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# Process Dynamics and Control

## 1. Mathematical Preliminaries and Dynamic Response

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### What are we up to?

Before we design controllers we need a language for **how processes respond in time**: models in deviation variables, the Laplace transform, and transfer functions whose **poles** tell us the shape of the response without solving for it.

| Section | Topic                                              | Key result                                                     |
|---------|----------------------------------------------------|----------------------------------------------------------------|
| 1.1     | process models, deviation variables, linearization | linear ODEs with zero initial conditions                       |
| 1.2     | Laplace transform                                  | table look-up, initial- and final-value theorems               |
| 1.3     | partial-fraction expansion                         | each pole gives one term of $f(t)$                             |
| 1.4     | transfer functions, poles and zeros                | poles fix the time dependence, zeros the coefficients          |
| 1.5–1.6 | first- and second-order systems                    | $\tau$ , $K$ , $\zeta$ ; overshoot, decay ratio, settling time |
| 1.7–1.9 | dead time, higher order, zeros                     | Padé, FOPDT approximation, inverse response                    |

Any linear model breaks into first-order terms by partial fractions, and complex-conjugate pairs combine into second-order terms, so first- and second-order systems are all we need to study in detail.




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## 1.1 Process Models and Deviation Variables




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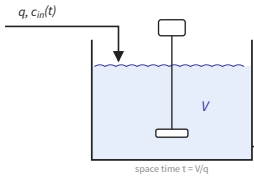
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## A simple differential-equation model

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Mass balance of a well-mixed tank:

$$\tau \frac{dc}{dt} = c_{in} - c, \quad c(0) = c_0, \quad \tau = \frac{V}{q}$$

Define deviation variables from the initial steady state:

$$\bar{c} = c - c_0, \quad \bar{c}_{in} = c_{in} - c_0.$$

$$\tau \frac{d\bar{c}}{dt} + \bar{c} = \bar{c}_{in}, \quad \bar{c}(0) = 0$$

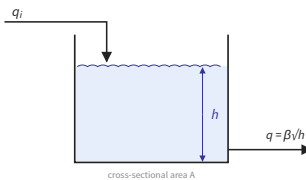
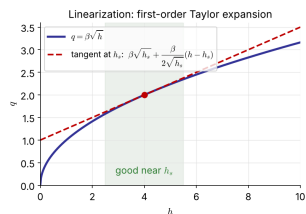
Why deviation variables? The initial condition becomes zero, so the Laplace transform of the model contains no initial-condition terms and gives a clean input-output ratio. For reference, the solution

$$\text{is the convolution } \bar{c}(t) = \frac{1}{\tau} \int_0^t \bar{c}_{in}(\alpha) e^{-(t-\alpha)/\tau} d\alpha.$$

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## Linearization of nonlinear models

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$$A \frac{dh}{dt} = q_i - \beta\sqrt{h}, \quad h(0) = h_s$$

Steady state:  $0 = q_{i,s} - \beta\sqrt{h_s}$ . Taylor-expand  $\sqrt{h}$  about  $h_s$ :

$$A \frac{dh}{dt} \simeq q_i - \beta \left[ \sqrt{h_s} + \frac{1}{2\sqrt{h_s}}(h - h_s) \right]$$

Subtract the steady state and use  $\bar{q}_i = q_i - q_{i,s}$ ,  $\bar{h} = h - h_s$ :

$$A \frac{d\bar{h}}{dt} + \frac{\beta}{2\sqrt{h_s}} \bar{h} = \bar{q}_i, \quad \bar{h}(0) = 0$$

General case (one state  $y$ , inputs  $u$ ): with  $y' = y - y_s$ ,  $u' = u - u_s$ ,

$$\frac{dy'}{dt} = \frac{\partial f}{\partial y} \Big|_s y' + \nabla_u^T f \Big|_s u'$$

The model is accurate only near the operating point.

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## Example 1.1 Linearize a mixed vessel

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$$V \frac{dc}{dt} = q c_i - q c \quad (q \text{ and } c_i \text{ both vary with time})$$

The product terms are nonlinear. Expand about  $(q_s, c_{i,s}, c_s)$ :

$$q c_i \simeq q_s c_{i,s} + c_{i,s} q' + q_s c'_i$$

$$q c \simeq q_s c_s + c_s q' + q_s c'$$

Subtract the steady state  $0 = q_s(c_{i,s} - c_s)$ :

$$V \frac{dc'}{dt} = q_s(c'_i - c') + (c_{i,s} - c_s) q'$$

At steady state  $c_s = c_{i,s}$ , so the flow term vanishes:

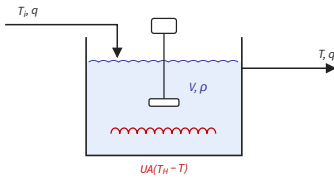
$$\frac{V}{q_s} \frac{dc'}{dt} + c' = c'_i$$

To first order, a flow change does not move the concentration; only  $c_i$  does.

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### Example 1.2 Stirred-tank heater

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$$\rho C_p V \frac{dT}{dt} = \rho C_p q (T_i - T) + UA(T_H - T)$$

With  $\tau = V/q$  and  $\kappa = UA/(\rho C_p q)$ , in deviation variables:

$$\tau \frac{d\bar{T}}{dt} + (1 + \kappa)\bar{T} = \bar{T}_i + \kappa\bar{T}_H$$

Pole-zero form

$$T(s) = \frac{K_i}{s + a} T_i(s) + \frac{K_H}{s + a} T_H(s)$$

$$a = \frac{1 + \kappa}{\tau}, K_i = \frac{1}{\tau}, K_H = \frac{\kappa}{\tau}$$

Time-constant form (used in control)

$$T(s) = \frac{K_d}{\tau_p s + 1} T_i(s) + \frac{K_p}{\tau_p s + 1} T_H(s)$$

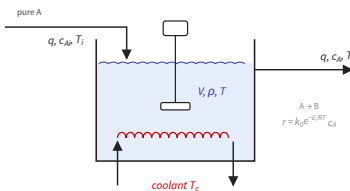
$$\tau_p = \frac{\tau}{1 + \kappa}, K_d = \frac{1}{1 + \kappa}, K_p = \frac{\kappa}{1 + \kappa}$$

One output, two inputs: the heater temperature  $T_H$  will be our manipulated variable, the inlet temperature  $T_i$  a disturbance. Both see the same pole  $-1/\tau_p$ .

