Process Dynamics and Control

2. Analysis of Single-Loop Control Systems

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What are we up to?

We finally launch into **controllers**. Two things come first:

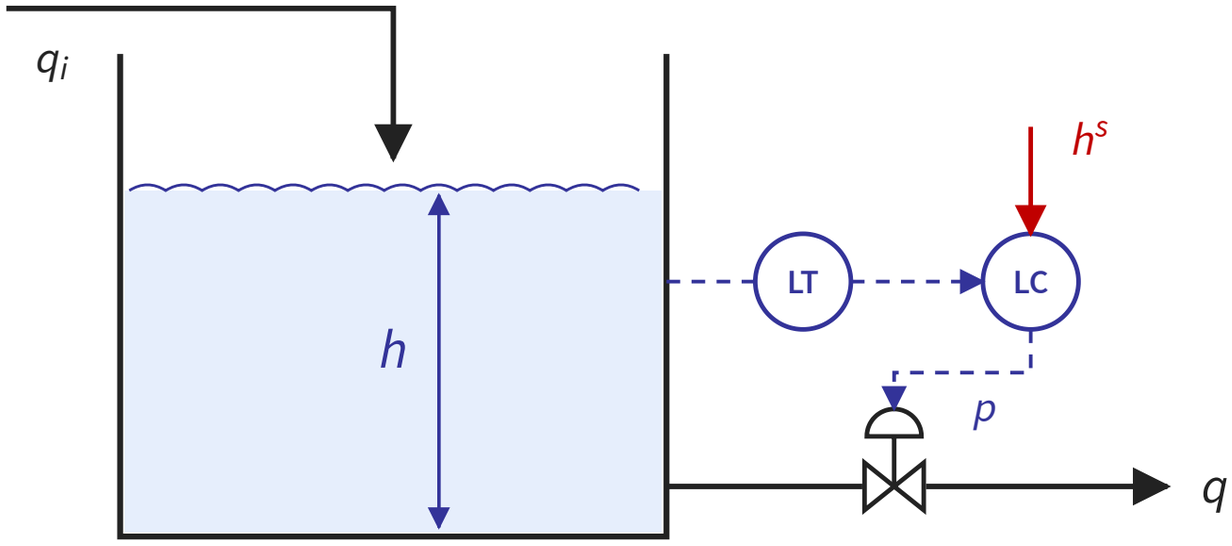
1. how to **set up** a control problem and derive the **closed-loop transfer functions** and the **characteristic equation**;
2. what the three modes of the classical **PID controller** do, and what each costs.

Section	Topic	Key result
2.1	the feedback loop, block diagrams, hardware	$C = G_{sp}R + G_{load}L$, characteristic equation $1 + G_{OL} = 0$
2.2	P, PI, PD and PID control	offset, reset windup, derivative kick
2.3	closed-loop response, Examples 2.1–2.6	how $K_c,\ \tau_I,\ \tau_D$ change gain, time constant and ζ
2.4	which controller for which variable	level, flow, temperature, pressure, composition

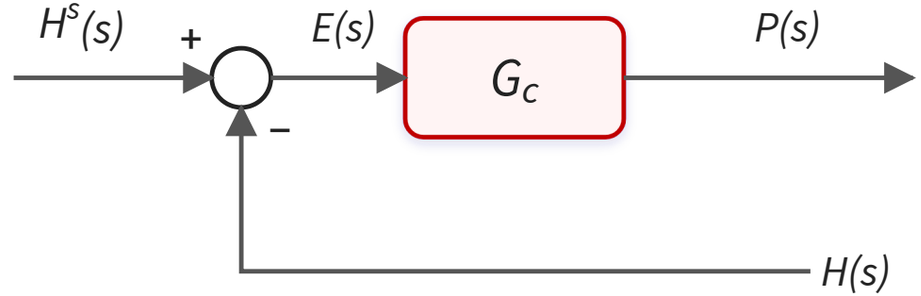
Feedback control is old: water-level regulators in Greece around 250 B.C., James Watt's fly-ball governor in 1788.

2.1 Classical Feedback Control System

A liquid-level loop



load: q_i · CV: h · MV: q



measured level fed back

- **set point** (reference) h^s: the level we want
- **controlled** = **measured variable**: level h
- **manipulated variable**: outlet flow q
- **disturbance** (load): inlet flow q_i

The controller acts on the **error**, the negative of the deviation:

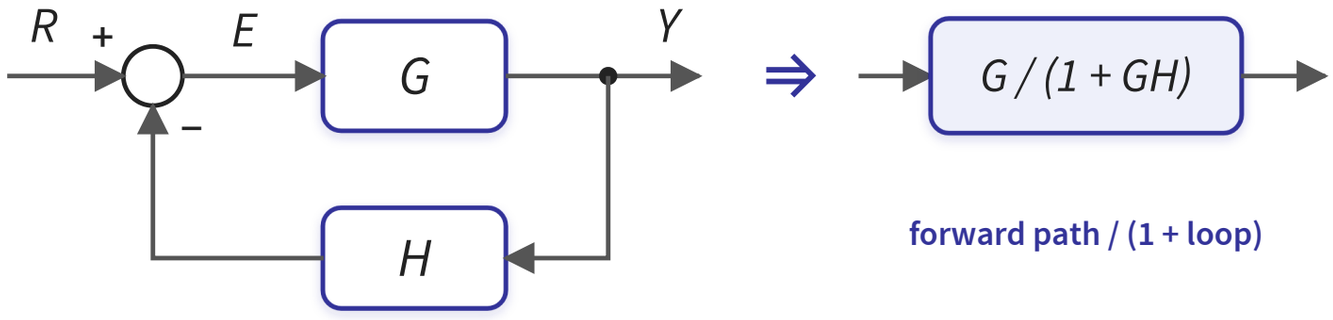
$$e(t) = h^s - h(t), \qquad p(t) = p^s + f\big[e(t)\big]$$

f is the **control law**; p^s is the **controller bias**, the output when $e = 0$.

A *system* means the process together with the controller and its accessories (sensor, transmitter, actuator).

Block-diagram algebra

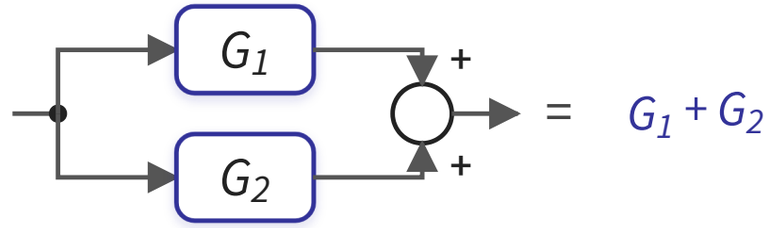
(a) negative feedback



forward path / (1 + loop)

(b) series

(c) parallel



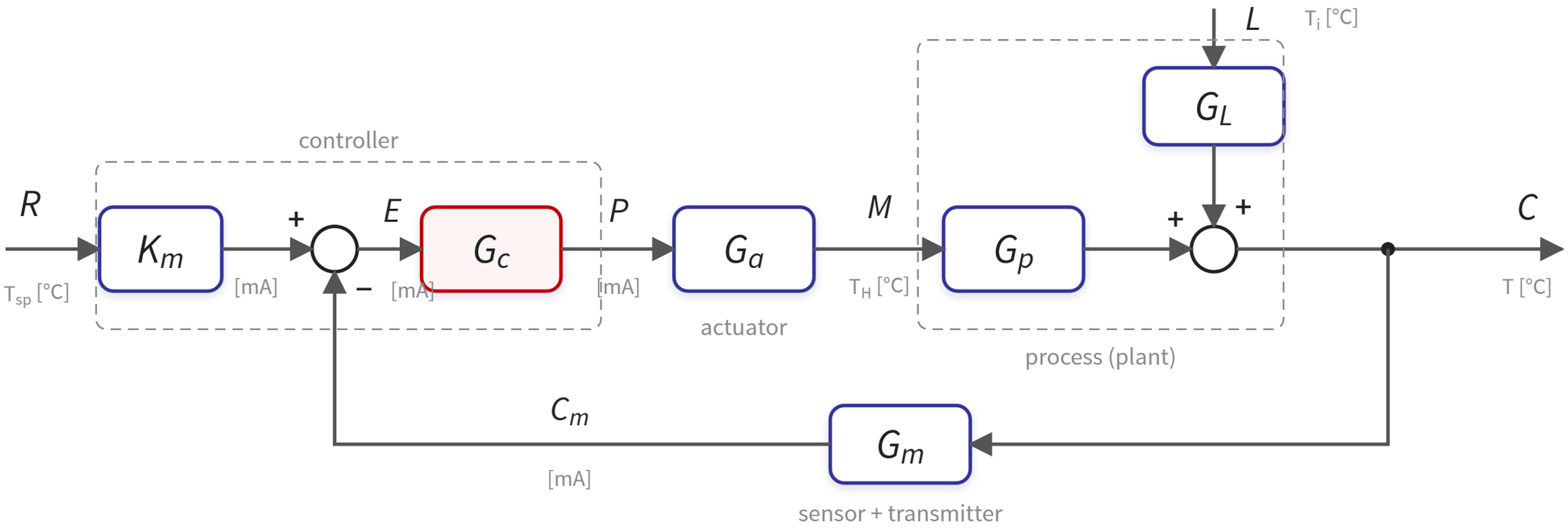
For negative feedback, $E = R - HY$ and $Y = GE$, so $Y = G(R - HY)$ and

$$\frac{Y}{R} = \frac{G}{1 + GH}$$

Rule. Closed-loop TF = (product of the blocks on the **forward path**) / (1 + product of **all** blocks around the loop).

