Process Dynamics and Control

3. Tuning of Single-Loop Control Systems

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What are we up to?

A PID controller needs only **three numbers**, K_c, τ_I and τ_D, yet choosing them is one of the most confusing parts of learning control.

Different tuning techniques give **similar but not identical** results. There is no “best” or “absolutely correct” answer, only trade-offs between speed, overshoot and robustness.

Approach	Experiment	Model	Methods
Empirical (3.1)	open-loop step test	FOPDT fit $\frac{Ke^{-\theta s}}{\tau s + 1}$	Ziegler–Nichols, Cohen–Coon, ITAE
	closed-loop test	K_{cu}, T_u only	ZN ultimate cycle
Model-based (3.2)	—	transfer function	direct synthesis, IMC

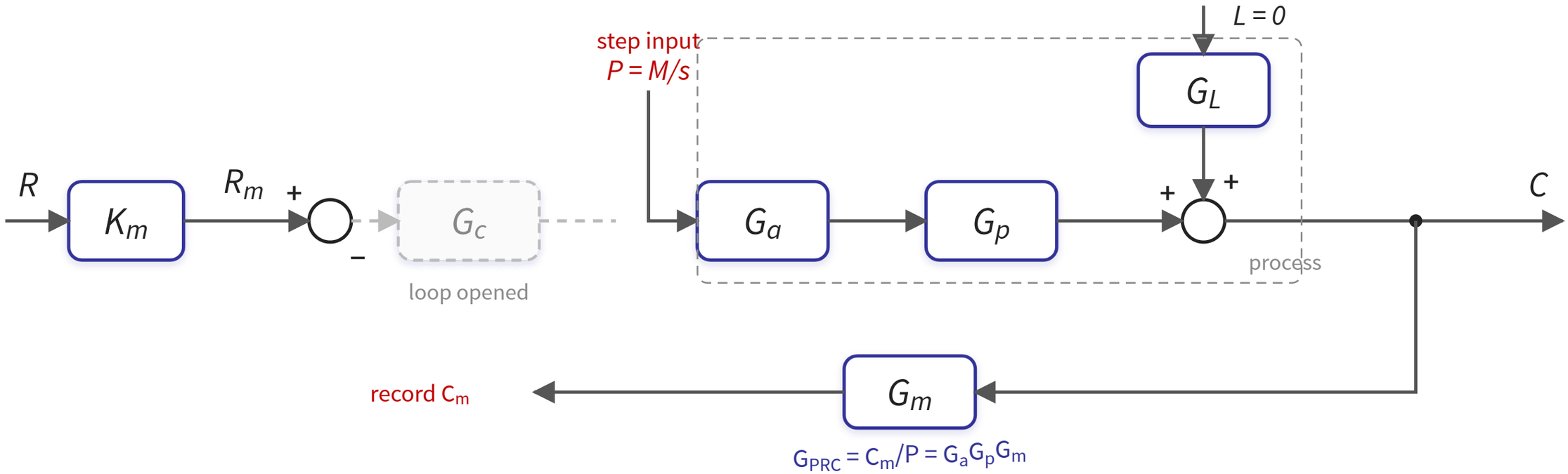
3.1 Tuning with Empirical Relations

Empirical tuning in practice

1. Run an **experiment**: an open-loop step test, or a closed-loop test.
2. **Fit** the response to a simple model.
3. Compute the settings from **empirical relations**.

The settings are a **quick-and-dirty starting point**. They are seldom optimal and usually need **field tuning** after installation to meet the actual response specifications.

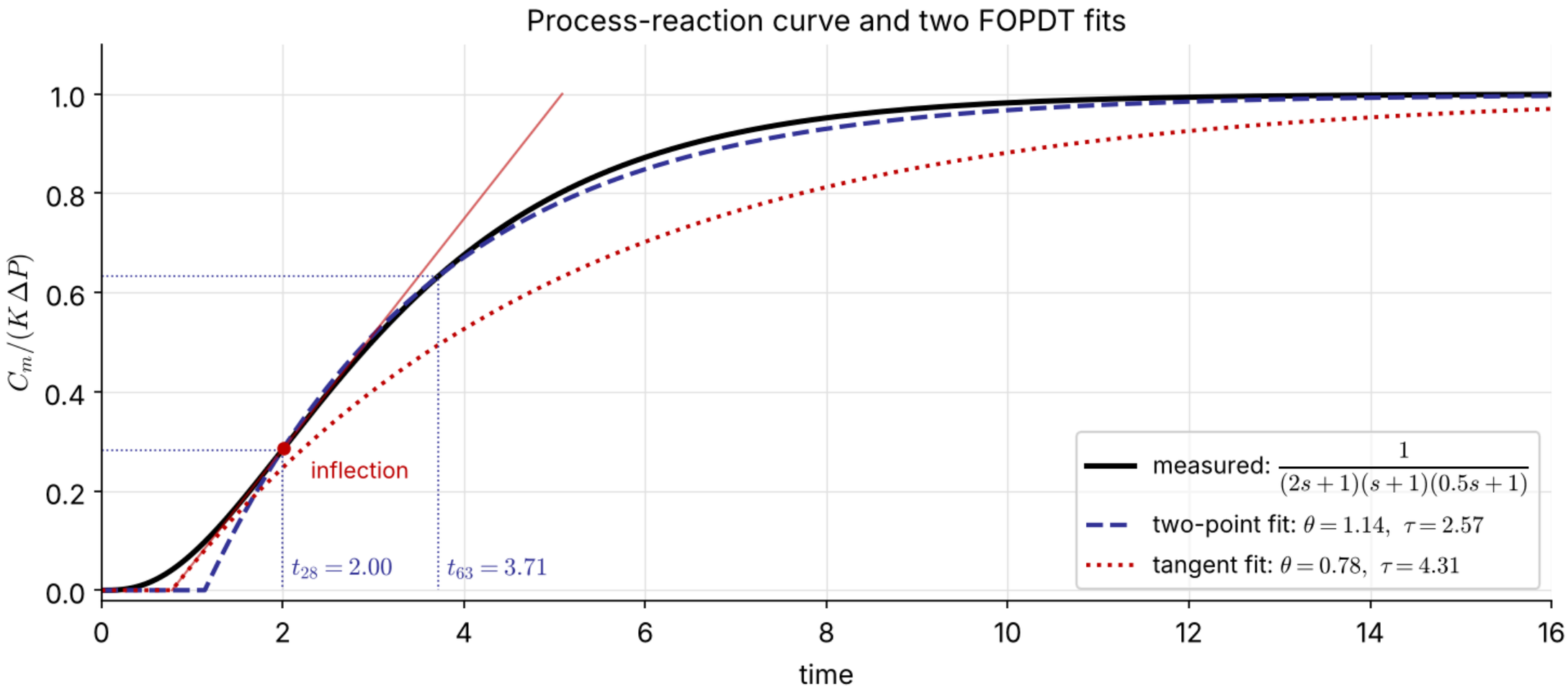
Process-reaction curve: the open-loop step test



