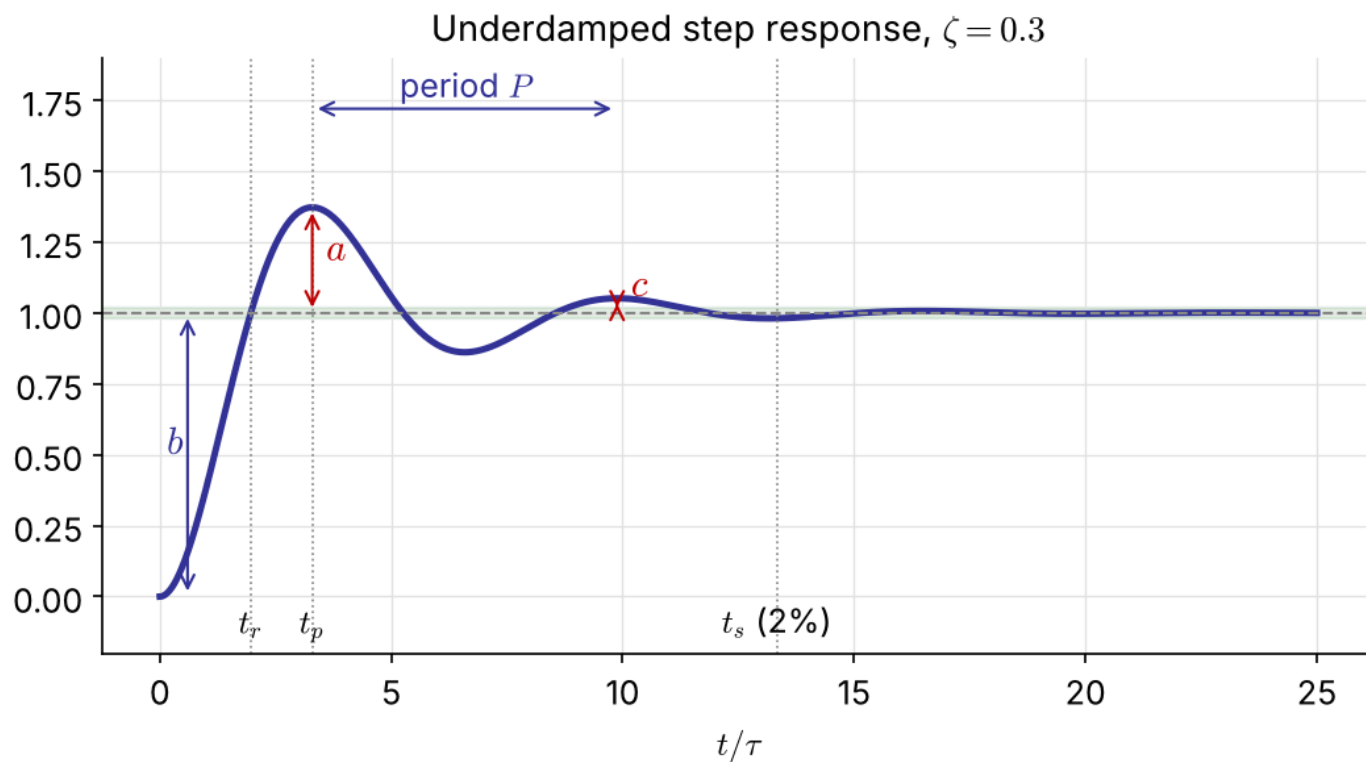
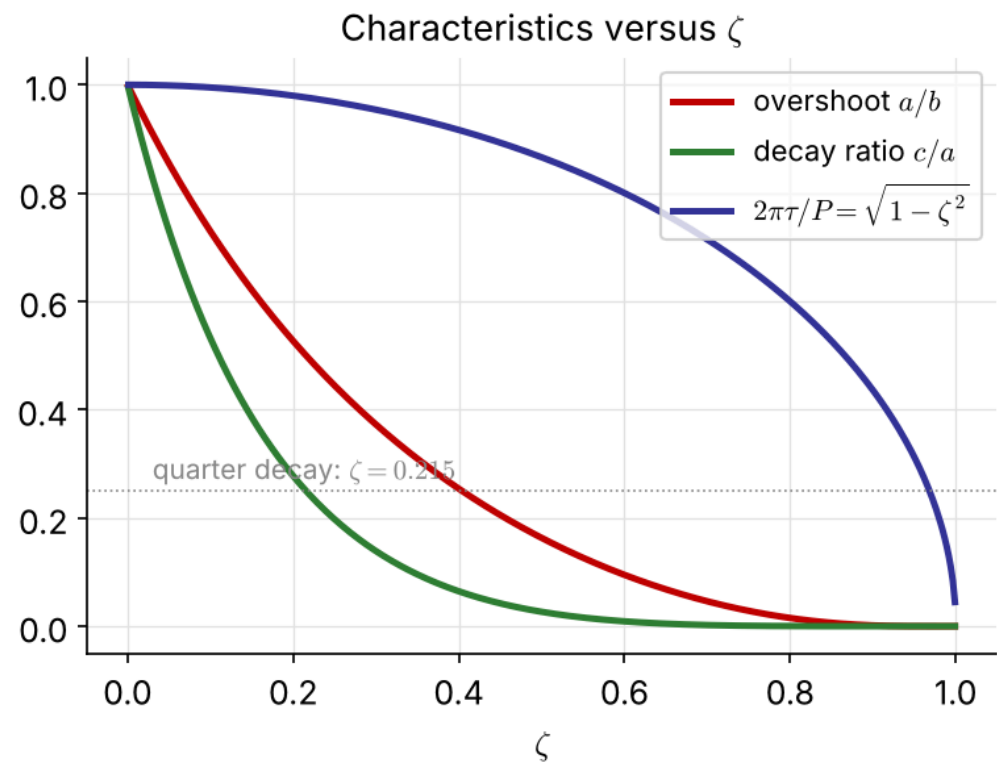
$$\zeta = 1: y = KM \left[1 - \left(1 + \frac{t}{\tau} \right) e^{-t/\tau} \right]$$