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### Example 1.3 Non-isothermal CSTR

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$$V \frac{dc_A}{dt} = q(c_{A,i} - c_A) - V k(T) c_A$$

$$\rho V C_p \frac{dT}{dt} = \rho q C_p (T_i - T) + (-\Delta H_R) V k(T) c_A + UA(T_c - T)$$

$k(T)c_A$  is nonlinear in both states. With  $q, c_{A,i}, T_i$  constant, linearize ( $k'_s = k_s E/RT_s^2$ ):

$$\frac{d}{dt} \begin{bmatrix} c'_A \\ T' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c'_A \\ T' \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} T'_c$$

$$a_{11} = -\frac{q}{V} - k_s, a_{12} = -k'_s c_{A,s}, a_{21} = \frac{(-\Delta H_R) k_s}{\rho C_p},$$

$$a_{22} = -\frac{q}{V} - \frac{UA}{\rho V C_p} + \frac{(-\Delta H_R) k'_s c_{A,s}}{\rho C_p}, b_2 = \frac{UA}{\rho V C_p}$$

$$\frac{C'_A(s)}{T'_c(s)} = \frac{a_{12} b_2}{(s - a_{11})(s - a_{22}) - a_{12} a_{21}}$$

second order: the two balances are coupled.

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### Problem #1 (homework)

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For the CSTR of Example 1.3, find all the approximate transfer functions relating the outputs  $c_A$  and  $T$  to the inputs  $c_{A,i}, T_i$  and  $T_c$ .

Show that all six transfer functions share **one** denominator, and explain why.

Hint: write the Laplace-transformed linear model as  $(s\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{B}\mathbf{u}$  and solve with Cramer's rule; the denominator is  $\det(s\mathbf{I} - \mathbf{A})$ .

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## 1.2 Laplace Transform

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### Definition and the look-up table approach

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$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st} dt = F(s), \quad f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} F(s)e^{st} ds$$

- In control we **never** use the inverse integral: we break  $F(s)$  into simple parts and read the answer from a **table**.

- $\mathcal{L}$  is a **linear operator**:

$$\mathcal{L}[af_1 + bf_2] = aF_1(s) + bF_2(s)$$

- A linear ODE becomes an **algebraic** equation in  $s$

The real goal: predict the **time response without the inverse transform**. The answer lies in the roots of the characteristic equation (§1.4).

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### Transforms of common functions

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| $f(t)$                                         | $F(s)$                                                              |
|------------------------------------------------|---------------------------------------------------------------------|
| $\delta(t)$ (unit impulse)                     | 1                                                                   |
| $u(t)$ (unit step)                             | $\frac{1}{s}$                                                       |
| $t$ (ramp)                                     | $\frac{1}{s^2}$                                                     |
| $t^{n-1}/(n-1)!$                               | $\frac{1}{s^n}$                                                     |
| $e^{-at}$                                      | $\frac{1}{s+a}$                                                     |
| $t^{n-1}e^{-at}/(n-1)!$                        | $\frac{1}{(s+a)^n}$                                                 |
| $\sin \omega t, \cos \omega t$                 | $\frac{\omega}{s^2 + \omega^2}, \frac{s}{s^2 + \omega^2}$           |
| $e^{-at} \sin \omega t, e^{-at} \cos \omega t$ | $\frac{\omega}{(s+a)^2 + \omega^2}, \frac{s+a}{(s+a)^2 + \omega^2}$ |

| Operation           | Transform                 |
|---------------------|---------------------------|
| $\frac{df}{dt}$     | $sF(s) - f(0)$            |
| $\frac{d^2f}{dt^2}$ | $s^2F(s) - sf(0) - f'(0)$ |
| $\int_0^t f dt$     | $\frac{F(s)}{s}$          |
| $f(t-t_0)u(t-t_0)$  | $e^{-st_0}F(s)$           |
| $e^{-at}f(t)$       | $F(s+a)$                  |

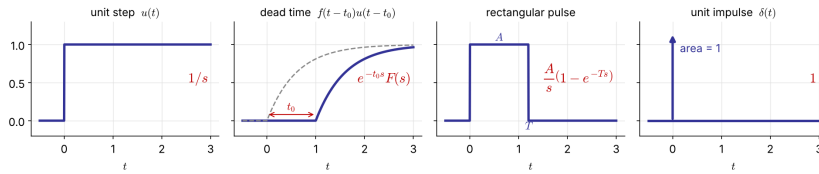
Derivation by integration by parts, e.g.  $\mathcal{L}[df/dt] = fe^{-st}|_0^{\infty} + s \int_0^{\infty} fe^{-st} dt = sF(s) - f(0)$ . With deviation variables all  $f^{(0)}(0) = 0$ , so  $d/dt \rightarrow s$  and  $\int dt \rightarrow 1/s$ .

Try: derive  $\mathcal{L}[\cos \omega t]$  from  $\cos \omega t = (e^{j\omega t} + e^{-j\omega t})/2$

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## Inputs common to control problems

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- Pulse:  $A[u(t) - u(t-T)] \Rightarrow \frac{A}{s}(1 - e^{-Ts})$
- Impulse:  $\delta(t) = \lim_{T \rightarrow 0} \frac{u(t) - u(t-T)}{T}$ , so  $\mathcal{L}[\delta] = \lim_{T \rightarrow 0} \frac{1 - e^{-Ts}}{Ts} = 1$
- $\int_{-\infty}^{\infty} \delta(t) dt = 1$ ,  $\int_{-\infty}^{\infty} f(t)\delta(t-t_0) dt = f(t_0)$
- Dead time (time shift) multiplies by  $e^{-st_0}$ : an exponential, not a polynomial, in  $s$  (§1.7).

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## Initial- and final-value theorems

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Values of  $f(t)$  at the two extremes **without** the inverse transform:

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s), \quad \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

Proof of the final value:

$$\lim_{s \rightarrow 0} \int_0^{\infty} \frac{df}{dt} e^{-st} dt = \int_0^{\infty} df = f(\infty) - f(0), \text{ and the left side is } \lim_{s \rightarrow 0} [sF(s) - f(0)].$$

The final-value theorem holds only if  $f(\infty)$  exists: all poles of  $sF(s)$  must lie in the left half-plane.