With the controller disconnected, step the controller output by $P = M/s$ and record the **measured** output. What we see is the lumped response of valve, process and sensor:

$$G_{PRC} = \frac{C_m}{P} = G_a G_p G_m \approx \frac{Ke^{-\theta s}}{\tau s + 1}$$

Fitting the FOPDT model



Two-point method (Smith): read t_{28} and t_{63} at 28.3 % and 63.2 % of the final change

$$\tau = 1.5\,(t_{63} - t_{28}), \qquad \theta = t_{63} - \tau, \qquad K = \frac{\Delta C_m}{\Delta P}$$

Tangent method: draw the tangent at the inflection point. Its intercepts give θ and τ.

Simple, but it **overestimates** τ and underestimates θ, so it gives aggressive settings.

Ziegler–Nichols and Cohen–Coon (open-loop test)

Controller		Cohen–Coon	Ziegler–Nichols
P	$K_c K$	$\frac{\tau}{\theta} + \frac{1}{3}$	$\frac{\tau}{\theta}$
PI	$K_c K$	$0.9\frac{\tau}{\theta} + \frac{1}{12}$	$0.9\frac{\tau}{\theta}$
	τ_I	$\theta\,\frac{30 + 3\theta/\tau}{9 + 20\theta/\tau}$	$3.3\,\theta$
PID	$K_c K$	$\frac{4}{3}\frac{\tau}{\theta} + \frac{1}{4}$	$1.2\frac{\tau}{\theta}$
	τ_I	$\theta\,\frac{32 + 6\theta/\tau}{13 + 8\theta/\tau}$	$2\,\theta$
	τ_D	$\theta\,\frac{4}{11 + 2\theta/\tau}$	$0.5\,\theta$

Both aim at a **quarter-decay ratio**, which is fast but oscillatory. Note $K_c \propto 1/K$ and $K_c \propto \tau/\theta$: the larger the dead time relative to the time constant, the gentler the controller.

Minimum error-integral criteria

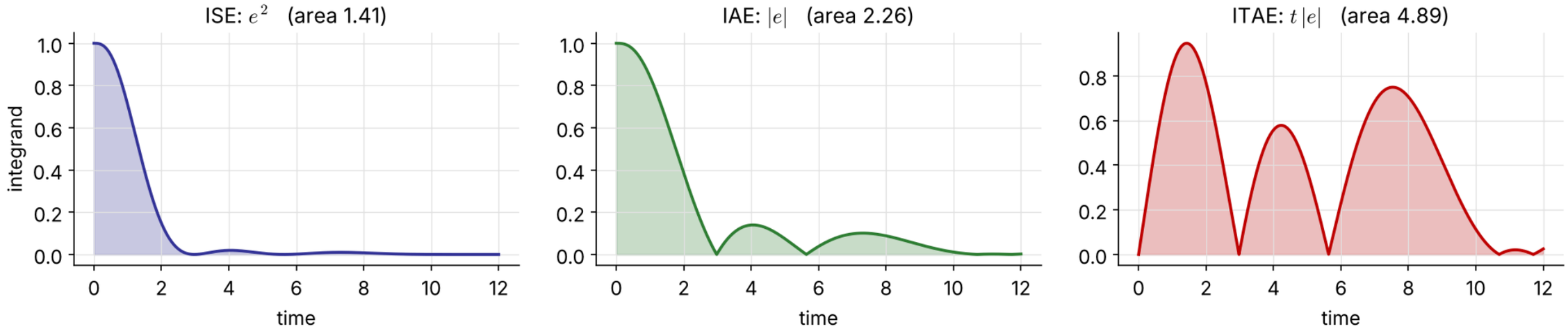
$$\mathrm{ISE} = \int_0^\infty e^2(t)\,dt$$

$$\mathrm{IAE} = \int_0^\infty |e(t)|\,dt$$

$$\mathrm{ITAE} = \int_0^\infty t\,|e(t)|\,dt$$

- **ISE** weights large errors heavily: fast initial response, but oscillatory.
- **IAE** weights all errors equally.
- **ITAE** weights errors that **persist**: it penalizes slow oscillation, so the settings are well damped and **conservative**. Recommended.