The denominator, set to zero, is the **characteristic equation** of the closed loop.

The standard loop: stirred-tank heater



Process model $T = G_L T_i + G_p T_H$. We manipulate the steam temperature T_H (M), so the inlet temperature T_i becomes the load L. Closing the loop:

$$C = \underbrace{\frac{K_m G_c G_a G_p}{1 + G_m G_c G_a G_p}}_{G_{sp}} R + \underbrace{\frac{G_L}{1 + G_m G_c G_a G_p}}_{G_{load}} L$$

K_m converts the set point (°C) into the same mA signal as the measurement, so K_m must equal the steady-state gain of G_m.

Servo and regulator problems

Servo problem — set-point change, $L = 0$:

$$C = G_{sp}R, \qquad \text{ideal: } C = R$$

(perfect tracking)

Regulator problem — disturbance, $R = 0$:

$$C = G_{load}L, \qquad \text{ideal: } C = 0$$

(perfect disturbance rejection)

Both transfer functions share **one** denominator, the **closed-loop characteristic equation**:

$$1 + G_m G_c G_a G_p = 1 + G_{OL} = 0$$

Dynamics and stability are therefore **the same** for set-point and load changes. Only the numerators, and so the size and shape of the response, differ.

Choosing the controlled and manipulated variables

Controlled variables: those that

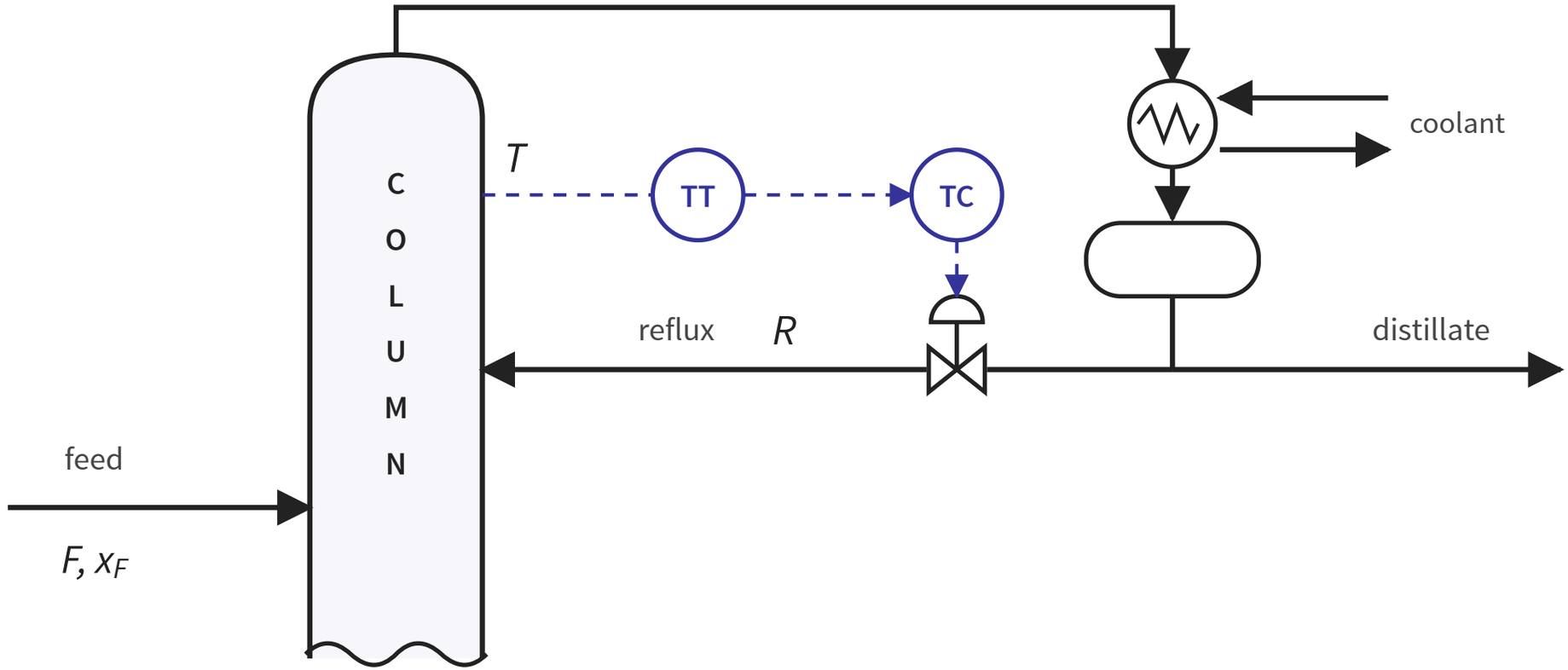
1. are dictated by the problem (the temperature of a refrigerator)
2. are not self-regulating (a liquid level)
3. may exceed equipment or process constraints
4. interact with other variables (reactor temperature affects yield)

Manipulated variables: those that

1. have a direct effect on the output
2. have a large steady-state gain (sensitivity)
3. have no dead time
4. do not interact with other loops

In a SISO system everything else becomes a **load** variable.

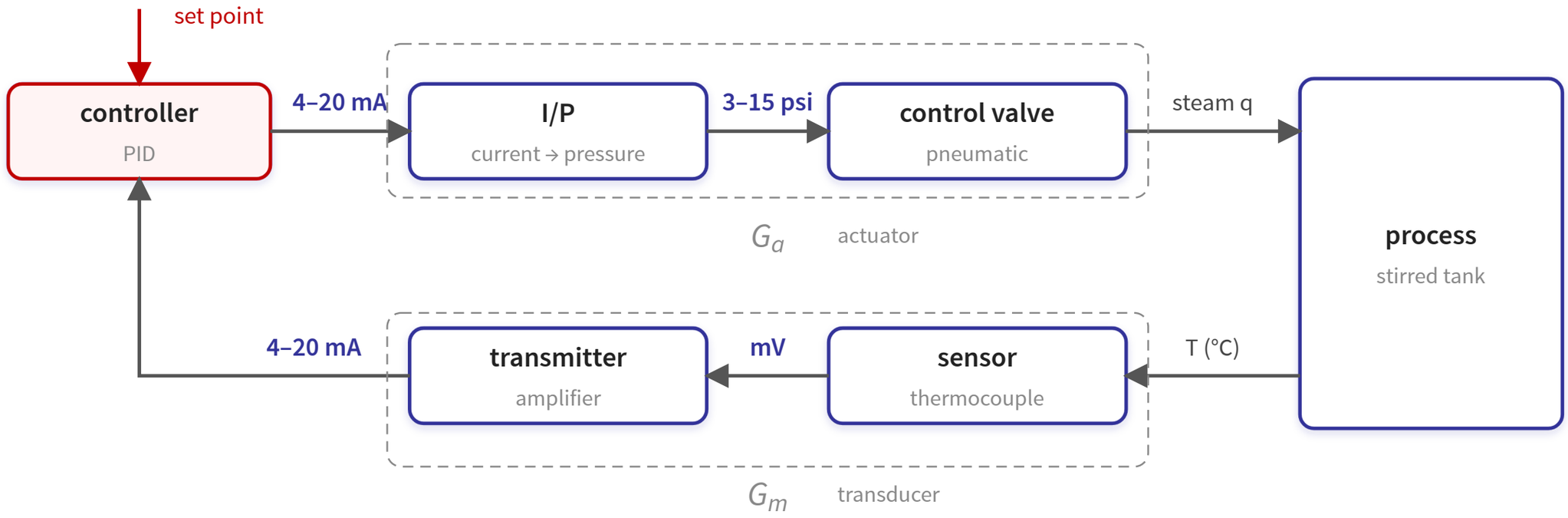
Problem #1 (homework)



The temperature T of a tray near the top of a distillation column is controlled by adjusting the reflux flow R. Draw a **block diagram** for this feedback loop. Treat both the feed flow rate F and the feed composition x_F as disturbances.

Hint: two load blocks, G_F and G_{x_F}, enter the same summing junction as G_p.

Hardware around the loop



Every block has **units**. Standard signal ranges let instruments from different vendors talk to each other: **4–20 mA** for electronic signals (4 mA is the live zero), **3–15 psi** for pneumatic valve actuators, with an **I/P transducer** in between.

Sensor and transmitter: G_m

$$G_m = \frac{T_m}{T} = \frac{K_m e^{-t_d s}}{\tau_m s + 1}$$

- K_m: gain of sensor plus transmitter (mA/°C)
- τ_m: sensor lag, t_d: transport lag

Thermowell, energy balance:

$$m_m C_m \frac{dT_m}{dt} = U_m A_m (T - T_m) \;\Rightarrow\; \tau_m = \frac{m_m C_m}{U_m A_m}$$

A fast sensor with no transport lag gives simply $G_m = K_m$.