**Example 1.4** What is  $f(\infty)$ ?

- $F = \frac{6(s-2)(s+2)}{s(s+1)(s+3)(s+4)}$ :  $sF(0) = \frac{6(-2)(2)}{(3)(4)} = -2$
- $F = \frac{1}{s-2}$ , i.e.  $f = e^{2t}$ :  $sF(0) = 0$ , but  $f \rightarrow \infty$ : theorem not applicable (pole at  $+2$ ).
- $F = \frac{6(s+3)}{s^3 + s^2 - 4s - 4} = \frac{6(s+3)}{(s+1)(s+2)(s-2)}$ : again a pole at  $+2$ ;  $f(\infty)$  does not exist.

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## 1.3 Partial-Fraction Expansion

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Breaking  $F(s)$  into table entries

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$$F(s) = \frac{q(s)}{p(s)} = \frac{\alpha_1}{s+a_1} + \frac{\alpha_2}{s+a_2} + \dots + \frac{\alpha_n}{s+a_n} \Rightarrow f(t) = \sum_i \alpha_i e^{-a_i t}$$

$-a_i$  are the roots of  $p(s) = 0$  (order  $n >$  order of  $q$ ). Heaviside's cover-up rule:

$$\alpha_i = (s + a_i) \frac{q(s)}{p(s)} \Big|_{s=-a_i}$$

1. The time dependence of  $f(t)$  comes entirely from the roots of the denominator.
2. The numerator affects only the coefficients  $\alpha_i$ .

3. Poles closer to the origin give terms that decay more slowly; they are dominant.
4. Three cases: distinct real, complex, and repeated roots.

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## Example 1.5 Case 1: distinct real roots

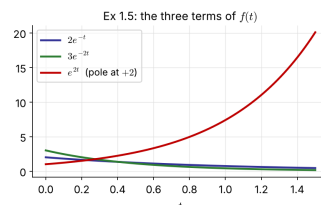
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$s^3 + s^2 - 4s - 4 = (s+1)(s+2)(s-2)$ . Two numerators, same denominator:

|                                              | $\frac{6s^2 - 12}{(s+1)(s+2)(s-2)}$ | $\frac{6s}{(s+1)(s+2)(s-2)}$ |
|----------------------------------------------|-------------------------------------|------------------------------|
| $\alpha_1 = \frac{q}{(s+2)(s-2)} \Big _{-1}$ | $\frac{-6}{-3} = 2$                 | $\frac{-6}{-3} = 2$          |
| $\alpha_2 = \frac{q}{(s+1)(s-2)} \Big _{-2}$ | $\frac{12}{4} = 3$                  | $\frac{-12}{4} = -3$         |
| $\alpha_3 = \frac{q}{(s+1)(s+2)} \Big _2$    | $\frac{12}{12} = 1$                 | $\frac{12}{12} = 1$          |

$$f(t) = 2e^{-t} + 3e^{-2t} + e^{2t} \quad 2e^{-t} - 3e^{-2t} + e^{2t}$$

Same three exponentials, different coefficients: the numerator changes only the weights.



The same procedure gives

$$\frac{6}{(s+1)(s+2)(s+3)} = \frac{3}{s+1} - \frac{6}{s+2} + \frac{3}{s+3}$$

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## Example 1.6 Case 2: complex roots

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$$F(s) = \frac{s+5}{s^2+4s+13}, \quad s^2+4s+13=0 \Rightarrow s=-2 \pm 3j$$

Method 1: cover-up with complex numbers

Method 2: complete the square (faster)

$$\alpha = \frac{s+5}{s-(-2-3j)} \Big|_{s=-2-3j} = \frac{3+3j}{6j} = \frac{1-j}{2}, \quad \alpha^* = \frac{1+j}{2} \quad \frac{s+5}{(s+2)^2+3^2} = \frac{s+2}{(s+2)^2+3^2} + \frac{3}{(s+2)^2+3^2}$$

$$f(t) = \frac{1-j}{2} e^{(-2+3j)t} + \frac{1+j}{2} e^{(-2-3j)t}$$

$$f(t) = e^{-2t} (\cos 3t + \sin 3t)$$

Euler's formula collects the terms into real functions.

Phase-lag form (the one we use):  $\cos 3t + \sin 3t = \sqrt{2} \sin(3t + \phi)$ ,  $\phi = \tan^{-1} 1 = \pi/4$ , so  $f(t) = \sqrt{2} e^{-2t} \sin(3t + \pi/4)$ . Roots  $-a \pm bj$  always give  $e^{-at}$  times an oscillation of frequency  $b$

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**Example 1.7 Case 3: repeated roots**

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$$\frac{2}{(s+1)^3(s+2)} = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{(s+1)^2} + \frac{\alpha_3}{(s+1)^3} + \frac{\alpha_4}{s+2}$$

$$\alpha_4 = \frac{2}{(s+1)^3} \Big|_{s=-2} = -2. \text{ Multiply by } (s+1)^3:$$

$$f(t) = 2 \left[ \left(1 - t + \frac{t^2}{2}\right) e^{-t} - e^{-2t} \right]$$

$$\frac{2}{s+2} = \alpha_1(s+1)^2 + \alpha_2(s+1) + \alpha_3 + \alpha_4 \frac{(s+1)^3}{s+2}$$

Set  $s = -1$ , then differentiate and set  $s = -1$  again:

$$\alpha_3 = 2, \quad \alpha_2 = -\frac{2}{(s+2)^2} \Big|_{s=-1} = -2, \quad \alpha_1 = \frac{1}{2} \frac{4}{(s+2)^3} \Big|_{s=-1}$$

In general

$$\mathcal{L}^{-1} \left[ \sum_{k=1}^n \frac{\alpha_k}{(s+a)^k} \right] = \left[ \alpha_1 + \alpha_2 t + \dots + \frac{\alpha_n t^{n-1}}{(n-1)!} \right] e^{-at}$$

Still based on the root  $-a$ , but the polynomial in  $t$  makes it decay more slowly.