Same response, three weightings: ITAE punishes errors that persist



Minimum-ITAE tuning relations

Load change

$$K_c = \frac{a_1}{K}\left(\frac{\tau}{\theta}\right)^{b_1}, \quad \tau_I = \frac{\tau}{a_2}\left(\frac{\theta}{\tau}\right)^{b_2}$$

	a_1	b_1	a_2	b_2	a_3	b_3
PI	0.859	0.977	0.674	0.680	—	—
PID	1.357	0.947	0.842	0.738	0.381	0.995

Set-point change

$$K_c = \frac{a_1}{K}\left(\frac{\tau}{\theta}\right)^{b_1}, \quad \tau_I = \frac{\tau}{a_2 - b_2\,\theta/\tau}$$

	a_1	b_1	a_2	b_2	a_3	b_3
PI	0.586	0.916	1.03	0.165	—	—
PID	0.965	0.855	0.796	0.147	0.308	0.929

both: $$\tau_D = a_3\,\tau\left(\frac{\theta}{\tau}\right)^{b_3}$$

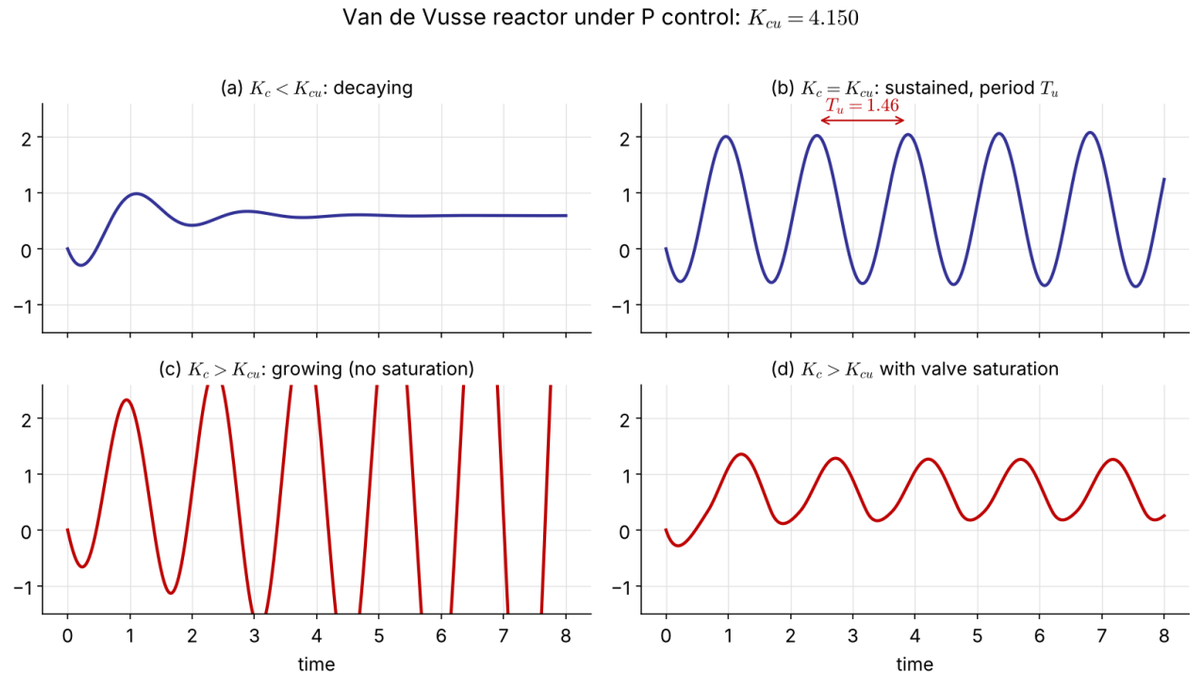
Load settings are more aggressive (higher K_c, shorter τ_I) than set-point settings. Choose the table for the disturbance you expect most.

Ziegler–Nichols ultimate-cycle method (closed-loop test)

For processes that are **open-loop stable** but can be destabilized by feedback.

1. P-only control: set τ_I to its maximum and τ_D to its minimum if they cannot be switched off.
2. Raise K_c slowly, with small set-point or load bumps, until the oscillation is **sustained**.
3. Record the **ultimate gain** K_{cu} and the **ultimate period** T_u.

These are exactly the K_{cu} and $\omega_u = 2\pi/T_u$ of Ch 4 (Routh, direct substitution) and Ch 5 (gain margin).



With valve saturation (d) the oscillation stays bounded and *looks* sustained even above K_{cu}, so check that the valve is not hitting its limits.

Ziegler–Nichols ultimate-cycle settings

Controller		K_c	τ_I	τ_D
P		$0.5\,K_{cu}$	—	—
PI		$0.455\,K_{cu}$	$0.833\,T_u$	—
PID	quarter decay	$0.6\,K_{cu}$	$0.5\,T_u$	$0.125\,T_u$
	some overshoot	$0.33\,K_{cu}$	$0.5\,T_u$	$0.333\,T_u$
	no overshoot	$0.2\,K_{cu}$	$0.5\,T_u$	$0.333\,T_u$

Advantages: no model is needed, and the test is done on the real loop. Drawbacks: the plant is pushed to the edge of instability, and it takes time.