$$0 \leq \zeta < 1: y = KM \left[1 - \frac{e^{-\zeta t/\tau}}{\sqrt{1 - \zeta^2}} \sin \left(\frac{\sqrt{1 - \zeta^2}}{\tau} t + \phi \right) \right],$$

$$\phi = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}$$

Rarely used directly; the underdamped one yields the characteristics on the next slide.

Features of the underdamped step response



overshoot

peak time

rise time

period

settling
time

decay ratio

$$\frac{a}{b} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \quad t_p = \frac{\pi\tau}{\sqrt{1-\zeta^2}} \quad t_r = \frac{\tau(\pi - \cos^{-1}\zeta)}{\sqrt{1-\zeta^2}} \quad P = \frac{2\pi\tau}{\sqrt{1-\zeta^2}} \quad t_s \approx \begin{cases} \frac{3\tau}{\zeta} & (5\%) \\ \frac{4\tau}{\zeta} & (2\%) \end{cases} \quad \frac{c}{a} = \text{OS}^2$$

Settling: the envelope $e^{-\zeta t/\tau}$ behaves like a first-order term with time constant τ/ζ . Smaller ζ : faster rise, larger overshoot, longer settling.

Second-order responses in code

Python

```

1 import numpy as np, control as ct, matplotlib.pyplot as plt
2
3 t = np.linspace(0, 15, 1501)
4 for zeta in (0.5, 1.0, 1.5):          #  $s^2 + 2 \zeta s + 1$  ( $\tau = 1$ )
5     G = ct.tf(1, [1, 2*zeta, 1])
6     plt.plot(t, ct.step_response(G, t).outputs, label=f'zeta = {zeta}')
7 plt.legend(); plt.xlabel('t')
8
9 info = ct.step_info(ct.tf(1, [1, 0.6, 1]))    # zeta = 0.3
10 print(info['Overshoot'], info['PeakTime'], info['SettlingTime'])    # 37.2 %, 3.29, ...

```

MATLAB

```

1 t = 0:0.01:15;
2 step(tf(1, [1 1 1]), tf(1, [1 2 1]), tf(1, [1 3 1]), t)    % zeta = 0.5, 1, 1.5
3 legend('\zeta = 0.5', '\zeta = 1', '\zeta = 1.5')
4
5 S = stepinfo(tf(1, [1 0.6 1]))                        % zeta = 0.3
6 % Simulink: Step -> three Transfer Fcn blocks in parallel -> Scope

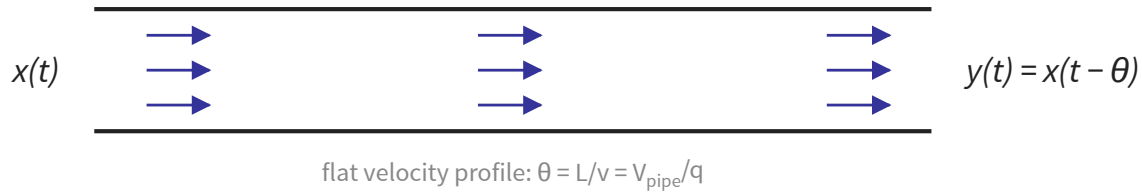
```

Problem #4 (homework)

- Show that the overdamped step response can be written with two exponentials e^{-t/τ_1} and e^{-t/τ_2} , and find τ_1, τ_2 in terms of τ and ζ .
- Starting from the underdamped solution, derive the phase-lag form and
$$\phi = \tan^{-1} \left(\sqrt{1 - \zeta^2} / \zeta \right).$$
- Derive the overshoot, peak time, rise time, period and decay ratio in the table.

1.7 Processes with Dead Time

Dead time (time delay, transport lag)



Caused by fluid travelling down a pipe or a slow analyzer: 시간지연 / 수송지연.

$$y(t) = x(t - t_d) \Rightarrow \frac{Y(s)}{X(s)} = e^{-t_d s}$$

$$\frac{K e^{-t_d s}}{\tau s + 1}, \quad \frac{K e^{-t_d s}}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Classical tools work with **polynomials** in s , so we use the **Padé approximation**, a ratio of polynomials that matches the Taylor series of $e^{-t_d s}$:

$$(1/1) \quad e^{-t_d s} \approx \frac{1 - \frac{t_d}{2}s}{1 + \frac{t_d}{2}s}$$

$$(2/2) \quad e^{-t_d s} \approx \frac{t_d^2 s^2 - 6t_d s + 12}{t_d^2 s^2 + 6t_d s + 12}$$

Each adds poles in the LHP and zeros in the RHP, which produce a small initial **inverse response** instead of a flat delay.