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**Partial fractions by computer**

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Python

```
1 import sympy as sp
2 s, t = sp.symbols('s t'); tp = sp.Symbol('t', positive=True)
3
4 for F in [(6*s**2 - 12)/(s**3 + s**2 - 4*s - 4), # Ex 1.5
5           (s + 5)/(s**2 + 4*s + 13), # Ex 1.6
6           2/((s + 1)**3*(s + 2))]: # Ex 1.7
7     print(sp.apart(F, s), ' -> ', sp.inverse_laplace_transform(F, s, tp))
8
9 import numpy as np, scipy.signal as sig
10 r, p, k = sig.residue([6, 0, -12], [1, 1, -4, -4]) # residues, poles, direct term
11 print(np.round(r, 4), np.round(p, 4)) # [3, 2, 1] at [-2, -1, 2]
```

MATLAB

```
1 [r, p, k] = residue([6 0 -12], [1 1 -4 -4]) % Ex 1.5: residues and poles
2 roots([1 1 -4 -4]) % 2, -2, -1
3
4 syms s t
5 ilaplace((s + 5)/(s**2 + 4*s + 13)) % Ex 1.6
6 ilaplace(2/((s + 1)**3*(s + 2))) % Ex 1.7
7 partfrac(2/((s + 1)**3*(s + 2)), s)
```

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**Problem #2 (homework)**

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a. Find  $f(t)$  for

$$F(s) = \frac{s + \beta}{(s + a)^2 + \omega^2}$$

b. Find  $f(t)$  for

$$F(s) = \frac{s \sin \phi + \omega \cos \phi}{s^2 + \omega^2}$$

Check your answers with `sympy.inverse_laplace_transform` or MATLAB `ilaplace`.

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## 1.4 Transfer Functions, Poles and Zeros

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### The transfer function

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For a linear ODE with zero initial conditions ( $n \geq m$ )

$$a_n y^{(n)} + \dots + a_1 y' + a_0 y = b_m x^{(m)} + \dots + b_1 x' + b_0 x$$

the Laplace transform gives the **transfer function**, the ratio of output to input:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0} = \frac{Q(s)}{P(s)}$$

Pole-zero form

$$G(s) = \frac{b_m (s - z_1) \dots (s - z_m)}{a_n (s - p_1) \dots (s - p_n)}$$

$P(s)$  is the characteristic polynomial; its roots are the poles, the roots of  $Q(s)$  the zeros.

Time-constant form (real roots)

$$G(s) = K \frac{(\tau_a s + 1) \dots (\tau_m s + 1)}{(\tau_1 s + 1) \dots (\tau_n s + 1)}$$

Steady-state gain: final change of  $y$  per unit step in  $x$ ,

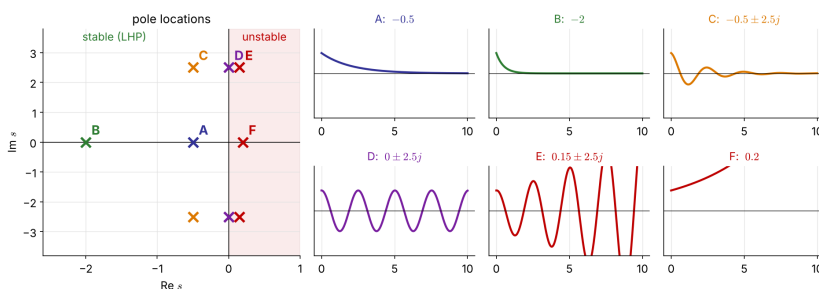
$$K = \lim_{s \rightarrow 0} s G(s) \frac{1}{s} = G(0) = \frac{b_0}{a_0}$$

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### Poles tell the shape of the response

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Each pole contributes a term  $e^{pt}$ ; the real part sets decay or growth, the imaginary part the oscillation



- real pole  $p \rightarrow e^{pt}$ ; repeated  $m$  times  $\rightarrow (a_1 + \dots + a_m t^{m-1}) e^{pt}$
- complex pair  $-\zeta/\tau \pm j\sqrt{1-\zeta^2}/\tau \rightarrow e^{-\zeta t/\tau} \sin(\sqrt{1-\zeta^2} t/\tau + \phi)$

- the discussion of poles is **independent of the input**; the actual  $y(t)$  also depends on  $x(t)$
- zeros change only the coefficients
- Stability**: all poles must have **negative real parts** (Ch 4).

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## Back to the first-order tank: step and impulse

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From §1.1,  $\tau \frac{d\bar{c}}{dt} + \bar{c} = \bar{c}_{in} \Rightarrow (\tau s + 1)C(s) = C_{in}(s) \Rightarrow G(s) = \frac{C(s)}{C_{in}(s)} = \frac{1}{\tau s + 1}$ , one pole at  $-1/\tau$ .

Unit step  $C_{in} = 1/s$ 

$$C = \frac{1}{(\tau s + 1)s} = \frac{1}{s} - \frac{1}{s + 1/\tau}$$

$$\bar{c}(t) = 1 - e^{-t/\tau}$$

Unit impulse  $C_{in} = 1$ 

$$C = \frac{1}{\tau s + 1} = \frac{1}{\tau} \frac{1}{s + 1/\tau}$$

$$\bar{c}(t) = \frac{1}{\tau} e^{-t/\tau}$$

Without specifying the input we already know  $\bar{c}(t)$  must contain  $e^{-t/\tau}$ : that is what the pole tells us.

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## 1.5 First-Order Systems

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## First-order model

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$$a_1 \frac{dy}{dt} + a_0 y = b x(t) \Rightarrow \tau \frac{dy}{dt} + y = K x(t), \quad G(s) = \frac{Y(s)}{X(s)} = \frac{K}{\tau s + 1}$$

$\tau = a_1/a_0$ : time constant (시간상수),  $K = b/a_0$ : steady-state gain (정상상태 이득), one real pole at  $-1/\tau$ .