Example 20–120 °C is transmitted linearly as 4–20 mA.

$$I = mT + I_0, \quad m = \frac{16\ \text{mA}}{100\ ^\circ\text{C}} = 0.16\ \text{mA}/^\circ\text{C}$$

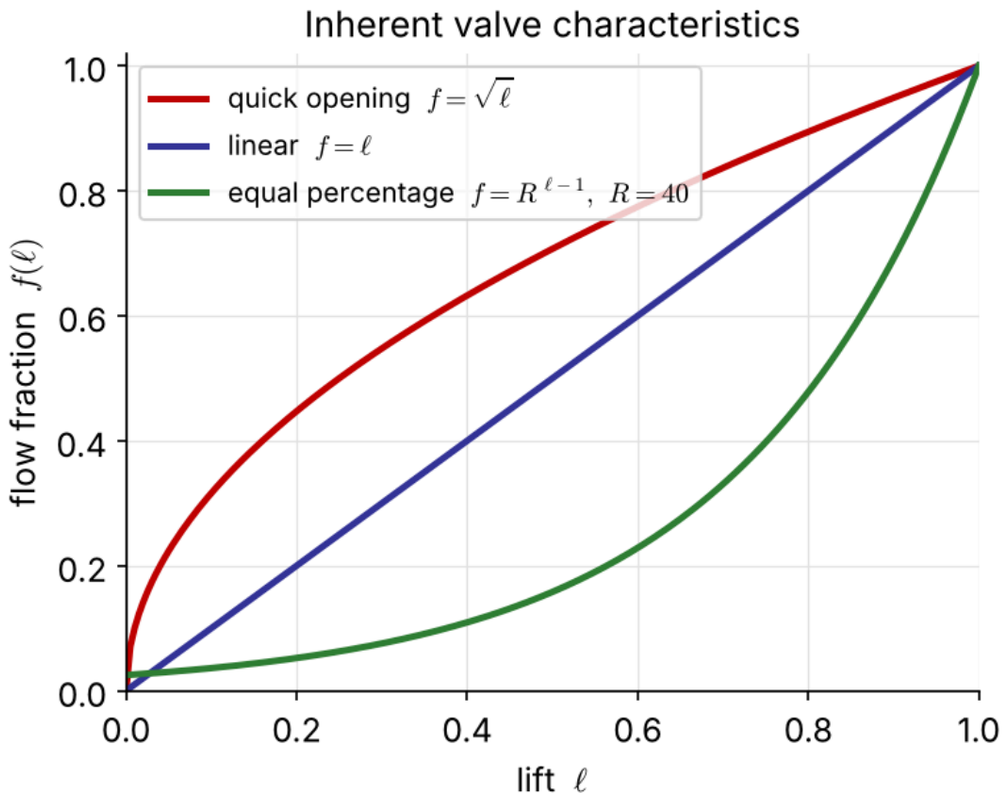
$I_0 = 4 - 20(0.16) = 0.8$ mA

- 66 °C $\rightarrow I = 0.16(66) + 0.8 =$ **11.36 mA**
- 6.5 mA $\rightarrow T = (6.5 - 0.8)/0.16 =$ **35.6 °C**

The transmitter gain is

$$K_m = \frac{\text{output span}}{\text{input span}} = 0.16\ \text{mA}/^\circ\text{C}$$

Actuator: the control valve, G_a



$$q = C_v f(\ell)\sqrt{\frac{\Delta P_v}{g_s}}$$

C_v: valve coefficient, ℓ: lift (0–1), $f(\ell)$: valve characteristic.

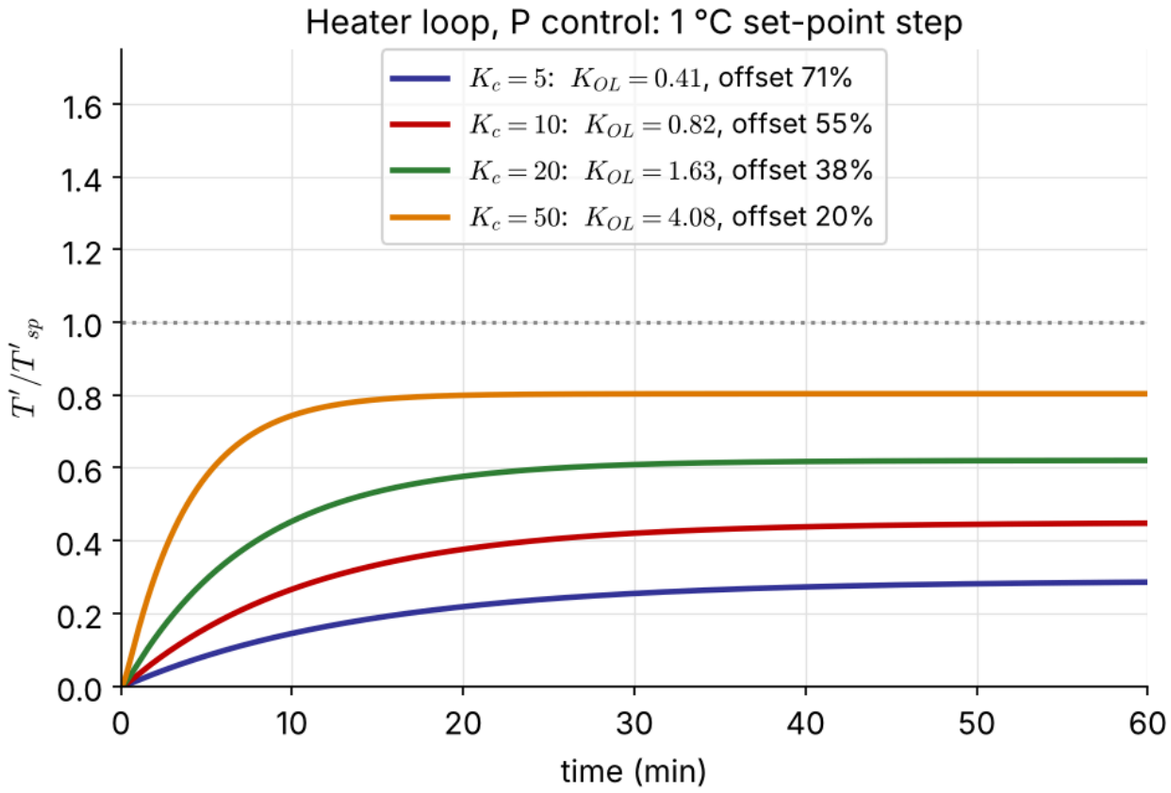
The actuator must cover the range of the manipulated variable and **respond faster than the process**.

Fail-safe action : *air-to-open* (fail closed) or *air-to-close* (fail open), chosen for safety when the air supply fails. It fixes the sign of the valve gain.

Lumping I/P, valve and steam coil:

$$G_a = G_v = \frac{K_v}{\tau_v s + 1}, \quad K_v = 0.6\ ^\circ\mathrm{C/mA},\ \tau_v = 0.2\ \mathrm{min}$$

The heater loop assembled



$$G_p = \frac{0.85}{20s + 1}, \quad G_v = \frac{0.6}{0.2s + 1}$$

$G_m = K_m = 0.16$ mA/°C (the transmitter above).

With P control, $G_c = K_c$ (mA/mA):

$$K_{OL} = K_c K_v K_p K_m = 0.0816\, K_c$$

$$\text{offset} = \frac{1}{1 + K_{OL}}$$

The loop gain is the product of **all** the gains, so a sluggish transmitter or valve gain must be made up by K_c. P control alone always leaves an offset, which motivates integral action.

2.2 PID Controllers

Proportional control

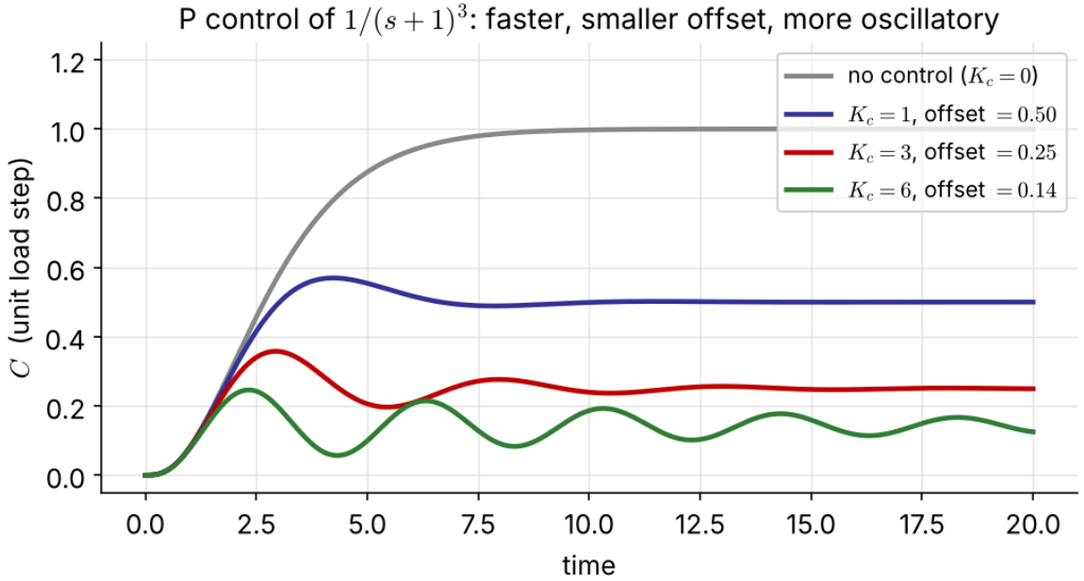
$$p(t) = p^s + K_c e(t) \;\Rightarrow\; G_c(s) = \frac{P(s)}{E(s)} = K_c$$

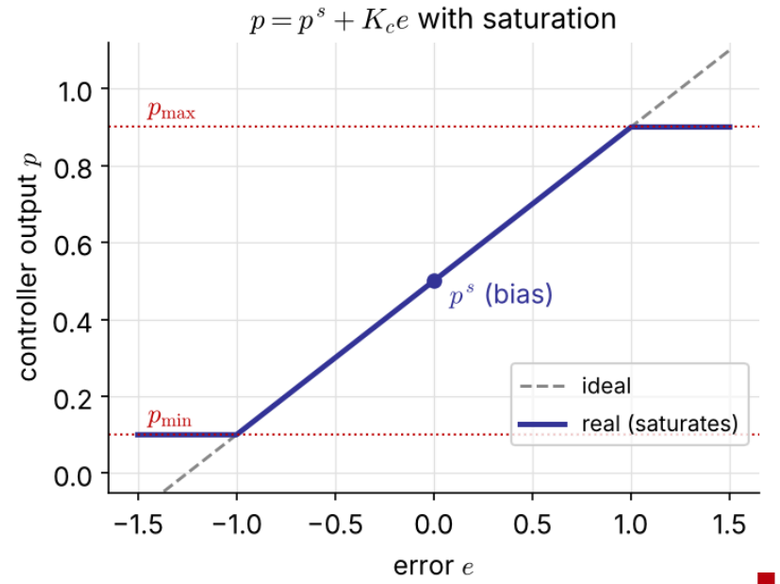
(in deviation variables, $e^s = 0$)

- K_c: **proportional gain** (controller sensitivity)
- **proportional band**: $\mathrm{PB} = \dfrac{100}{K_c}\,\%$

Action: *reverse acting* ($K_c > 0$): output falls as the measurement rises. *Direct acting* ($K_c < 0$) is needed when the process gain is negative, so that $K_c K_p > 0$.