Padé $P_M^N(x) = \sum_0^N a_n x^n / \sum_0^M b_n x^n$, $b_0 = 1$: the $N + M + 1$ coefficients follow from the Taylor coefficients c_n by $a_k = \sum_{j=0}^k c_{k-j} b_j$.

Example 1.10 Padé approximation of $e^{-3s}/(10s + 1)$

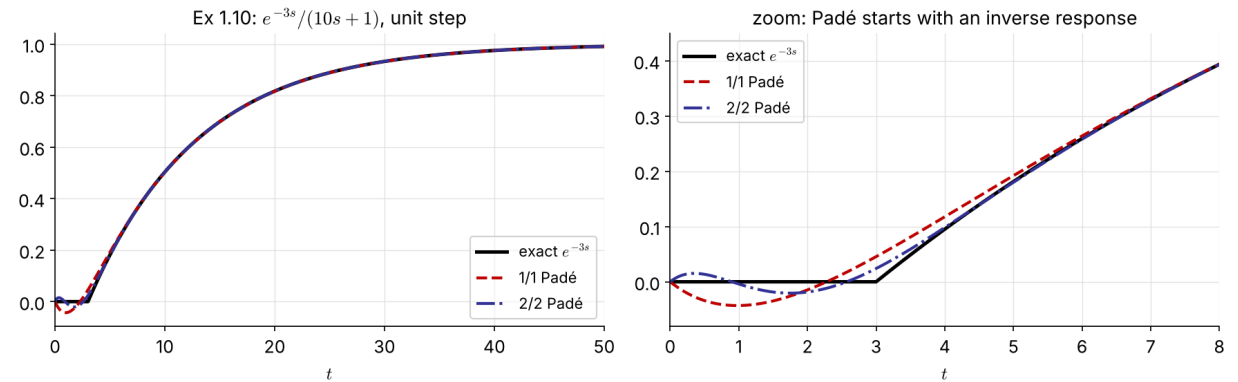
$$\frac{Y}{X} \approx \frac{-1.5s + 1}{(10s + 1)(1.5s + 1)}$$

Python

```
1 import numpy as np, control as ct
2 G = ct.tf(1, [10, 1])
3 num, den = ct.pade(3, 1)          # 1/1 Padé
4 P1 = ct.tf(num, den)
5 t = np.linspace(0, 50, 5001)
6 y1 = ct.step_response(G*P1, t).outputs
7 y0 = ct.step_response(G, t).outputs
8 ye = np.interp(t-3, t, y0, left=0) # exact
```

MATLAB

```
1 G = tf(1, [10 1]);
2 P1 = tf([-1.5 1], [1.5 1]); % pade(3,1)
3 step(G*P1); hold on
4 step(G*exp(-3*tf('s')))) % exact delay
5 legend('Padé 1/1', 'exact')
```



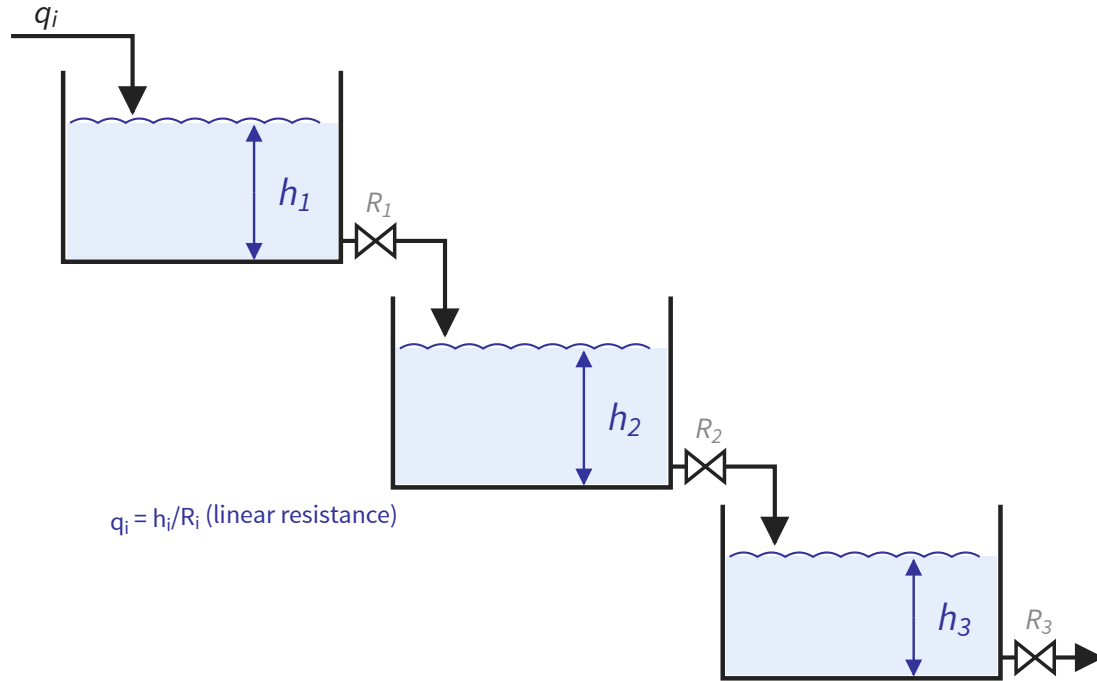
Far from $t = 0$ both approximations are excellent; at early times 1/1 dips to -0.04 , and 2/2 wiggles around zero.