Example: the Van de Vusse reactor

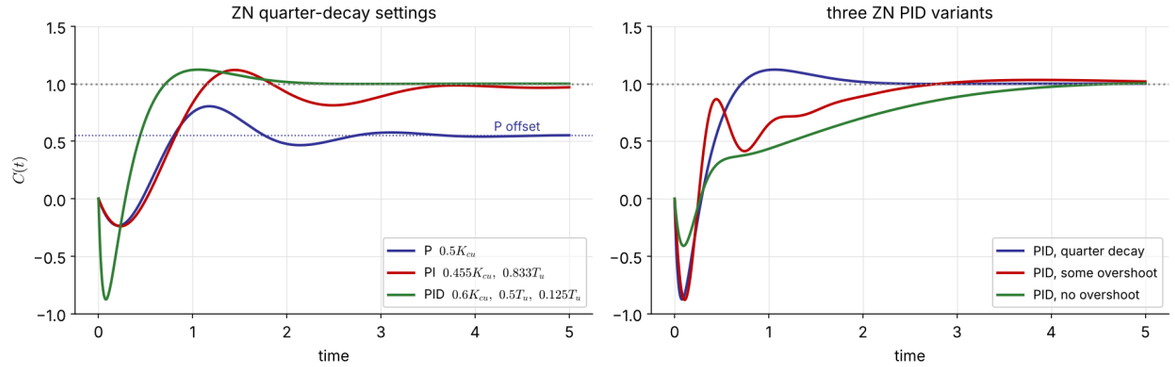
$$G_p(s) = \frac{0.5858\,(-0.3549s + 1)}{0.1858s^2 + 0.8627s + 1} \qquad \text{(RHP zero: inverse response)}$$

Raising K_c under P control (ultimate-cycle figure, two slides back):

$$K_{cu} = 4.150, \qquad T_u = 1.46$$

(the original read $T_u = 1.45$ off the plot; exact: $\omega_u = 4.297, T_u = 2\pi/\omega_u = 1.462$)

	K_c	τ_I	τ_D
P	2.07	—	—
PI	1.89	1.22	—
PID	2.49	0.73	0.18



- P: offset, steady state 0.55
- PI: no offset, slow and oscillatory
- PID: fastest, but the deep **inverse response** (-0.88) is the price of high gain on an RHP-zero process

Van de Vusse in code

Python

```python
import numpy as np, control as ct
from tuning import pid_tf          # ideal PID with filtered derivative (alpha = 0.1)

Gp = ct.tf(0.5858*np.array([-0.3549, 1]), [0.1858, 0.8627, 1])
Kcu, _, wu, _ = ct.margin(Gp)       # 4.150 — no trial and error needed
Tu = 2*np.pi/wu                     # 1.462

settings = {'P':   (0.5*Kcu,),
            'PI':  (0.455*Kcu, 0.833*Tu),
            'PID': (0.6*Kcu, 0.5*Tu, 0.125*Tu)}
t = np.linspace(0, 5, 2001)
for name, pars in settings.items():
    T = ct.feedback(pid_tf(*pars)*Gp, 1)
    plt.plot(t, ct.step_response(T, t).outputs, label=name)
```

MATLAB

```matlab
Gp = tf(0.5858*[-0.3549 1], [0.1858 0.8627 1]);
[Kcu, ~, wu] = margin(Gp);   Tu = 2*pi/wu;      % 4.150, 1.462

C_P   = 0.5*Kcu;
C_PI  = pidstd(0.455*Kcu, 0.833*Tu);
C_PID = pidstd(0.6*Kcu, 0.5*Tu, 0.125*Tu, 10);  % N = 10 <-> alpha = 0.1
step(feedback(C_P*Gp,1), feedback(C_PI*Gp,1), feedback(C_PID*Gp,1), 5)
legend('P','PI','PID')
```