- larger K_c: faster response, smaller offset
- too large K_c: oscillation, eventually **instability**
- the output **saturates** at $p_{\min}, p_{\max}$
- **offset** always remains: to hold $e = 0$ at a new steady state, p must move away from p^s, but $p - p^s = K_c e$ needs $e \neq 0$

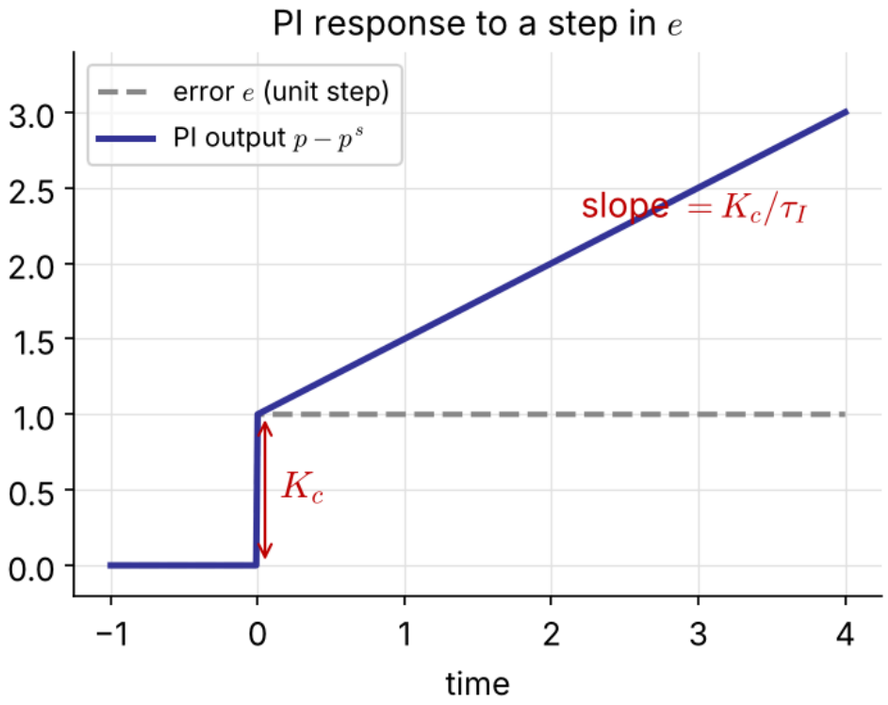


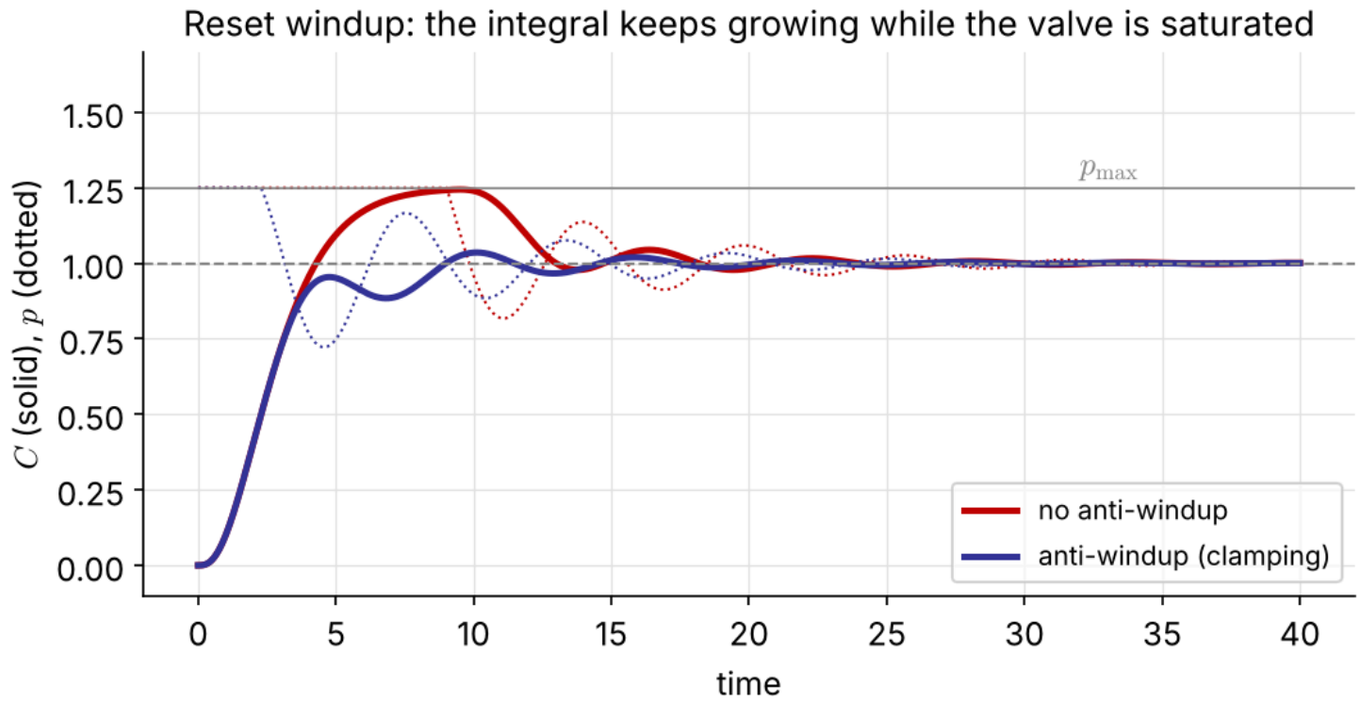


Proportional-integral control

$$p(t) = p^s + K_c \left[e(t) + \frac{1}{\tau_I} \int_0^t e\, dt \right] \Rightarrow G_c(s) = K_c \left[1 + \frac{1}{\tau_I s} \right]$$

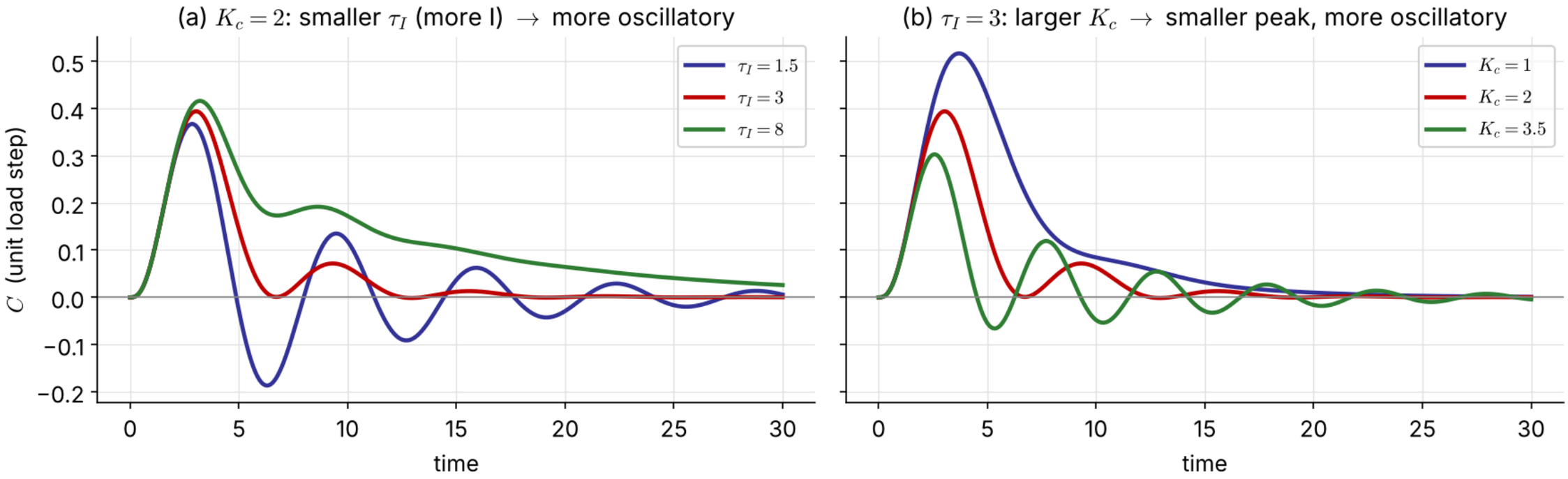
τ_I: **integral (reset) time**; $1/\tau_I$ is the reset rate. Integral action is never used alone. The integral keeps changing as long as $e \neq 0$, so it can supply the new output that a steady state needs while $e = 0$: **no offset**.