Problem #5 (homework)

- a. Show that the 1/1 Padé approximation is more accurate than the first-order Taylor expansion $e^{-t_d s} \approx 1 - t_d s$. (Compare the Taylor series of both.)
- b. Derive the 2/2 Padé approximation and find its poles and zeros for $t_d = 3$.

1.8 Higher-Order Processes and Approximations

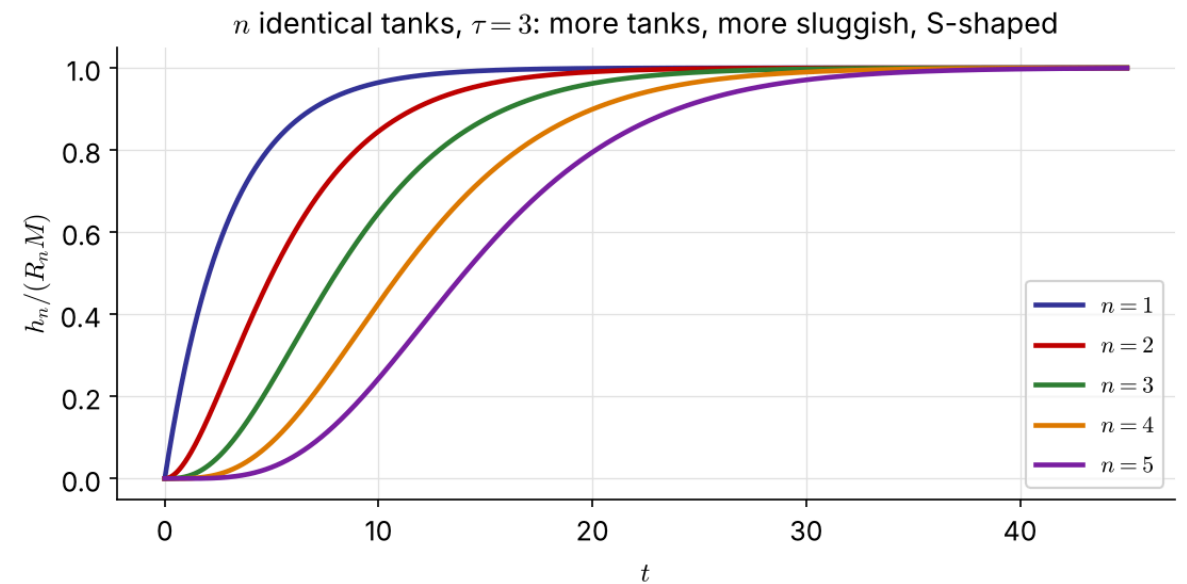
Non-interacting tanks in series



$$A_i \frac{dh_i}{dt} = q_{i-1} - q_i, \quad q_i = \frac{h_i}{R_i} \text{ (linear resistance):}$$

$$\frac{H_i(s)}{H_{i-1}(s)} = \frac{K_i}{\tau_i s + 1}, \quad \tau_i = A_i R_i, \quad K_i = \frac{R_i}{R_{i-1}}$$

$$\frac{H_n(s)}{Q_1(s)} = \frac{R_n}{(\tau_1 s + 1) \cdots (\tau_n s + 1)}$$



The more tanks, the longer the n -th tank waits to “see” a change: the response looks like a **delay** followed by a lag.

Example 1.11 Low-order approximation with dead time

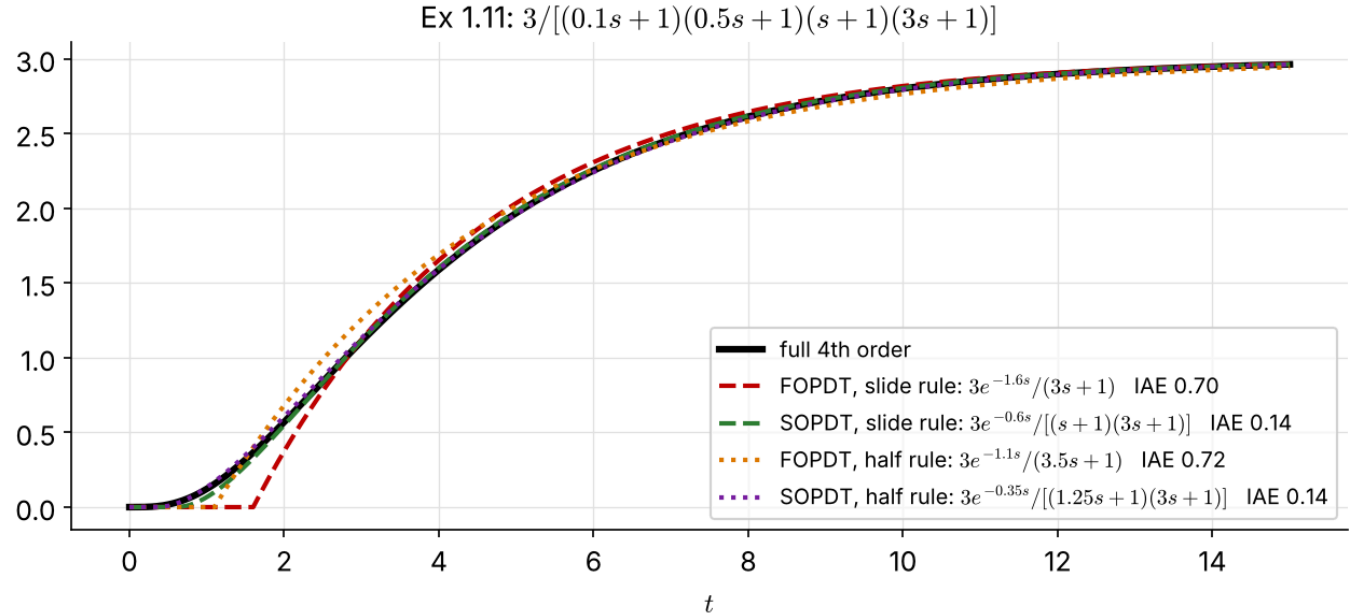
Keep the **dominant** (large) time constants and lump the small ones into a dead time:

$$G(s) \approx \frac{Ke^{-t_d s}}{(\tau_1 s + 1)(\tau_2 s + 1)}, \quad t_d \approx \sum_{i \neq 1,2} \tau_i$$

$$\frac{Y}{X} = \frac{3}{(0.1s + 1)(0.5s + 1)(s + 1)(3s + 1)}$$

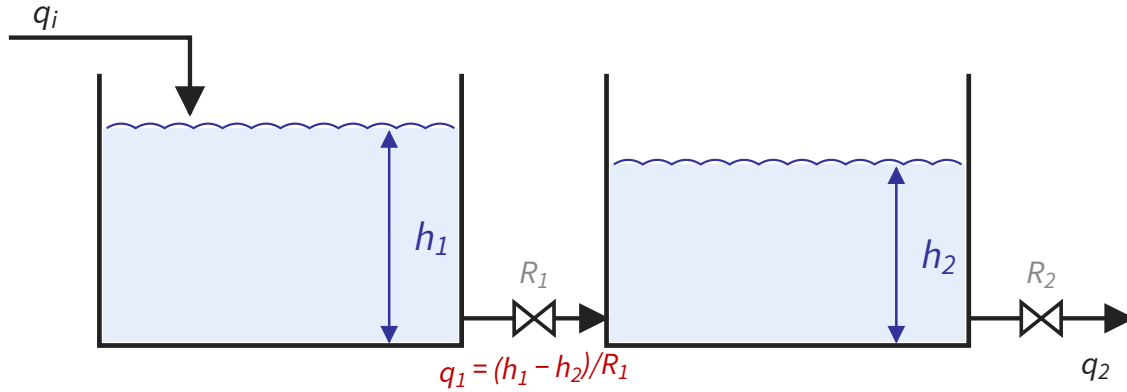
- FOPDT: $\frac{3e^{-1.6s}}{3s + 1}$
- SOPDT: $\frac{3e^{-0.6s}}{(s + 1)(3s + 1)}$

Half rule (Skogestad) : split the largest neglected time constant, half to the lag, half to the delay, e.g. $\frac{3e^{-0.35s}}{(1.25s + 1)(3s + 1)}$.



The second-order approximation is clearly better (IAE 0.14 vs 0.70). FOPDT models are the basis of the empirical tuning rules of Ch 3.