Step  $x = Mu(t)$ :

$$Y = \frac{KM}{s(\tau s + 1)} \Rightarrow y(t) = KM(1 - e^{-t/\tau})$$

Impulse  $x = M\delta(t)$ :

$$y(t) = \frac{KM}{\tau} e^{-t/\tau}$$

Final value check:  $\lim_{s \rightarrow 0} s \frac{KM}{s(\tau s + 1)} = KM$ .

| $t$     | $y/KM = 1 - e^{-t/\tau}$ |
|---------|--------------------------|
| $\tau$  | 0.632                    |
| $2\tau$ | 0.865                    |
| $3\tau$ | 0.950                    |
| $4\tau$ | 0.982                    |
| $5\tau$ | 0.993                    |

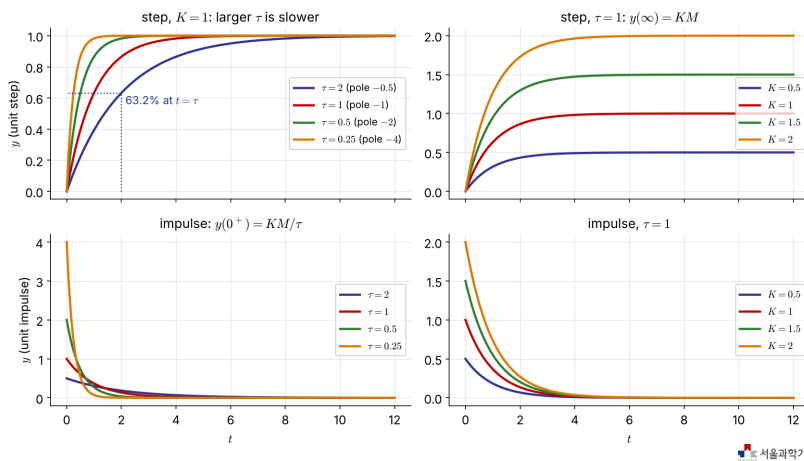
After  $3\tau$  within 5%, after  $5\tau$  within 1%: "settled".

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Effect of  $\tau$  and  $K$ 

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The larger the time constant, the slower the response. The larger the gain, the more sensitive the output to the input.



## Ramp input and the integrating process

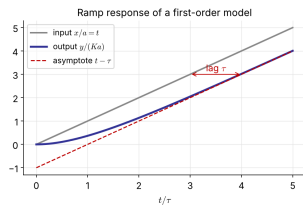
29

Ramp  $x = at$ ,  $X = a/s^2$ :

$$Y = \frac{Ka}{s^2(\tau s + 1)} = \frac{Ka\tau^2}{\tau s + 1} - \frac{Ka\tau}{s} + \frac{Ka}{s^2}$$

$$y(t) = Ka\tau(e^{-t/\tau} - 1) + Kat$$

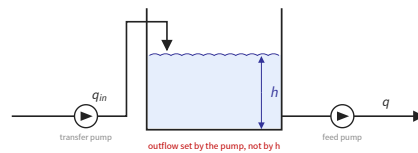
→  $Ka(t - \tau)$  for  $t \gg \tau$ : the output lags the input by  $\tau$ .



Integrating process ( $a_0 = 0$ ):

$$\frac{dy}{dt} = K_I x(t) \Rightarrow G(s) = \frac{K_I}{s}$$

a pole at the origin.

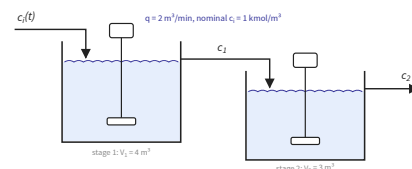


$$A \frac{dh}{dt} = \bar{q}_{in} - \bar{q} \Rightarrow H(s) = \frac{1}{A} \frac{Q_{in}(s) - Q(s)}{s}$$

A step input gives a ramp; an impulse is not forgotten, the level keeps whatever was added. Not self-regulating: levels need control.

## Example 1.8 Two stirred tanks in series

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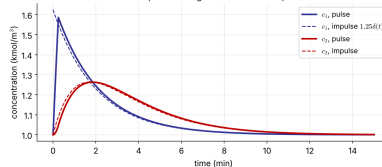


Feed:  $c_1 = 6$  for  $0 < t < 0.25$  min, otherwise 1, so  $\bar{c}_1$  is a pulse of height 5:

$$C_1(s) = \frac{5}{s} (1 - e^{-s/4}) \quad \text{or, as an impulse, } C_1(s) = \frac{5}{4}$$

$$C_2(s) = \frac{5(1 - e^{-s/4})}{s(2s + 1)(1.5s + 1)} \quad \text{or} \quad \frac{5/4}{(2s + 1)(1.5s + 1)}$$

Ex 1.8: a 0.25-min pulse of height 5 is almost an impulse of area 1.25



The original MATLAB script used `pulseLength = 1.25`, a pulse 5x too long.

**Problem #3 (homework)**

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For Example 1.8, carry out the inverse transforms:

- the impulse response  $\bar{e}_2(t)$  for  $C_i = 5/4$ ;
- the pulse response  $\bar{e}_2(t)$ , using the dead-time property for the factor  $e^{-s/4}$ .

Plot both together with  $\bar{e}_1(t)$  and compare with the figure on the previous slide.

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## 1.6 Second-Order Systems

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**Second-order model**

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$$\tau^2 \frac{d^2 y}{dt^2} + 2\zeta\tau \frac{dy}{dt} + y = Kx(t) \Rightarrow G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\tau$ : natural period,  $\omega_n = 1/\tau$ : natural frequency,  $\zeta$ : damping ratio (감쇠계수),  $K$ : steady-state gain.  
Poles:

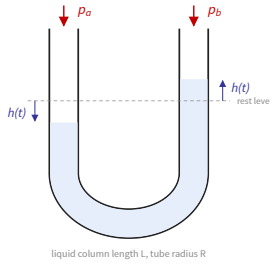
$$p_{1,2} = -\frac{\zeta}{\tau} \pm \frac{\sqrt{\zeta^2 - 1}}{\tau} \quad (\text{stable only if } \zeta > 0, \text{ since } \tau > 0)$$

| $\zeta$         | poles                                                      | name              | response                                                             |
|-----------------|------------------------------------------------------------|-------------------|----------------------------------------------------------------------|
| $\zeta > 1$     | two distinct real,<br>$-1/\tau_1, -1/\tau_2$               | overdamped        | sluggish,<br>$G = K/[(\tau_1 s + 1)(\tau_2 s + 1)]$                  |
| $\zeta = 1$     | repeated real, $-1/\tau$                                   | critically damped | fastest without oscillation                                          |
| $0 < \zeta < 1$ | complex pair<br>$-\zeta/\tau \pm j\sqrt{1 - \zeta^2}/\tau$ | underdamped       | overshoot, decaying<br>oscillation; $\tau$ is not a time<br>constant |
| $\zeta = 0$     | purely imaginary $\pm j/\tau$                              | undamped          | sustained oscillation at $\omega_n$                                  |

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### Example 1.9 Oscillations of a damped manometer