Reset windup: when e persists (start-up, large set-point changes) the integral grows until the valve saturates, then must "unwind", which causes a long overshoot. **Anti-windup** stops integrating while the output is saturated.

The price of integral action



- removing the offset usually costs a **more oscillatory** response: short rise time, but more overshoot or a longer settling time
- so K_c is usually **lowered** when integral action is added
- more integral action (small τ_I) is faster but less damped

Derivative action

$$p(t) = p^s + K_c \left[e + \tau_D \frac{de}{dt} \right] \;\Rightarrow\; G_c = K_c[1 + \tau_D s]$$

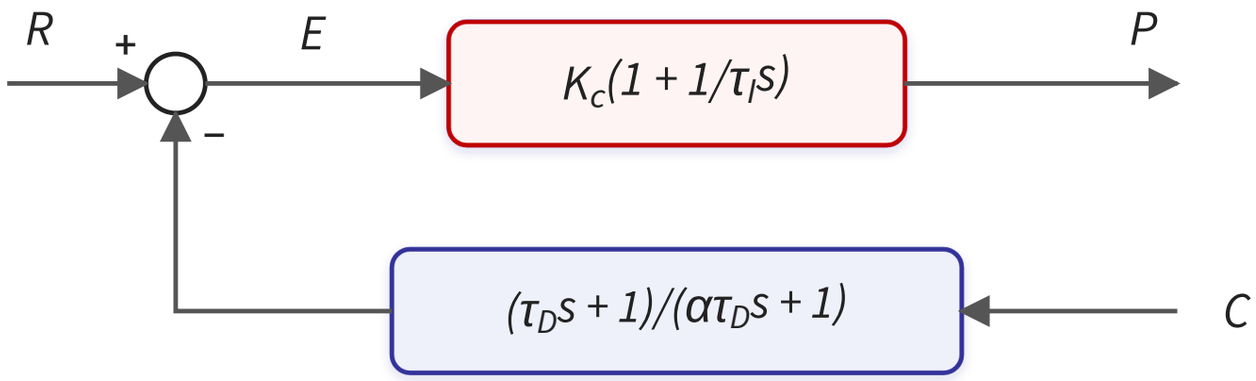
- P and I act on the present and the past; D anticipates from **how fast** e changes.
- D does nothing while e is constant, so it is never used alone.
- D opposes rapid change: it can damp the overshoot that integral action causes.
- D amplifies **noise**; PD is not useful with large dead time or noisy signals.

Ideal derivative action cannot be built (more zeros than poles), so a **lead-lag** element is used:

$$G_c = K_c \frac{\tau_D s + 1}{\alpha \tau_D s + 1}, \qquad 0.05 \leq \alpha \leq 0.2$$

This adds a fast pole at $s = -1/(\alpha \tau_D)$ that limits the high-frequency gain to K_c/α.

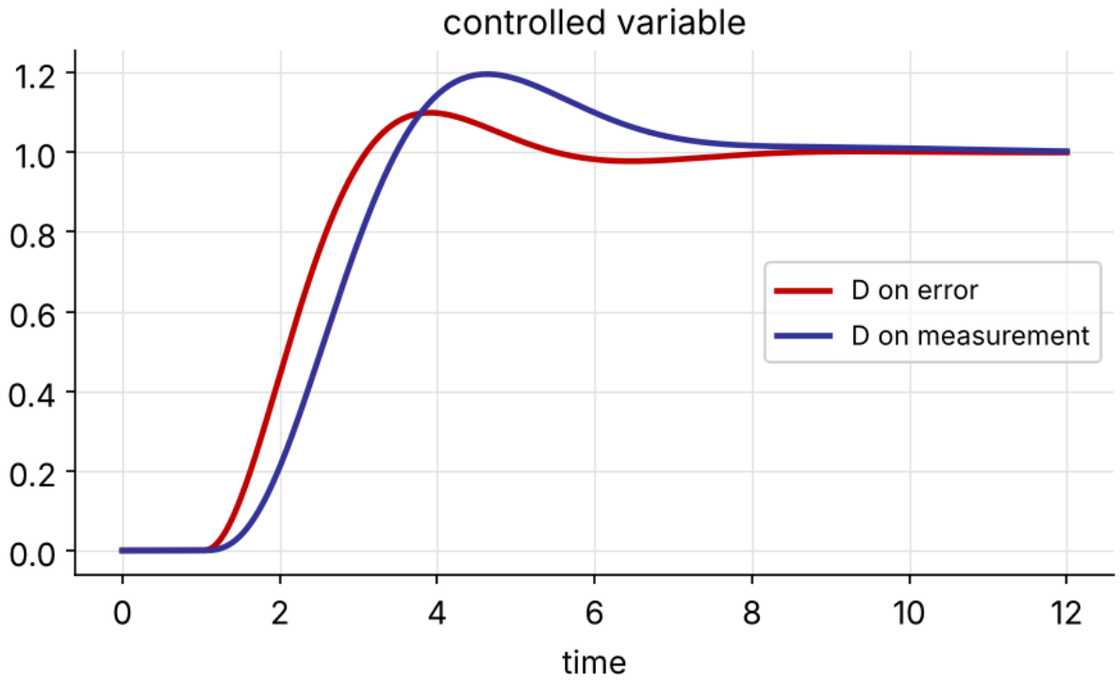
Derivative kick

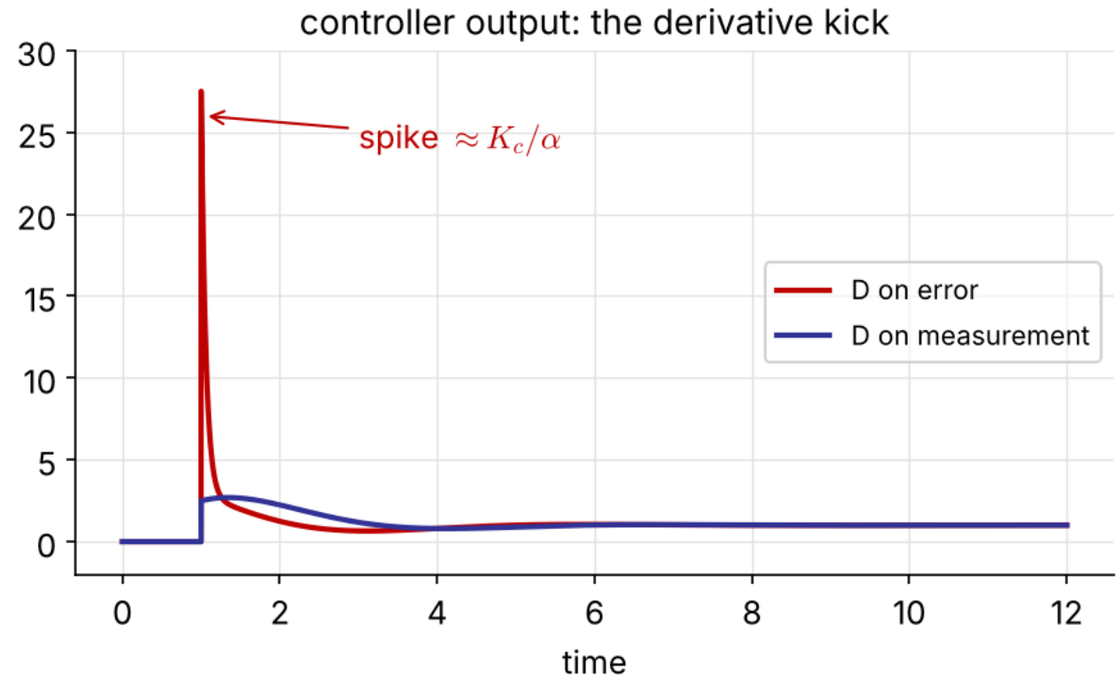


derivative action on the measured variable

A set-point step makes de/dt infinite, and the filtered D produces a spike of about K_c/α in the controller output.

Acting on the **measurement** instead of the error removes it: R reaches the output only through the PI part. For a load change (R constant) nothing changes, since then $de/dt = -dC/dt$.