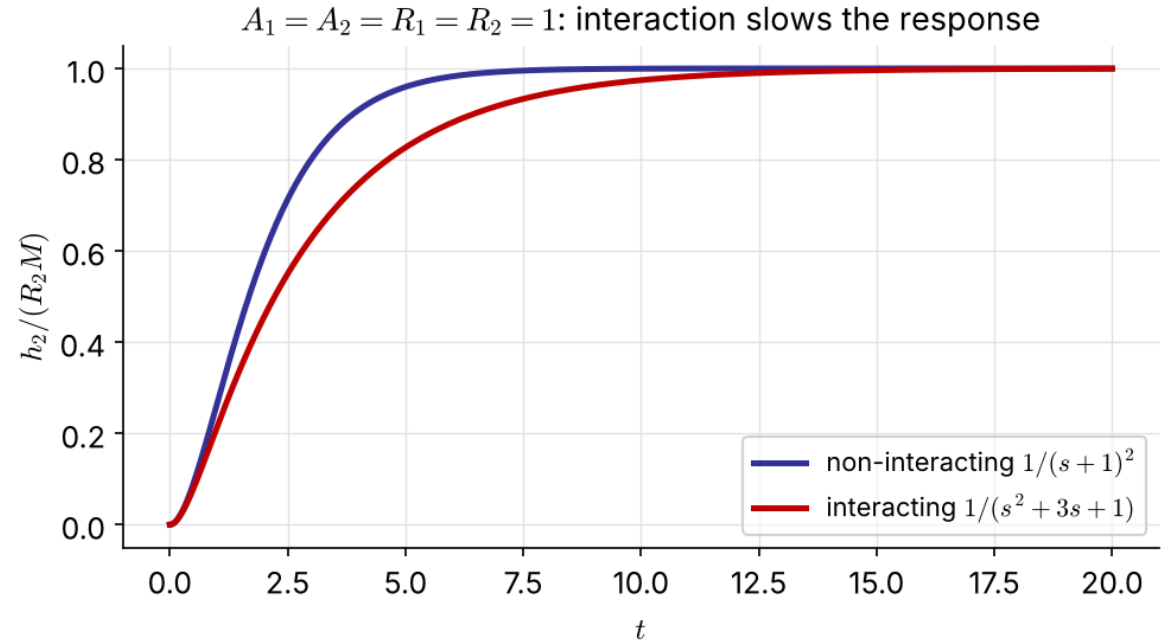
Interacting tanks



$$A_1 \frac{dh_1}{dt} = q_i - \frac{h_1 - h_2}{R_1}, \quad A_2 \frac{dh_2}{dt} = \frac{h_1 - h_2}{R_1} - \frac{h_2}{R_2}$$

$$\frac{H_1}{Q_i} = \frac{R_1 \tau_2 s + R_1 + R_2}{p(s)}, \quad \frac{H_2}{Q_i} = \frac{R_2}{p(s)}$$

$$p(s) = \tau_1 \tau_2 s^2 + \left(\tau_1 + \tau_2 + \tau_1 \frac{R_2}{R_1} \right) s + 1$$



1. Even the first tank is second order (with a zero).
2. Interaction adds $\tau_1 R_2 / R_1$ to the middle coefficient; the poles stay **real and negative**.

Which damps inlet-flow disturbances better? The interacting pair: the extra term pushes one pole closer to the origin, so the response is slower and smoother.

Problem #6 (homework) Blending tank and transfer pipe

A nearly perfectly mixed blending tank (first order) feeds a downstream unit through a long transfer pipe. A test shows that the pipe behaves as if its volume were split into **five** equal perfectly stirred tanks in series.

$$V_{\text{tank}} = 2 \text{ m}^3, \quad V_{\text{pipe}} = 0.1 \text{ m}^3, \quad q_{\text{total}} = 1 \text{ m}^3/\text{min}$$

- What is the most accurate transfer function $C'_{\text{out}}(s)/C'_{\text{in}}(s)$ you can develop?
- What approximate low-order transfer function could represent this process?
- What do you conclude about the need to model the pipe's mixing accurately?
- Simulate the approximate and full-order responses to a step change in C_{in} .

1.9 Effect of Zeros

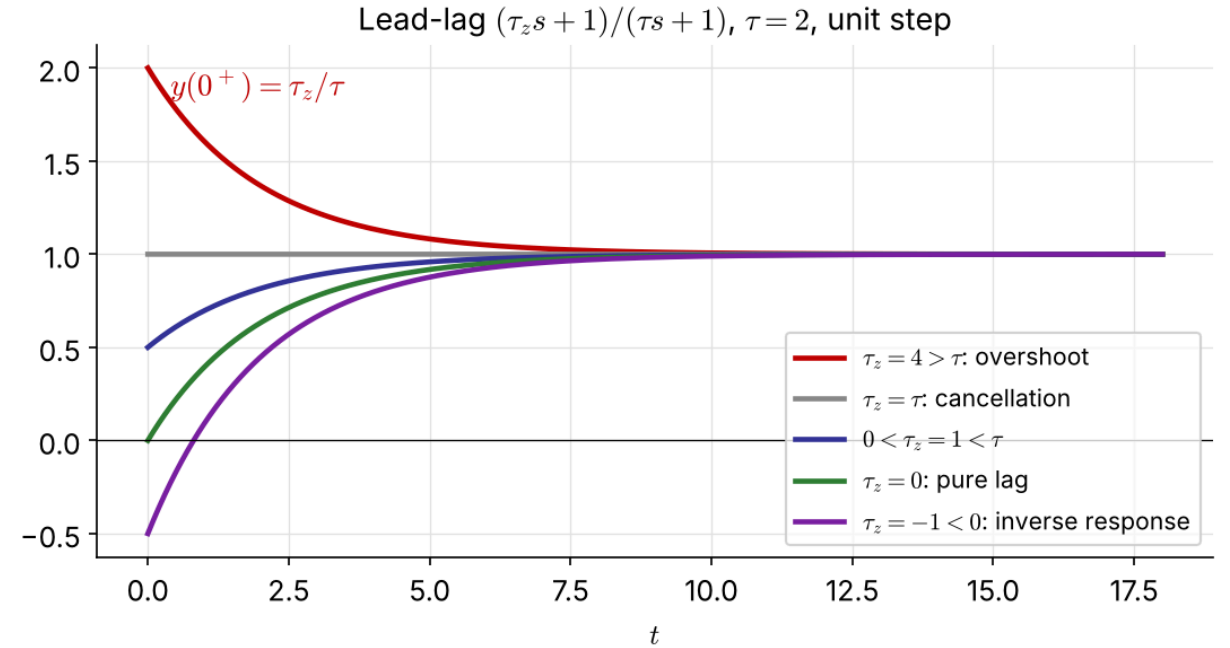
Lead-lag element

Poles govern the inherent dynamics; zeros add finer “fingerprints” by changing the coefficients.