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A pressure step  $p_a - p_b$  sets the liquid in motion (laminar, isothermal). A momentum balance on the liquid column gives

$$\frac{4}{3}\pi R^2 L \rho \frac{d^2 h}{dt^2} = -8\pi L \mu \frac{dh}{dt} - 2\pi R^2 \rho g h + \pi R^2 (p_a - p_b)$$

$$\frac{d^2 h}{dt^2} + \frac{6\mu}{R^2 \rho} \frac{dh}{dt} + \frac{3g}{2L} h = \frac{3}{4} \frac{p_a - p_b}{\rho L}$$

Compare with  $s^2 + 2\zeta\omega_n s + \omega_n^2$ :

$$\omega_n = \sqrt{\frac{3g}{2L}}, \quad \zeta = \frac{3\mu}{R^2 \rho \omega_n}$$

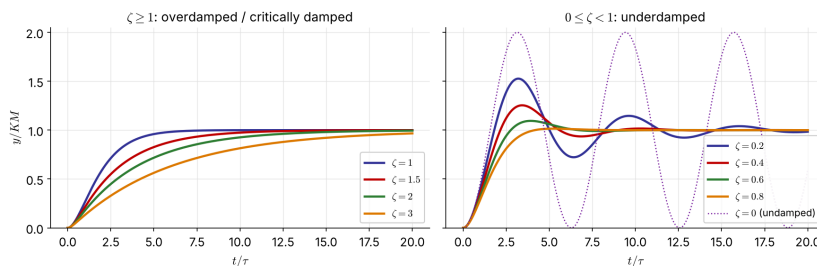
Critical damping  $\zeta = 1$  at  $R_{cr} = \left(\frac{6\mu^2 L}{\rho^2 g}\right)^{1/4}$ . Wider tubes oscillate, narrower tubes creep.

Bird, Stewart & Lightfoot, Transport Phenomena, 2nd ed., pp. 219-221.

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### Step response of a second-order model

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$\zeta > 1$ :

$$y = KM \left[ 1 - e^{-\alpha/\tau} \left( \cosh \frac{\sqrt{\zeta^2 - 1}}{\tau} t + \frac{\zeta}{\sqrt{\zeta^2 - 1}} \sinh \frac{\sqrt{\zeta^2 - 1}}{\tau} t \right) \right]$$

(simpler with  $e^{-t/\tau_1}$ ,  $e^{-t/\tau_2}$ )

$$\zeta = 1: y = KM \left[ 1 - \left( 1 + \frac{t}{\tau} \right) e^{-t/\tau} \right]$$

$$0 \leq \zeta < 1: y = KM \left[ 1 - \frac{e^{-\alpha/\tau}}{\sqrt{1 - \zeta^2}} \sin \left( \frac{\sqrt{1 - \zeta^2}}{\tau} t + \phi \right) \right]$$

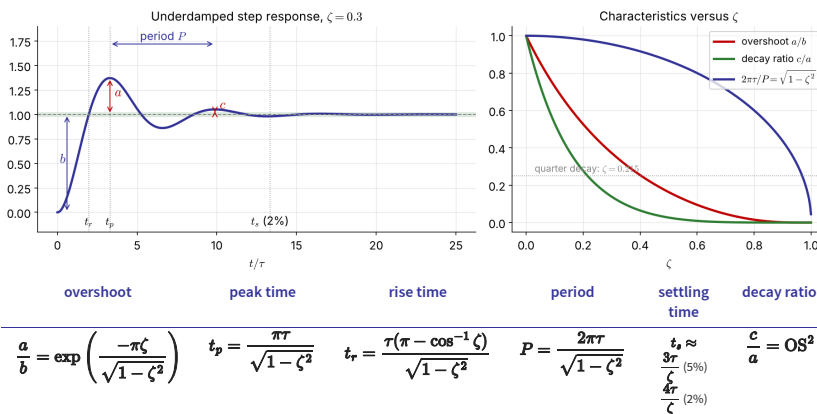
$$\phi = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}$$

Rarely used directly; the underdamped one yields the characteristics on the next slide.

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### Features of the underdamped step response

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Settling: the envelope  $e^{-\alpha/\tau}$  behaves like a first-order term with time constant  $\tau/\zeta$ . Smaller  $\zeta$ , faster rise, larger overshoot, longer settling.

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## Second-order responses in code

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Python

```

1 import numpy as np, control as ct, matplotlib.pyplot as plt
2
3 t = np.linspace(0, 15, 1501)
4 for zeta in (0.5, 1.0, 1.5):
5     G = ct.tf(1, [1, 2*zeta, 1])
6     plt.plot(t, ct.step_response(G, t).outputs, label=f'zeta = {zeta}')
7     plt.legend(); plt.xlabel('t')
8
9 info = ct.step_info(ct.tf(1, [1, 0.6, 1])) # zeta = 0.3
10 print(info['Overshoot'], info['PeakTime'], info['SettlingTime']) # 37.2 %, 3.29, ...

```

MATLAB

```

1 t = 0:0.01:15;
2 step(tf(1, [1 1]), tf(1, [1 2 1]), tf(1, [1 3 1]), t) % zeta = 0.5, 1, 1.5
3 legend('\zeta = 0.5', '\zeta = 1', '\zeta = 1.5')
4
5 S = stepinfo(tf(1, [1 0.6 1])) % zeta = 0.3
6 % Simulink: Step -> three Transfer Fcn blocks in parallel -> Scope

```

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## Problem #4 (homework)

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- Show that the overdamped step response can be written with two exponentials  $e^{-t/\tau_1}$  and  $e^{-t/\tau_2}$ , and find  $\tau_1, \tau_2$  in terms of  $\tau$  and  $\zeta$ .
- Starting from the underdamped solution, derive the phase-lag form and  $\phi = \tan^{-1} \left( \sqrt{1 - \zeta^2} / \zeta \right)$ .
- Derive the overshoot, peak time, rise time, period and decay ratio in the table.

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## 1.7 Processes with Dead Time

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## Dead time (time delay, transport lag)

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Caused by fluid travelling down a pipe or a slow analyzer: 시간지연 / 수송지연.

$$y(t) = x(t - t_d) \Rightarrow \frac{Y(s)}{X(s)} = e^{-t_d s}$$

$$\frac{K e^{-t_d s}}{\tau s + 1}, \quad \frac{K e^{-t_d s}}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Classical tools work with **polynomials** in  $s$ , so we use the **Padé approximation**, a ratio of polynomials that matches the Taylor series of  $e^{-t_d s}$ :

$$(1/1) \quad e^{-t_d s} \approx \frac{1 - \frac{t_d}{2}s}{1 + \frac{t_d}{2}s}$$

$$(2/2) \quad e^{-t_d s} \approx \frac{t_d^2 s^2 - 6t_d s + 12}{t_d^2 s^2 + 6t_d s + 12}$$

Each adds poles in the LHP and zeros in the RHP, which produce a small initial inverse response instead of a flat delay.