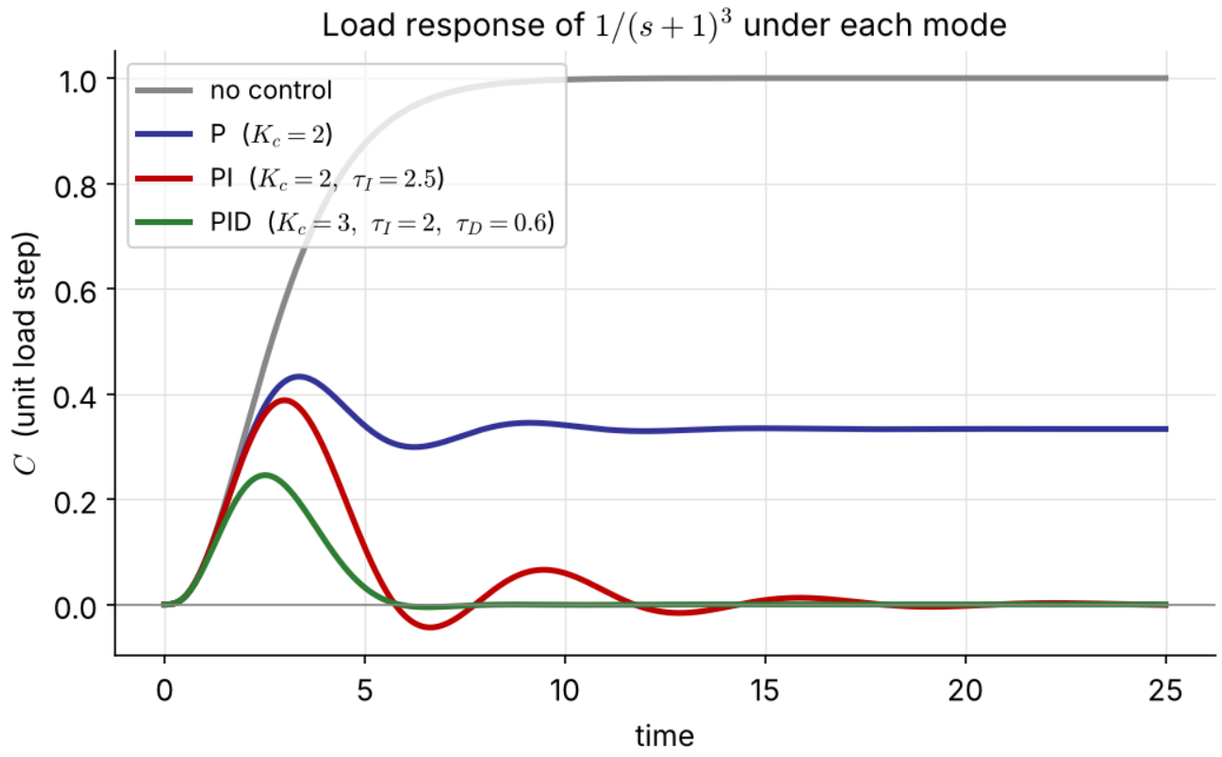


PID control and its forms

Form	$G_c(s)$	Where
ideal (ISA)	$K_c\left[1+\dfrac{1}{\tau_I s}+\tau_D s\right]=K_c\dfrac{\tau_I\tau_D s^2+\tau_I s+1}{\tau_I s}$	analysis
parallel	$K_c+\dfrac{K_I}{s}+K_D s,\ K_I=\dfrac{K_c}{\tau_I},\ K_D=K_c\tau_D$	Simulink, textbooks
real (filtered D)	$K_c\left[1+\dfrac{1}{\tau_I s}+\dfrac{\tau_D s}{\alpha\tau_D s+1}\right]$	most industrial controllers
series (interacting)	$K_c\left[\dfrac{\tau_I s+1}{\tau_I s}\right]\left[\dfrac{\tau_D s+1}{\alpha\tau_D s+1}\right]$	pneumatic, older analog

- The ideal PID has a numerator of higher order than its denominator, so it is **not physically realizable**. We still use it in analysis for its clean algebra and the insight it gives.
- Simulink's PID block: $P+I\dfrac{1}{s}+D\dfrac{N}{1+N/s}$, so $N=1/(\alpha\tau_D)$.
- **Check the form** before copying settings from one controller to another.

The three modes side by side

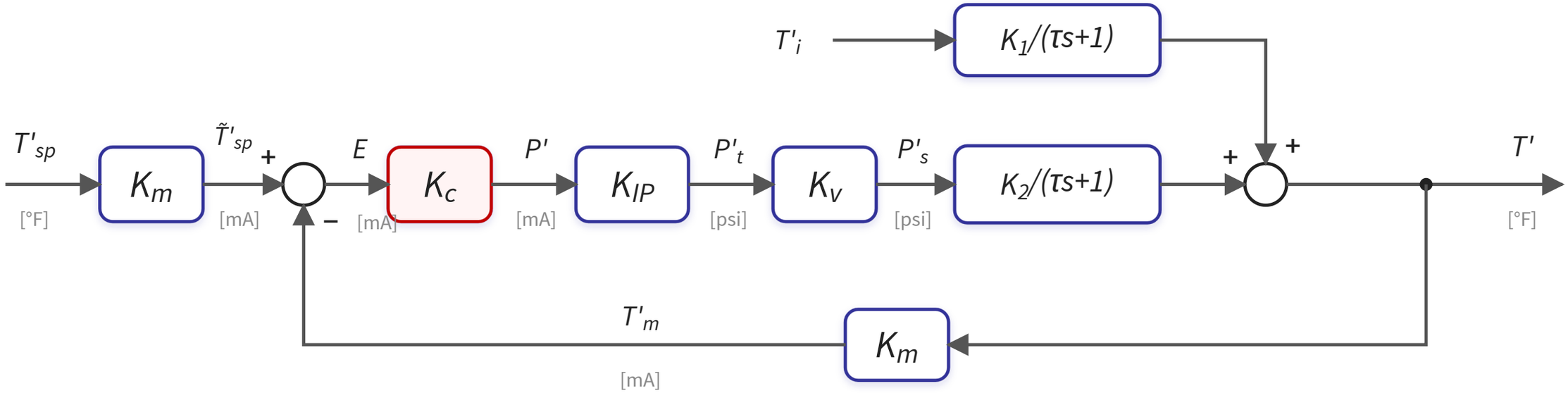


	P	I	D
speed	↑	↑	–
offset	↓ but stays	**removed**	–
overshoot	↑	↑	↓
stability margin	↓	↓	↑ (usually)
noise sensitivity	–	–	↑

Arrows: effect of increasing K_c, $1/\tau_I$ and τ_D.

Start simple: P, then PI; add D only if it helps.

Problem #2 (homework)

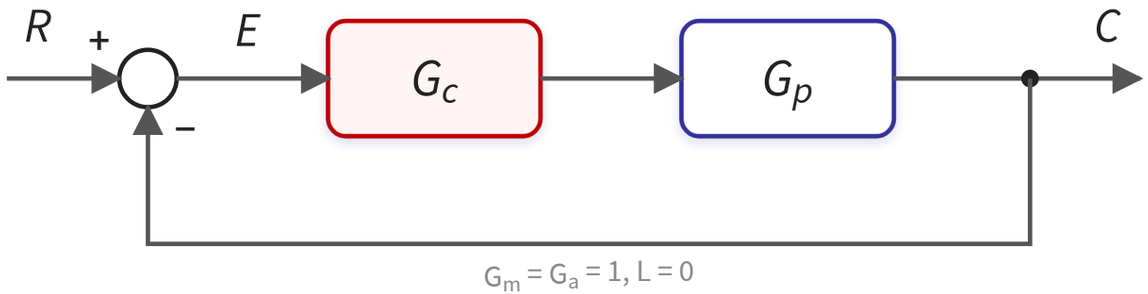


Proportional-only control of the stirred-tank heater above. The transmitter has a span of 50 °F and a zero of 55 °F. Nominal conditions are $\bar{T} = 80$ °F and $\bar{T}_i = 65$ °F. $K_c = 5$, $K_v = 1.2$ (dimensionless), $K_{IP} = 0.75$ psi/mA, $\tau = 5$ min; valve and transmitter dynamics are negligible. After the set point is changed from 80 to 85 °F, the tank temperature settles at 84.14 °F.

(a) What is the offset? (b) What is the process gain K_2? (c) What is the pressure signal p_t to the valve at the final steady state?

2.3 Closed-Loop System Response

Unity-feedback servo loop



$$\frac{C}{R} = \frac{G_c G_p}{1 + G_c G_p}$$

1. Choose G_c (P, PI, PD, PID) and G_p and substitute.
2. Rearrange into standard first- or second-order form, so the **closed-loop gain**, **time constant** and **damping ratio** can be read off.
3. The final-value theorem gives the **offset** for a unit step: $e(\infty) = 1 - \lim_{s \to 0} \frac{C}{R}$.

$G_m = G_a = 1$ and $L = 0$ keep the algebra light. The ideas carry over to the full loop.

Example 2.1 P control, first-order process

$G_c = K_c, \quad G_p = \dfrac{K_p}{\tau_p s + 1}$ What is $C(\infty)$ after a unit set-point step?

$$\frac{C}{R} = \frac{K_c K_p}{\tau_p s + 1 + K_c K_p} = \frac{K_{cl}}{\tau_{cl} s + 1}, \qquad K_{cl} = \frac{K_c K_p}{1 + K_c K_p}, \quad \tau_{cl} = \frac{\tau_p}{1 + K_c K_p}$$

- still **first order**
- faster: $\tau_{cl} < \tau_p$
- $C(\infty) = K_{cl} < 1$: offset

$$e(\infty) = \frac{1}{1 + K_c K_p}$$

Larger K_c shrinks both τ_{cl} and the offset, but the offset vanishes only as $K_c \to \infty$.

A first-order process under P control **cannot** oscillate or go unstable for $K_c K_p > 0$; real loops do because of the lags and dead time this model ignores.

Example 2.2 P control, second-order overdamped process

$G_p = \dfrac{K_p}{(\tau_1 s + 1)(\tau_2 s + 1)}$, $\tau_1 = 2$, $\tau_2 = 4$ min, $K_p = 1$. Which K_c gives $\zeta = 0.7$?

$$\frac{C}{R} = \frac{K_c K_p}{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2) s + 1 + K_c K_p} = \frac{K_{cl}}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

$$\tau = \sqrt{\frac{\tau_1 \tau_2}{1 + K_c K_p}}, \qquad \zeta = \frac{\tau_1 + \tau_2}{2\sqrt{\tau_1 \tau_2 (1 + K_c K_p)}}$$

$$\zeta = \frac{3}{\sqrt{8(1 + K_c)}} = 0.7 \;\Rightarrow\; 1 + K_c = 2.296$$

$K_c = 1.30$, offset $= \dfrac{1}{1 + K_c} = 0.44$: large!

Raising K_c lowers ζ: the closed loop becomes underdamped although the open loop is overdamped. Speed and damping trade off, and the offset at $\zeta = 0.7$ is unacceptable.