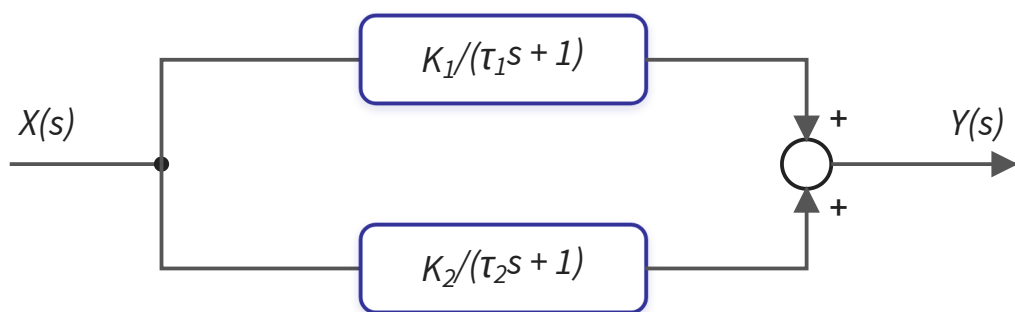
$$\frac{Y}{X} = \frac{\tau_z s + 1}{\tau s + 1} \xrightarrow{X=1/s} Y = \frac{1}{s} + \frac{\tau_z - \tau}{\tau s + 1}$$

$$y(t) = 1 - \left(1 - \frac{\tau_z}{\tau}\right)e^{-t/\tau}, \quad y(0^+) = \frac{\tau_z}{\tau}$$

- the exponential still comes from the pole $-1/\tau$
- $\tau_z > \tau$: jump above 1, **overshoot**
- $0 < \tau_z < \tau$: jump, then rise
- $\tau_z < 0$ (RHP zero): **inverse response**
- $\tau_z = \tau$: pole-zero cancellation, $y = 1$



Transfer functions in parallel



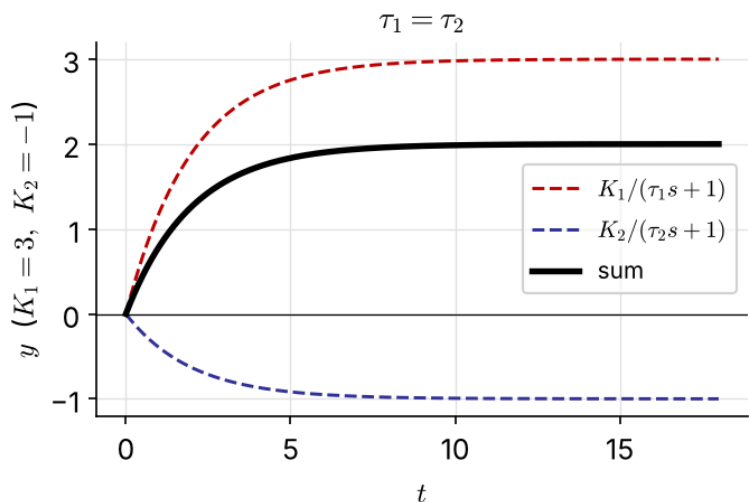
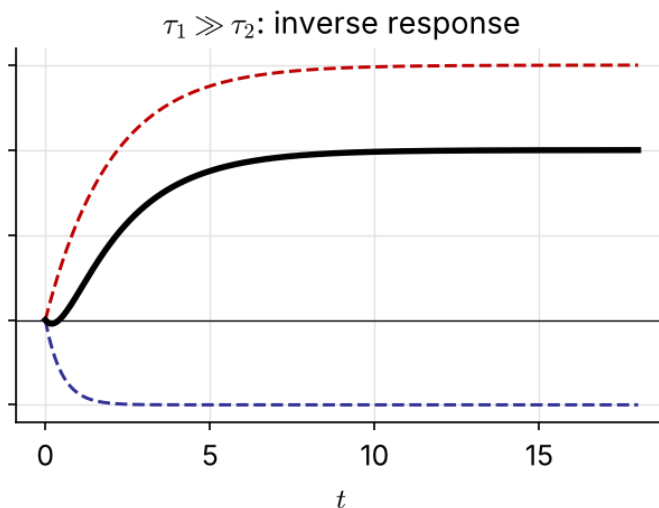
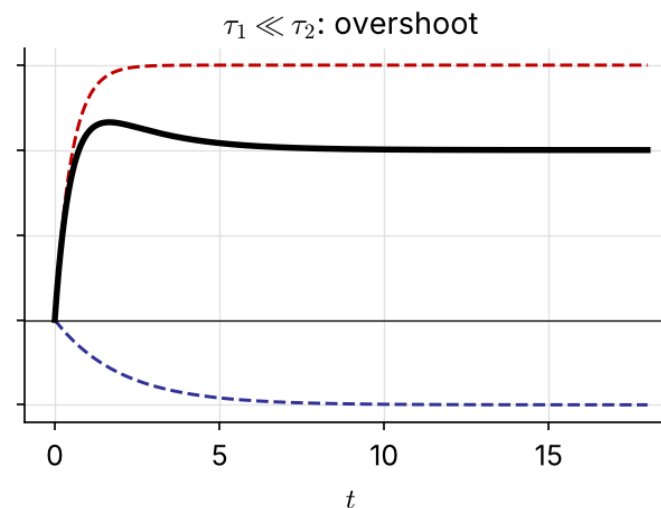
With **opposing** effects ($K_2 < 0$), superposition gives a rough picture right away.