Padé:  $P_M^N(s) = \sum_{k=0}^M a_k s^k / \sum_{k=0}^N b_k s^k$ ,  $b_0 = 1$  the  $N+M+1$  coefficients follow from the Taylor coefficients  $c_n$  by  $a_k b_N = \sum_{j=0}^N c_{k+j} b_j$

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Example 1.10 Padé approximation of  $e^{-3s}/(10s+1)$ 

41

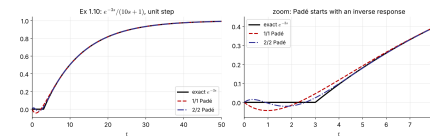
$$\frac{Y}{X} \approx \frac{-1.5s + 1}{(10s + 1)(1.5s + 1)}$$

Python

```
1 import numpy as np, control as ct
2 G = ct.tf(1, [10, 1])
3 num, den = ct.pade(3, 1) # 1/1 Padé
4 P1 = ct.tf(num, den)
5 t = np.linspace(0, 50, 5001)
6 y1 = ct.step_response(G*P1, t).outputs
7 y0 = ct.step_response(G, t).outputs
8 ye = np.interp(t-3, t, y0, left=0) # exact
```

MATLAB

```
1 G = tf(1, [10 1]);
2 P1 = tf([-1.5 1], [2.5 1]); % pade(3,1)
3 step(G*P1); hold on
4 step(G*exp(-3*tf('s'))); % exact delay
5 legend('Padé 1/1', 'exact')
```



Far from  $t = 0$  both approximations are excellent; at early times 1/1 dips to  $-0.04$ , and 2/2 wiggles around zero.

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## Problem #5 (homework)

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- Show that the 1/1 Padé approximation is more accurate than the first-order Taylor expansion  $e^{-t_d s} \approx 1 - t_d s$ . (Compare the Taylor series of both.)
- Derive the 2/2 Padé approximation and find its poles and zeros for  $t_d = 3$ .

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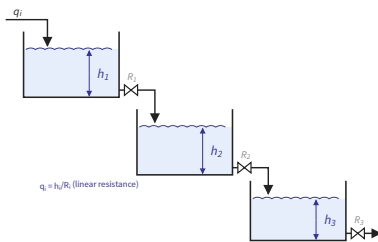
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## 1.8 Higher-Order Processes and Approximations

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### Non-interacting tanks in series

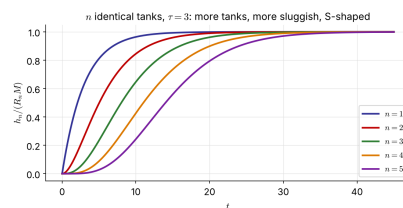
44



$$A_i \frac{dh_i}{dt} = q_{i-1} - q_i, \quad q_i = \frac{h_i}{R_i} \text{ (linear resistance):}$$

$$\frac{H_i(s)}{H_{i-1}(s)} = \frac{K_i}{\tau_i s + 1}, \quad \tau_i = A_i R_i, \quad K_i = \frac{R_i}{R_{i-1}}$$

$$\frac{H_n(s)}{Q_1(s)} = \frac{R_n}{(\tau_1 s + 1) \cdots (\tau_n s + 1)}$$



The more tanks, the longer the  $n$ -th tank waits to "see" a change: the response looks like a **delay** followed by a lag.

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### Example 1.11 Low-order approximation with dead time

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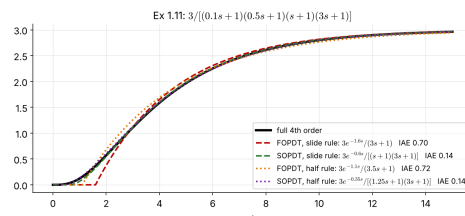
Keep the **dominant** (large) time constants and lump the small ones into a dead time:

$$G(s) \approx \frac{K e^{-t_d s}}{(\tau_1 s + 1)(\tau_2 s + 1)}, \quad t_d \approx \sum_{i \neq 1,2} \tau_i$$

$$\frac{Y}{X} = \frac{3}{(0.1s + 1)(0.5s + 1)(s + 1)(3s + 1)}$$

- FOPDT:  $\frac{3e^{-1.6s}}{3s + 1}$
- SOPDT:  $\frac{3e^{-0.6s}}{(s + 1)(3s + 1)}$

**Half rule** (Skogestad): split the largest neglected time constant, half to the lag, half to the delay, e.g.  $\frac{3e^{-0.35s}}{(1.25s + 1)(3s + 1)}$

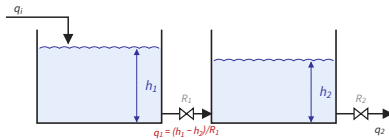


The second-order approximation is clearly better (IAE 0.14 vs 0.70). FOPDT models are the basis of the empirical tuning rules of Ch 3.

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## Interacting tanks

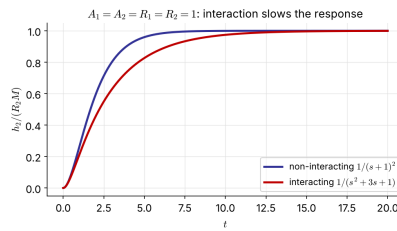
46



$$A_1 \frac{dh_1}{dt} = q_i - \frac{h_1 - h_2}{R_1}, \quad A_2 \frac{dh_2}{dt} = \frac{h_1 - h_2}{R_1} - \frac{h_2}{R_2}$$

$$\frac{H_1}{Q_i} = \frac{R_1 \tau_2 s + R_1 + R_2}{p(s)}, \quad \frac{H_2}{Q_i} = \frac{R_2}{p(s)}$$

$$p(s) = \tau_1 \tau_2 s^2 + \left( \tau_1 + \tau_2 + \tau_1 \frac{R_2}{R_1} \right) s + 1$$



1. Even the first tank is second order (with a zero).
2. Interaction adds  $\tau_1 R_2 / R_1$  to the middle coefficient; the poles stay real and negative.