Example 2.3 PI control, first-order process

$G_c = K_c\left(1 + \dfrac{1}{\tau_I s}\right)$, $G_p = \dfrac{K_p}{\tau_p s + 1}$. What is the offset?

$$\frac{C}{R} = \frac{K_c K_p(\tau_I s + 1)}{\tau_I \tau_p s^2 + \tau_I(1 + K_c K_p)s + K_c K_p} = \frac{\tau_I s + 1}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

$$\tau = \sqrt{\frac{\tau_I \tau_p}{K_c K_p}}, \qquad \zeta = \frac{1}{2}(1 + K_c K_p)\sqrt{\frac{\tau_I}{K_c K_p \tau_p}}$$

$\lim_{s\to 0} C/R = 1$: **no offset**.

The integrator raises the order by one: a first-order process can now **oscillate** (small τ_I, large K_c).

The closed loop also has a **zero** at $s = -1/\tau_I$, which adds overshoot beyond what ζ alone predicts.

Example 2.4 PD control, first-order process

$$G_c = K_c(1+\tau_D s),\; G_p = \frac{K_p}{\tau_p s + 1}$$

$$\frac{C}{R} = \frac{K_c K_p(\tau_D s + 1)}{(\tau_p + K_c K_p \tau_D)s + 1 + K_c K_p} = K_{cl}\,\frac{\tau_D s + 1}{\tau_{cl} s + 1}$$

$$K_{cl} = \frac{K_c K_p}{1 + K_c K_p}, \qquad \tau_{cl} = \frac{\tau_p + K_c K_p \tau_D}{1 + K_c K_p}$$

- same closed-loop gain as P: **offset remains**
- still first order; D adds a lead (zero)

The ideal D makes C jump at $t = 0^+$ to $K_{cl}\tau_D/\tau_{cl}$. On a first-order process derivative action brings little; it pays off on higher-order processes, where it adds damping.

Example 2.5 P control, integrating process

$G_c = K_c,\ G_p = \dfrac{K_p}{s}$ (e.g. a level with a pumped outlet). What is the offset?

Servo

$$\frac{C}{R} = \frac{K_c K_p}{s + K_c K_p} = \frac{1}{\tau s + 1},\quad \tau = \frac{1}{K_c K_p}$$

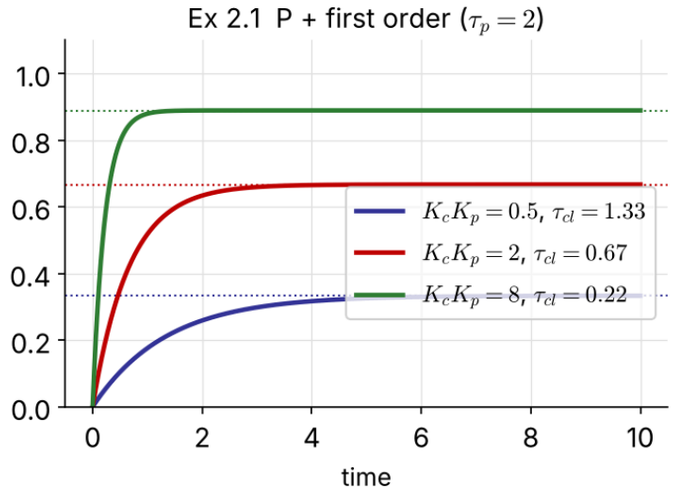
$C(\infty) = 1$: **no offset** for set-point changes. The process already contains an integrator.

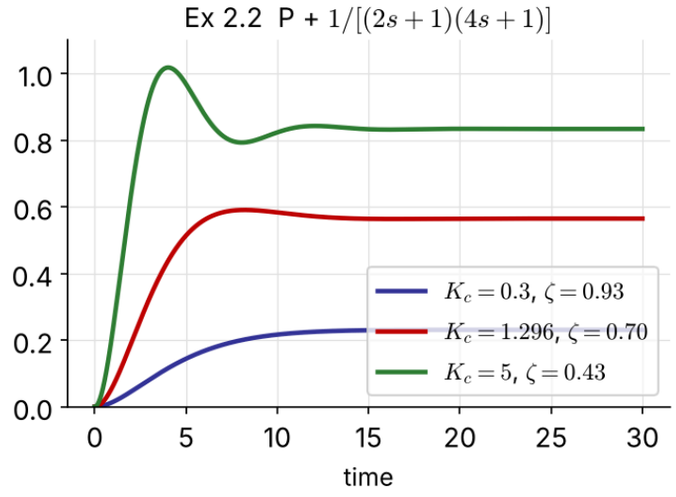
Regulator (load at the process input, $G_L = G_p$)

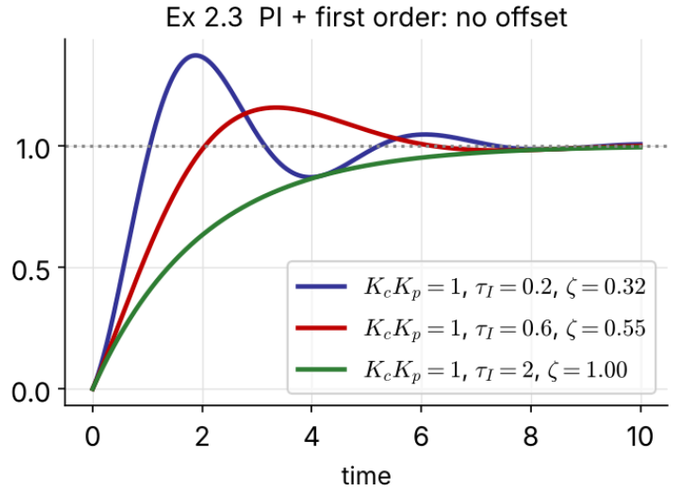
$$\frac{C}{L} = \frac{K_p}{s + K_c K_p} \Rightarrow C(\infty) = \frac{1}{K_c}$$

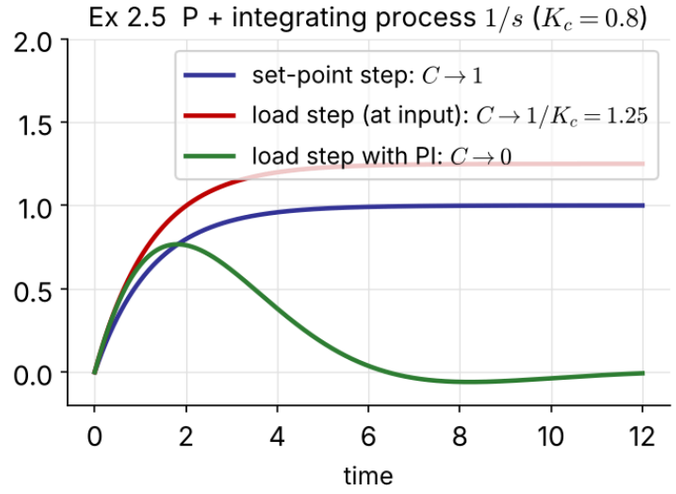
A step **disturbance** still leaves an offset. Only an integrator in the **controller** (PI) removes it.

Examples 2.1–2.5 at a glance









Ex	G_c	G_p	CL order	offset
2.1	P	1st	1	$\dfrac{1}{1+K_cK_p}$
2.2	P	2nd	2	$\dfrac{1}{1+K_cK_p}$
2.3	PI	1st	2	0
2.4	PD	1st	1	$\dfrac{1}{1+K_cK_p}$
2.5	P	K_p/s	1	0 (servo)

Offset disappears only when the loop contains an **integrator** between the error and the disturbance.

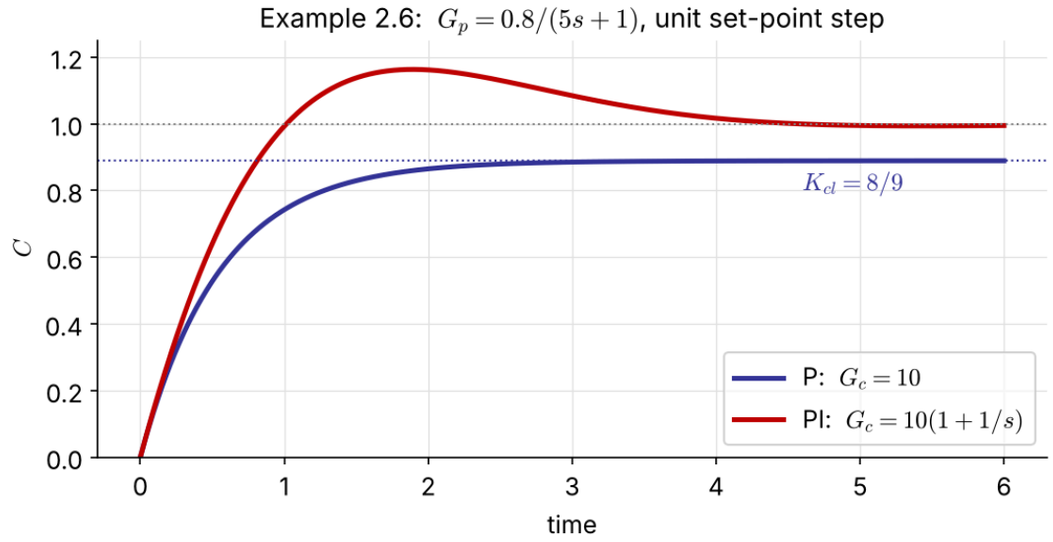
Example 2.6 Closed-loop simulation

Python

```python
import numpy as np, control as ct
import matplotlib.pyplot as plt

Gp = ct.tf(0.8, [5, 1])
t = np.linspace(0, 6, 601)

Gc = 10                          # P
Gcl = ct.feedback(Gp*Gc, 1)      # 8/9
plt.plot(t, ct.step_response(Gcl, t).outputs)

Gc = ct.tf([10, 10], [1, 0])     # PI
Gcl = ct.feedback(Gp*Gc, 1)
plt.plot(t, ct.step_response(Gcl, t).outputs)
```