The fast element wins first, the large-gain element wins at the end:
 $\tau_1 \gg \tau_2$ gives $\tau_z < 0$ and an inverse response.

Two competing effects with different time constants:

$$\frac{Y}{X} = \frac{K_1}{\tau_1 s + 1} + \frac{K_2}{\tau_2 s + 1} = \frac{K(\tau_z s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

$$K = K_1 + K_2, \quad \tau_z = \frac{K_1 \tau_2 + K_2 \tau_1}{K_1 + K_2}$$



Problem #7 (homework)

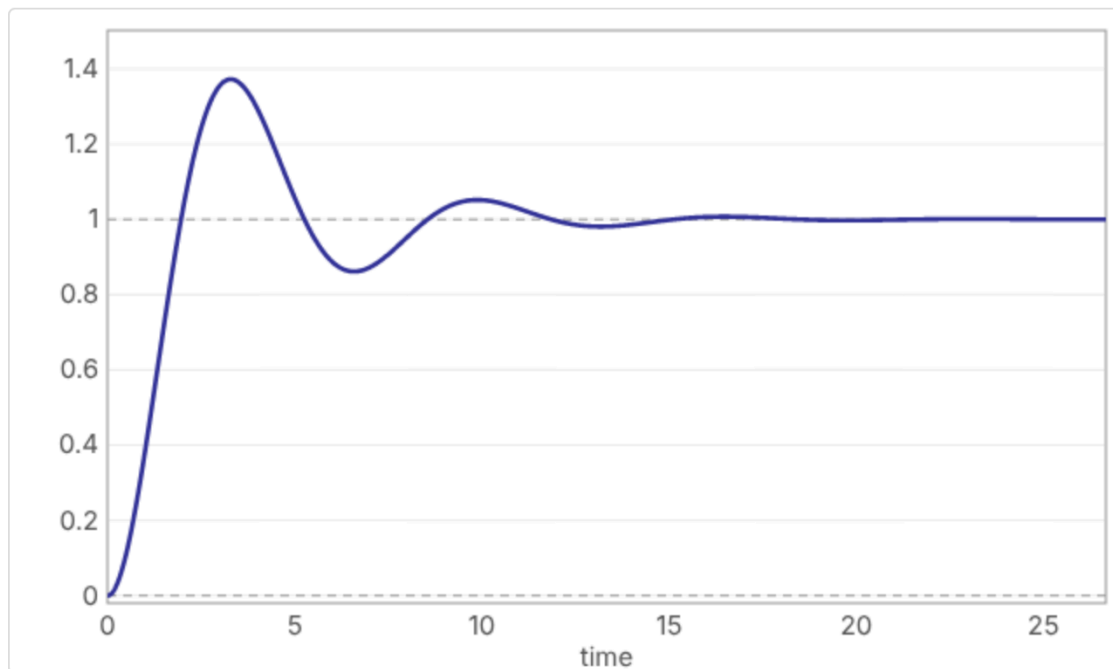
$$\frac{Y}{X} = \frac{K(\tau_z s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)}, \quad \tau_1 > \tau_2$$

Calculate the response to a step of magnitude M and show that

- a. $y(t)$ has an extremum only if $\frac{1 - \tau_z/\tau_2}{1 - \tau_z/\tau_1} > 1$;
- b. overshoot occurs only for $\tau_z/\tau_1 > 1$;
- c. inverse response occurs only for $\tau_z < 0$;
- d. if an extremum exists, find the time at which it occurs.

Interactive: poles, zeros, dead time

model input $K = 1.00$ $\tau = 1.00$
 $\zeta = 0.30$ zero $\tau_z = 0.00$ dead time $\theta = 0.00$



poles: $-0.300 + 0.954j$, $-0.300 - 0.954j$

zero: none

final value $K = 1.00$

overshoot **37.2%** at $t = 3.29$

theory: OS = **37.2%**, DR = **0.139**

2% settling time **11.23**

$G(s) = K(\tau_z s + 1)e^{-\theta s} / (\tau s + 1)$ or $/(\tau^2 s^2 + 2\zeta\tau s + 1)$

Interactive version: seoultechpse.github.io/ProCon/ch1/ch1_dynamics.html — first or second order, K , τ , ζ , zero τ_z , dead time θ .

Summary

Tools

- deviation variables $\rightarrow$ zero initial conditions
- linearize about the operating point
- Laplace transform + table; final value only if stable
- partial fractions: one term per pole
- $G(s) = Q(s)/P(s)$, gain $K = G(0)$

Responses

- first order: τ (63.2%), K ; integrator K_I/s
- second order: ζ sets over/critical/under damping; OS, DR = OS^2 , $t_s \approx 4\tau/\zeta$
- dead time $e^{-t_d s} \rightarrow$ Padé; high order $\rightarrow$ FOPDT/SOPDT
- zeros: overshoot, inverse response (RHP zero)

Next (Ch 2): close the loop with a controller and see how K_c, τ_I, τ_D move these poles.

Resources

- **Slides (interactive HTML):** seoultechpse.github.io/ProCon/ch1/ch1_dynamics.html
- **PDF:** [slides](#) · [handout with note space](#)
- **Python notebook:** [Open in Colab](#) — every transform (sympy), calculation and figure of this chapter
- **Response explorer:** [standalone page](#)