Example 3.6 Same plant, five tuning rules

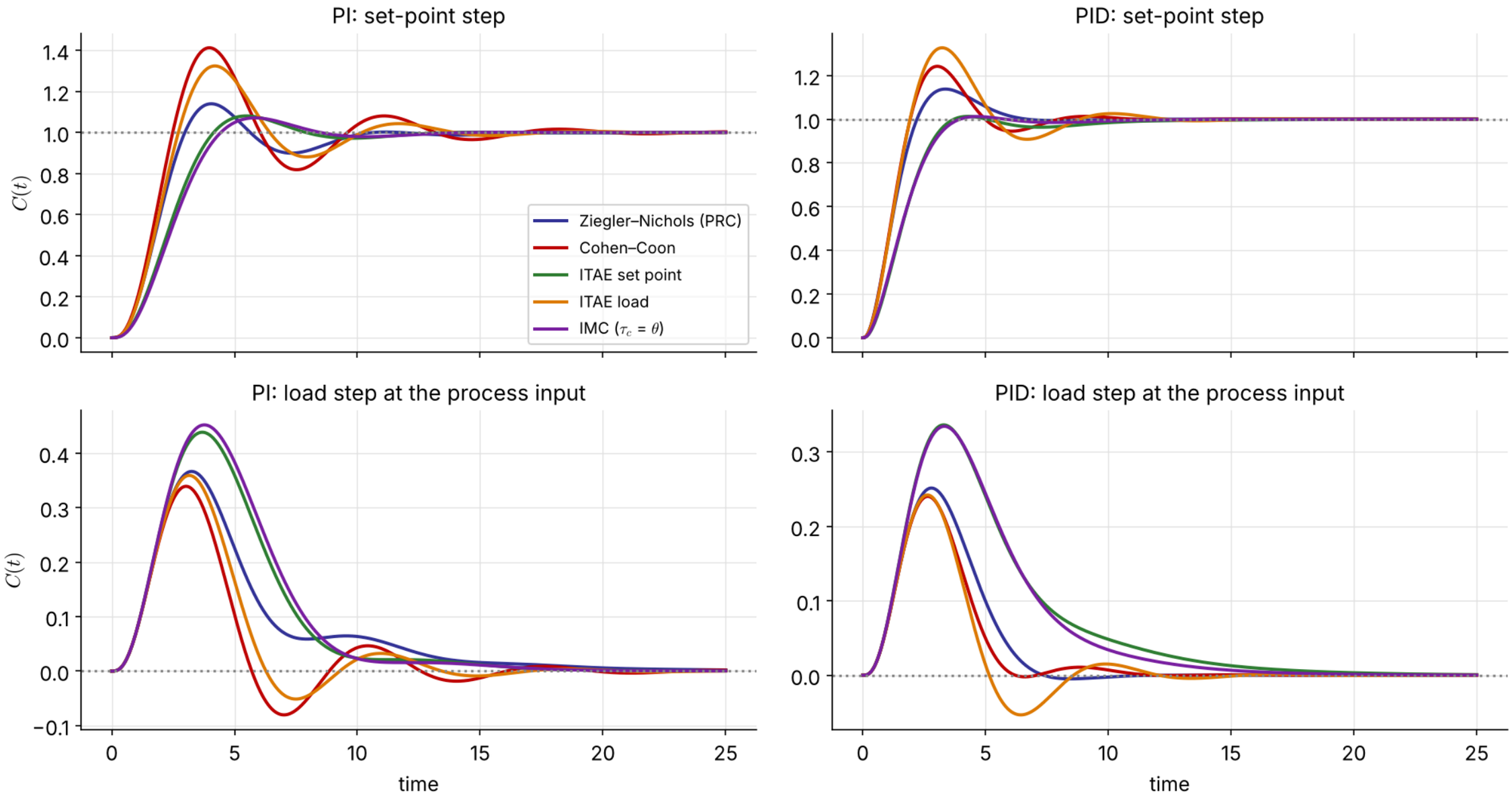
"True" process $\dfrac{1}{(2s+1)(s+1)(0.5s+1)}$, two-point FOPDT fit

$K = 1,\ \tau = 2.57,\ \theta = 1.14\ (\theta/\tau = 0.44)$

PID	K_c	τ_I	τ_D	OS	IAE sp	IAE load	GM	PM
Ziegler–Nichols	2.71	2.28	0.57	14%	1.48	0.87	21	55°
Cohen–Coon	3.26	2.39	0.38	24%	1.67	**0.74**	21	44°
ITAE set point	1.93	3.52	0.37	**1%**	1.84	1.82	37	67°
ITAE load	2.93	1.68	0.44	33%	2.07	0.81	23	38°
IMC, $\tau_c = \theta$	1.84	3.14	0.47	**1%**	1.74	1.71	37	68°

The aggressive rules (CC, ITAE load) reject loads best and overshoot most. The gentle ones (ITAE set point, IMC) are almost overshoot-free and have twice the phase margin. **No rule wins on every criterion.**

Example 3.6: responses



Tuning rules in code

Python

```python
import numpy as np, control as ct
from tuning import RULES, pid_tf, fit_two_point     # tuning.py in ch3/

G = ct.tf(1, np.polymul(np.polymul([2, 1], [1, 1]), [0.5, 1]))
t = np.linspace(0, 30, 30001)
K, tau, theta = fit_two_point(t, ct.step_response(G, t).outputs)   # 1, 2.57, 1.14

for name, rule in RULES.items():
    Kc, tI, tD = rule('PID', K, tau, theta)
    T = ct.feedback(pid_tf(Kc, tI, tD)*G, 1)
    gm, pm, *_ = ct.margin(pid_tf(Kc, tI, tD)*G)
    print(f'{name:22s} Kc={Kc:.2f} tI={tI:.2f} tD={tD:.2f}  GM={gm:.1f} PM={pm:.0f}')
```