MATLAB

```matlab
Gp = tf(0.8, [5 1]);

Gc  = 10;                        % P
Gcl = feedback(Gp*Gc, 1);
step(Gcl); hold on

Gc  = tf([10 10], [1 0]);        % PI: 10(1 + 1/s)
Gcl = feedback(Gp*Gc, 1);
step(Gcl)
```

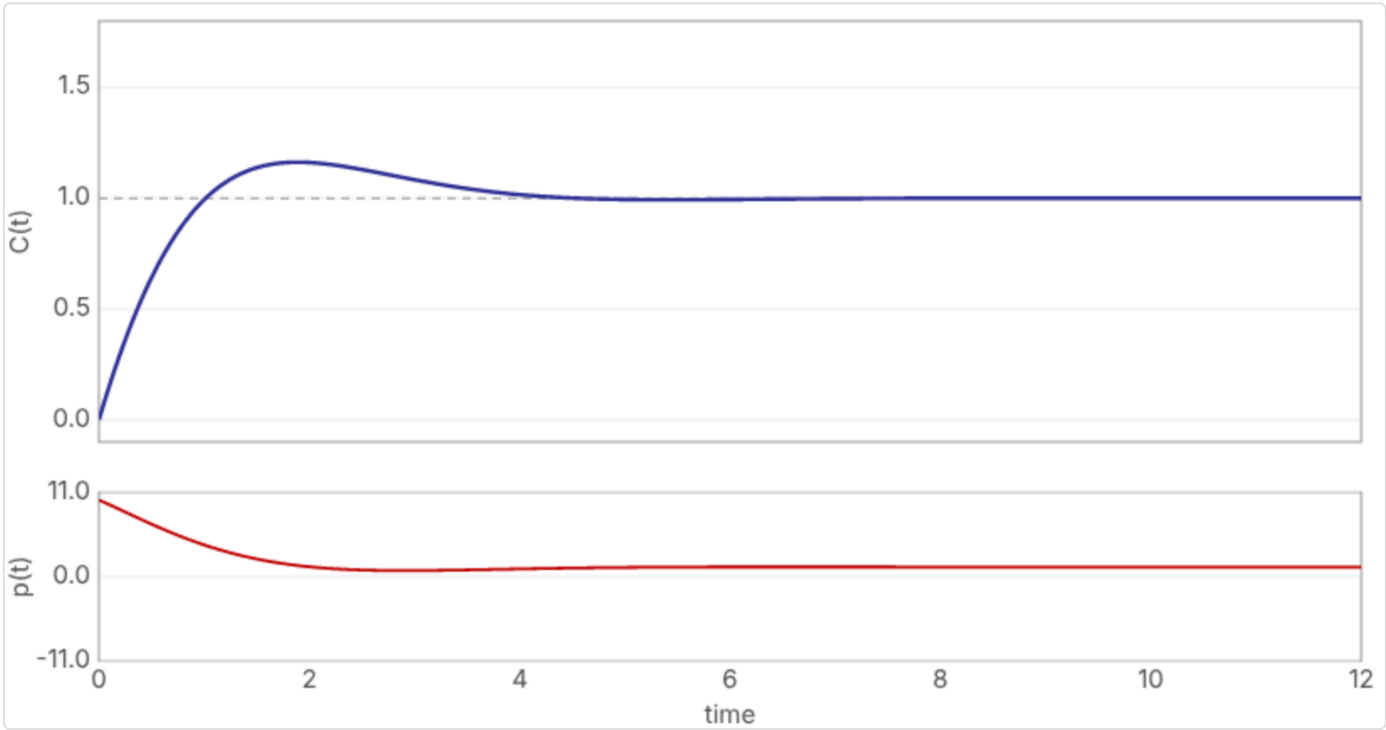


P: $K_{cl} = 8/9$, offset 0.11. PI: no offset, $\zeta = 0.71$, 16% overshoot. In Simulink the same loop is a Step, a Sum, two gains with an integrator (PI) and a Transfer Fcn block.

Interactive: try the modes yourself

process 0.8/(5s+1) — Ex 2.6 mode PI input set-point step D on measurement

K_c = **10.00** τ_I = **1.00** τ_D = —



final value **1.000**
offset $r - C(\infty)$ = **0.000 none**
peak **1.163** (overshoot 16%)
max $|p|$ = **10.02**

load enters at the process input ($G_L = G_p$);
derivative filter α = 0.1

Interactive version: seoultechpse.github.io/ProCon/ch2/ch2_analysis.html — process, mode, K_c, τ_I, τ_D, set-point or load step.

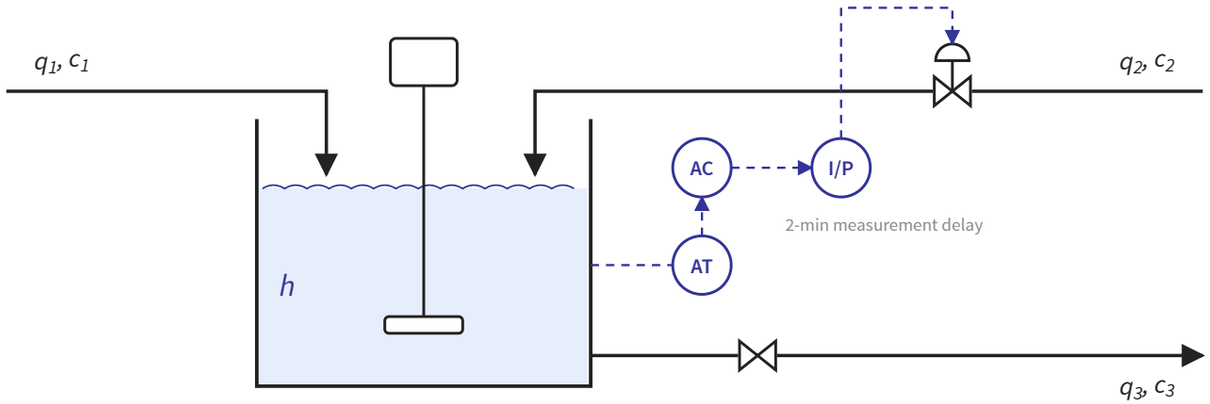
2.4 Choice of Controllers

Which mode for which variable?

Variable	Usual choice	Why
level	P	only needs to stay within a range around the set point
flow	PI	fast, noisy loop: low to moderate K_c, small τ_I for fast tracking
temperature	PID	heat-transfer lags make it sluggish; offsets are rarely acceptable; PI alone is slow
pressure, flash-drum vapor	PI	fast and tight, like flow control
pressure, column top (via condenser cooling)	P, or PID if speed matters	slow heat-exchanger dynamics rule out PI
composition, fast analyzer (pH, UV/IR)	PI or PID	decide from the process dynamics
composition, slow analyzer (GC)	sampled-data control	large measurement delay

Know the hardware (actuators, sensors) from handbooks and manuals. Prevalent time delay is what makes chemical process control harder than many mechanical or electrical systems.

Problem #3 (homework) Blending tank



The exit concentration c_3 is controlled with stream 2.

(a) Draw the block diagram of the composition loop. (b) Derive each transfer function and substitute numbers. (c) The PI controller was tuned at the nominal conditions. Should it be retuned when (i) $\bar{c}_2$ changes to 8.5 lb/ft^3; (ii) the transmitter span becomes 3–14 lb/ft^3; (iii) its zero is moved so that it reads 4–10 lb/ft^3? Justify briefly.

1. The tank is perfectly mixed; all densities are equal and constant.
2. q_2 and c_2 vary with time; stream 1 is constant.
3. Exit valve: $q_3 = C_v\sqrt{h}$.
4. A 2-min delay in the composition measurement; the transmitter output varies linearly from 4 to 20 mA as c_3 goes from 3 to 9 lb/ft^3.
5. Pneumatic valve, negligible dynamics:

p_t (psi)	6	9	12
q_2 (gal/min)	20	15	10

6. Electronic, direct-acting PI controller.
7. I/P transducer: negligible dynamics, 0.3 psi/mA.
8. Nominal: $\rho = 75$ lb/ft^3, $\bar{q}_1 = 10$, $\bar{q}_2 = 15$ gal/min, $\bar{c}_3 = 5$, $\bar{c}_2 = 7$ lb/ft^3, $C_v = 12.5$ gal/min/ft$^{1/2}$, tank diameter 4 ft.

Summary

Setting up

- $C = G_{sp}R + G_{load}L$
- forward path / (1 + loop)
- one characteristic equation $1 + G_{OL} = 0$ for servo and regulator
- every block has units; K_m in front of the comparator equals the gain of G_m

PID modes

- **P**: faster, but leaves an offset
- **I**: removes the offset, costs damping; beware windup
- **D**: adds damping, amplifies noise; filter it and put it on the measurement
- closed-loop order, τ and ζ follow from substituting G_c and G_p

Next (Ch 3): how to choose K_c, τ_I and τ_D. Then (Ch 4–5): when the loop goes unstable.

Resources

- **Slides (interactive HTML):** seoultechpse.github.io/ProCon/ch2/ch2_analysis.html
- **PDF:** slides · handout with note space
- **Python notebook:** Open in Colab — every derivation (sympy), calculation and figure of this chapter
- **Closed-loop explorer:** standalone page