Which damps inlet-flow disturbances better? The interacting pair: the extra term pushes one pole closer to the origin, so the response is slower and smoother.

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## Problem #6 (homework) Blending tank and transfer pipe

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A nearly perfectly mixed blending tank (first order) feeds a downstream unit through a long transfer pipe. A test shows that the pipe behaves as if its volume were split into five equal perfectly stirred tanks in series.

$$V_{\text{tank}} = 2 \text{ m}^3, \quad V_{\text{pipe}} = 0.1 \text{ m}^3, \quad q_{\text{total}} = 1 \text{ m}^3/\text{min}$$

- a. What is the most accurate transfer function  $C'_{\text{out}}(s)/C'_{\text{in}}(s)$  you can develop?
- b. What approximate low-order transfer function could represent this process?
- c. What do you conclude about the need to model the pipe's mixing accurately?
- d. Simulate the approximate and full-order responses to a step change in  $C_{\text{in}}$ .

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## 1.9 Effect of Zeros

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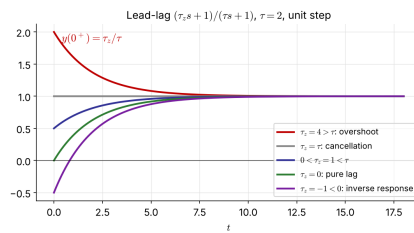
## Lead-lag element

Poles govern the inherent dynamics; zeros add finer "fingerprints" by changing the coefficients.

$$\frac{Y}{X} = \frac{\tau_z s + 1}{\tau s + 1} \xrightarrow{X=1/s} Y = \frac{1}{s} + \frac{\tau_z - \tau}{\tau s + 1}$$

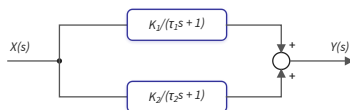
$$y(t) = 1 - \left(1 - \frac{\tau_z}{\tau}\right)e^{-t/\tau}, \quad y(0^+) = \frac{\tau_z}{\tau}$$

- the exponential still comes from the pole  $-1/\tau$
- $\tau_z > \tau$ : jump above 1, **overshoot**
- $0 < \tau_z < \tau$ : jump, then rise
- $\tau_z < 0$  (RHP zero): **inverse response**
- $\tau_z = \tau$ : pole-zero cancellation,  **$y = 1$**



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## Transfer functions in parallel



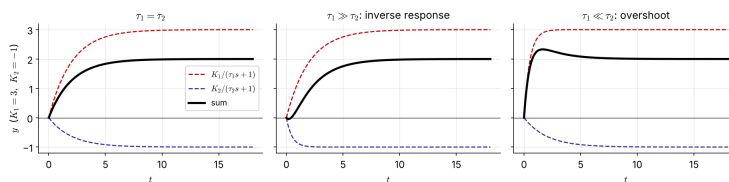
With **opposing effects** ( $K_2 < 0$ ), superposition gives a rough picture right away.

The fast element wins first, the large-gain element wins at the end:  
 $\tau_1 \gg \tau_2$  gives  $\tau_z < 0$  and an inverse response.

Two competing effects with different time constants:

$$\frac{Y}{X} = \frac{K_1}{\tau_1 s + 1} + \frac{K_2}{\tau_2 s + 1} = \frac{K(\tau_z s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

$$K = K_1 + K_2, \quad \tau_z = \frac{K_1 \tau_2 + K_2 \tau_1}{K_1 + K_2}$$



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## Problem #7 (homework)

$$\frac{Y}{X} = \frac{K(\tau_z s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)}, \quad \tau_1 > \tau_2$$

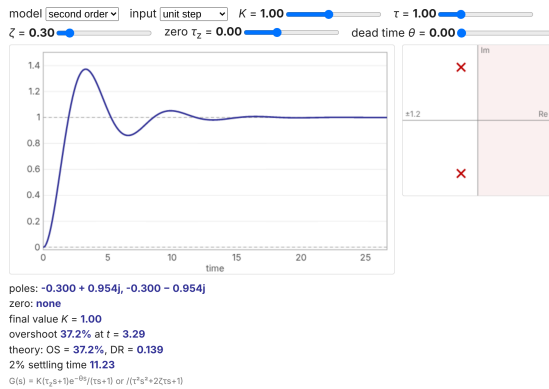
Calculate the response to a step of magnitude  $M$  and show that

- $y(t)$  has an extremum only if  $\frac{1 - \tau_z/\tau_2}{1 - \tau_z/\tau_1} > 1$ ;
- overshoot occurs only for  $\tau_z/\tau_1 > 1$ ;
- inverse response occurs only for  $\tau_z < 0$ ;
- if an extremum exists, find the time at which it occurs.

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## Interactive: poles, zeros, dead time

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Interactive version: [seoultechpse.github.io/ProCon/ch1/ch1\\_dynamics.html](https://seoultechpse.github.io/ProCon/ch1/ch1_dynamics.html) — first or second order,  $K$ ,  $\tau$ ,  $\zeta$ , zero  $\tau_z$ , dead time  $\theta$ .

## Summary

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## Tools

- deviation variables  $\rightarrow$  zero initial conditions
- linearize about the operating point
- Laplace transform + table; final value only if stable
- partial fractions: one term per pole
- $G(s) = Q(s)/P(s)$ , gain  $K = G(0)$

## Responses

- first order:  $\tau$  (63.2%),  $K$ ; integrator  $K_I/s$
- second order:  $\zeta$  sets over/critical/under damping; OS, DR =  $OS\%$ ,  $t_s \approx 4\tau/\zeta$
- dead time  $e^{-t_d s} \rightarrow$  Padé; high order  $\rightarrow$  FOPDT/SOPDT
- zeros: overshoot, inverse response (RHP zero)

Next (Ch 2): close the loop with a controller and see how  $K_c$ ,  $\tau_I$ ,  $\tau_D$  move these poles.

## Resources

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- Slides (interactive HTML): [seoultechpse.github.io/ProCon/ch1/ch1\\_dynamics.html](https://seoultechpse.github.io/ProCon/ch1/ch1_dynamics.html)
- PDF: [slides](#) · [handout with note space](#)
- Python notebook: [Open in Colab](#) — every transform (sympy), calculation and figure of this chapter
- Response explorer: [standalone page](#)