MATLAB

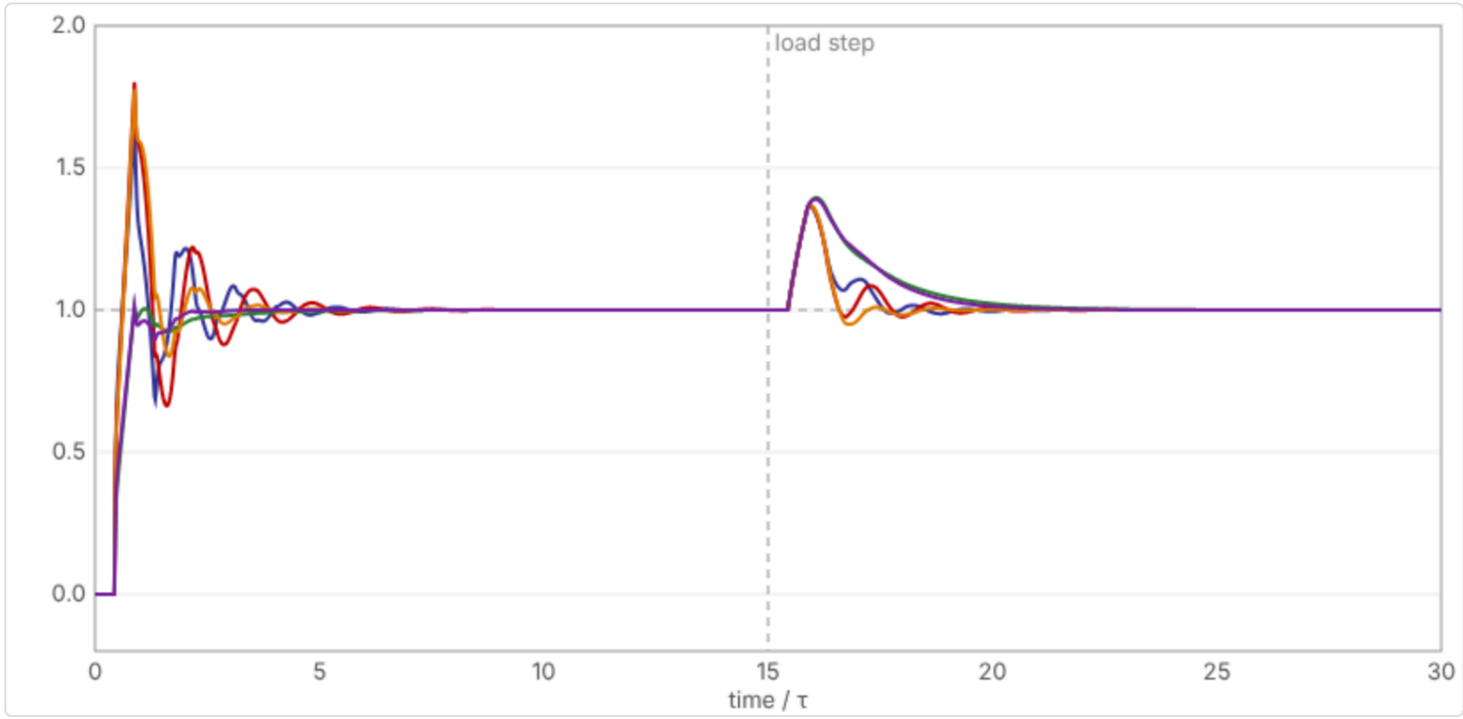
```matlab
G = tf(1, conv(conv([2 1],[1 1]),[0.5 1]));
[y, t] = step(G, 30);
t28 = t(find(y >= 0.283, 1));   t63 = t(find(y >= 0.632, 1));
tau = 1.5*(t63 - t28);   theta = t63 - tau;   K = y(end);   % 2.57, 1.14, 1

r = theta/tau;                                  % ITAE set point, PID
Kc = 0.965/K*(1/r)^0.855;  tI = tau/(0.796 - 0.147*r);  tD = 0.308*tau*r^0.929;
C = pidstd(Kc, tI, tD, 10);
figure, step(feedback(C*G, 1)),  figure, margin(C*G)
```

Interactive: compare the rules yourself

controller PID ▾ θ/τ = **0.44** IMC τ_c/θ = **1.00**

⚠ = still oscillating at t = 30τ · plant $e^{-\theta s}/(\tau s+1)$, τ = 1 · set-point step at 0, load step at t = 15τ



	K_cK	τ_I/τ	τ_D/τ	IAE
☑ Ziegler–Nichols	2.73	0.88	0.22	1.26
☑ Cohen–Coon	3.28	0.92	0.15	1.44
☑ ITAE set point	1.95	1.37	0.14	1.40
☑ ITAE load	2.95	0.65	0.17	1.20
☑ IMC (τ_c=1.00θ)	1.85	1.22	0.18	1.32

Interactive version: seoultechpse.github.io/ProCon/ch3/ch3_tuning.html — vary θ/τ, PI/PID and the IMC τ_c.

3.2 Direct Synthesis and Internal Model Control

Design the closed-loop response first

A different philosophy: **specify the desired closed-loop response**, then solve for the controller. The result depends on the process model, so this is a **model-based design**.

Servo problem ($L = 0, G_m = G_a = 1$):

$$\frac{C}{R} = \frac{G_c G_p}{1 + G_c G_p} \quad \Longrightarrow \quad G_c = \frac{1}{G_p}\left(\frac{C/R}{1 - C/R}\right) \qquad (3.2\text{-}1)$$

$C/R = 1$ would need infinite gain. Instead ask for a first-order response with **unit gain** (no offset):

$$\frac{C}{R} = \frac{1}{\tau_c s + 1} \;\Rightarrow\; G_c = \frac{1}{G_p}\,\frac{1}{\tau_c s}$$

(3.2-2), (3.2-3)

τ_c is the **design parameter**: the closed-loop time constant. The closed-loop pole is $s = -1/\tau_c$ whatever G_p is.

It looks like magic. In reality, saturation limits how small τ_c can be, and there is a more serious catch (next slide).

The catch, and two more target responses

$G_c = 1/(G_p \tau_c s)$ **cancels** the process: the poles of G_c are the zeros of G_p. An RHP zero (Van de Vusse) gives an **unstable controller**, and the cancellation is internally unstable even though C/R looks fine. Dead time would need $e^{+\theta s}$, which requires knowing the future. IMC fixes both.

Dead time in the target (cannot be removed):

$$\frac{C}{R} = \frac{e^{-\theta s}}{\tau_c s + 1} \qquad (3.2\text{-}4)$$

$$G_c = \frac{1}{G_p} \frac{e^{-\theta s}}{(\tau_c s + 1) - e^{-\theta s}} \approx \frac{1}{G_p} \frac{e^{-\theta s}}{(\tau_c + \theta) s}$$

using $e^{-\theta s} \approx 1 - \theta s$ in the denominator.

Underdamped target:

$$\frac{C}{R} = \frac{1}{\tau^2 s^2 + 2\zeta \tau s + 1} \qquad (3.2\text{-}6)$$

$$G_c = \frac{1}{G_p} \frac{1}{\tau^2 s^2 + 2\zeta \tau s} \qquad (3.2\text{-}7)$$

Example 3.1 First-order process → PI

$$G_p = \frac{K_p}{\tau_p s + 1}, \qquad \frac{C}{R} = \frac{1}{\tau_c s + 1}$$

$$G_c = \frac{\tau_p s + 1}{K_p} \frac{1}{\tau_c s} = \frac{\tau_p}{K_p \tau_c} \left(1 + \frac{1}{\tau_p s} \right)$$

PI controller: $K_c = \dfrac{\tau_p}{K_p \tau_c}, \qquad \tau_I = \tau_p$

K_c is inversely proportional to the process gain, and a faster target (small τ_c) means a larger gain. The integral time simply equals the process time constant.

Example 3.2 Second-order overdamped process → PID

$$G_p = \frac{K_p}{(\tau_1 s + 1)(\tau_2 s + 1)} \quad \Rightarrow \quad G_c = \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{K_p \tau_c s}$$

Ideal PID: expand the numerator and factor out $(\tau_1 + \tau_2)$:

$$G_c = \frac{\tau_1 + \tau_2}{K_p \tau_c}\left[1 + \frac{1}{(\tau_1 + \tau_2)s} + \frac{\tau_1 \tau_2}{\tau_1 + \tau_2}\, s\right]$$

Series (interacting) PID, $\tau_1 > \tau_2$:

$$G_c = \frac{\tau_1}{K_p \tau_c}\left(1 + \frac{1}{\tau_1 s}\right)(\tau_2 s + 1)$$

Same controller, two parameterizations: industrial controllers come in both forms, so check which one before entering the numbers.

Example 3.3 FOPDT process → PI

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$$G_p = \frac{K_p e^{-\theta s}}{\tau_p s + 1}, \qquad \frac{C}{R} = \frac{e^{-\theta s}}{\tau_c s + 1} \quad (3.2\text{-}4)$$

$$G_c \approx \frac{\tau_p s + 1}{K_p e^{-\theta s}} \frac{e^{-\theta s}}{(\tau_c + \theta) s} = \frac{\tau_p s + 1}{K_p (\tau_c + \theta) s}$$

PI controller: $K_c = \dfrac{\tau_p}{K_p(\tau_c + \theta)}, \qquad \tau_I = \tau_p$

The dead time cancels exactly and acts like extra τ_c, so K_c stays finite even for $\tau_c \to 0$.

Example 3.4 Overdamped process, underdamped target

(left blank in the original)

$$G_p = \frac{K_p}{(\tau_1 s + 1)(\tau_2 s + 1)}, \qquad \frac{C}{R} = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1} \qquad (3.2\text{-}6)$$

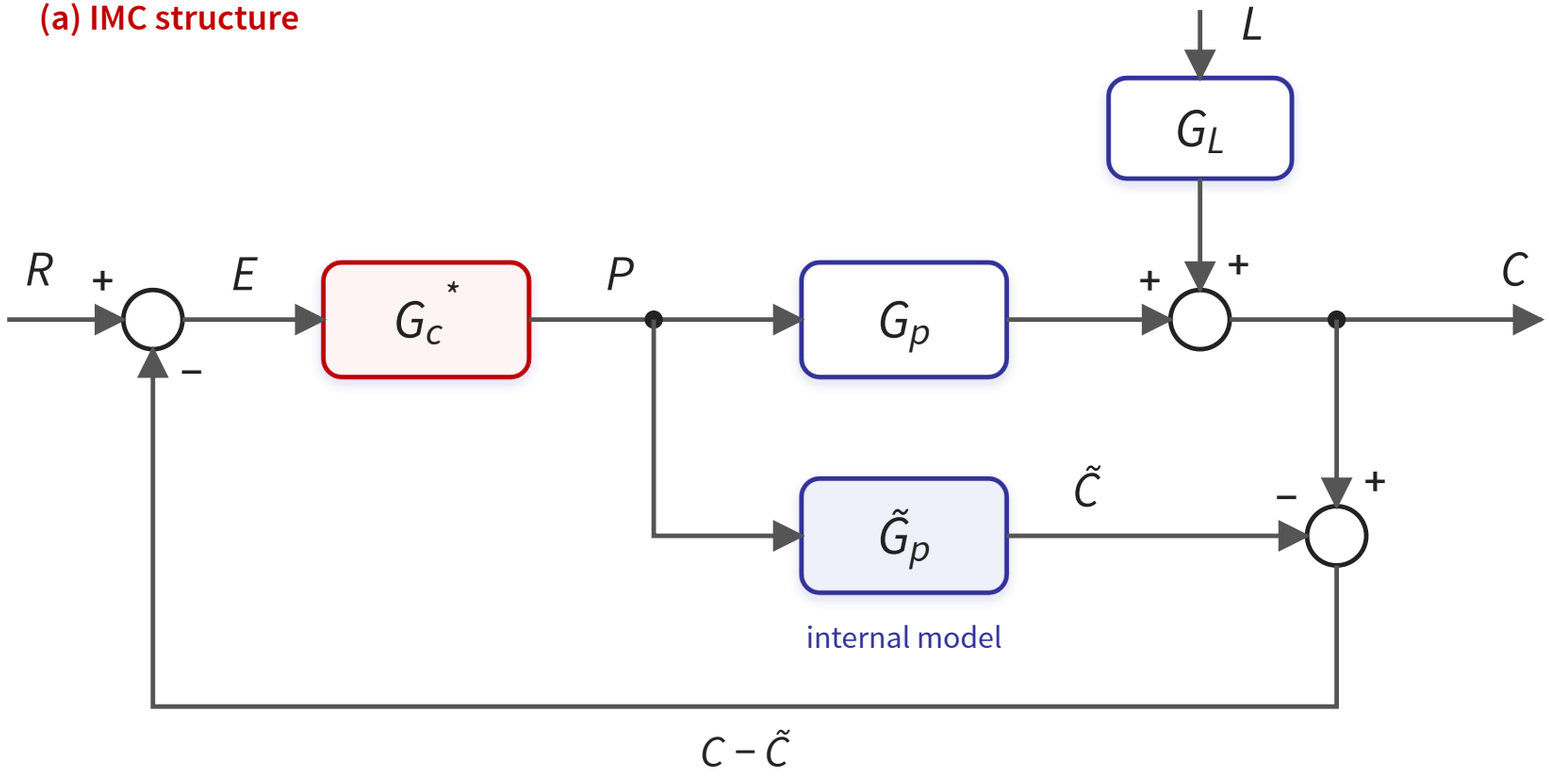
$$G_c = \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{K_p\, s\,(\tau^2 s + 2\zeta\tau)} = \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{K_p\, 2\zeta\tau\, s\left(\dfrac{\tau}{2\zeta} s + 1\right)}$$

PID with a derivative filter of time constant $\tau/2\zeta$:

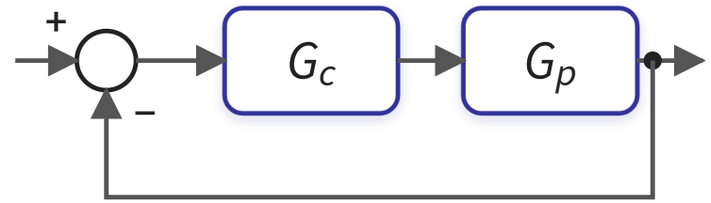
$$K_c = \frac{\tau_1 + \tau_2}{2\zeta\tau K_p}, \qquad \tau_I = \tau_1 + \tau_2, \qquad \tau_D = \frac{\tau_1 \tau_2}{\tau_1 + \tau_2}$$

Internal model control (IMC)

(a) IMC structure



(b) equivalent loop



$G_c = G_c^* / (1 - G_c^* \tilde{G}_p)$

The controller G_c^* runs alongside a **model** $\tilde{G}_p$. Only the **mismatch** $C - \tilde{C}$ (model error plus disturbance) is fed back.

$$E = R - (C - \tilde{C}), \quad P = G_c^* E, \quad \tilde{C} = \tilde{G}_p P$$

$$P = \frac{G_c^*}{1 - G_c^* \tilde{G}_p}(R - C) \Rightarrow \boxed{G_c = \frac{G_c^*}{1 - G_c^* \tilde{G}_p}} \quad (3.2\text{-}8)$$

An IMC design is implemented as a conventional controller G_c.

IMC closed loop and the model controller

$$C = \left[\frac{\left(1 - G_c^* \tilde{G}_p\right) G_L}{1 + G_c^* \left(G_p - \tilde{G}_p\right)}\right] L + \left[\frac{G_p G_c^*}{1 + G_c^* \left(G_p - \tilde{G}_p\right)}\right] R$$

With a perfect model ($\tilde{G}_p = G_p$), $C/R = G_p G_c^*$: an open-loop design, which is stable whenever G_p and G_c^* are.

Split the model so the inverse never contains an RHP zero or a dead time:

$$\tilde{G}_p = \tilde{G}_{p+}\, \tilde{G}_{p-}, \qquad G_c^* = \frac{1}{\tilde{G}_{p-}} \underbrace{\left[\frac{1}{\tau_c s + 1}\right]^r}_{\text{IMC filter}}, \quad r = 1, 2, \ldots \qquad (3.2\text{-}9)$$

- $\tilde{G}_{p+}$: all RHP zeros and dead time, scaled to steady-state gain 1
- $\tilde{G}_{p-}$: the invertible rest

Perfect model: $\dfrac{C}{R} = \dfrac{\tilde{G}_{p+}}{(\tau_c s + 1)^r}$. The non-invertible part is **kept**, not cancelled. Choose r so that G_c^* is proper.

Example 3.5 IMC for a FOPDT process

$$\tilde{G}_p = \frac{K_p e^{-\theta s}}{\tau_p s + 1} \approx \underbrace{\frac{K_p}{(\tau_p s + 1)\left(\frac{\theta}{2} s + 1\right)}}_{\tilde{G}_{p-}} \underbrace{\left(-\frac{\theta}{2} s + 1\right)}_{\tilde{G}_{p+}} \qquad \text{(1st-order Padé)}$$

$r = 1$:

$$G_c^* = \frac{(\tau_p s + 1)\left(\frac{\theta}{2} s + 1\right)}{K_p(\tau_c s + 1)} \qquad (3.2\text{-}11)$$

Into (3.2-8):

$$G_c = \frac{(\tau_p s + 1)(\theta s + 2)}{K_p(\theta + 2\tau_c)\, s}$$

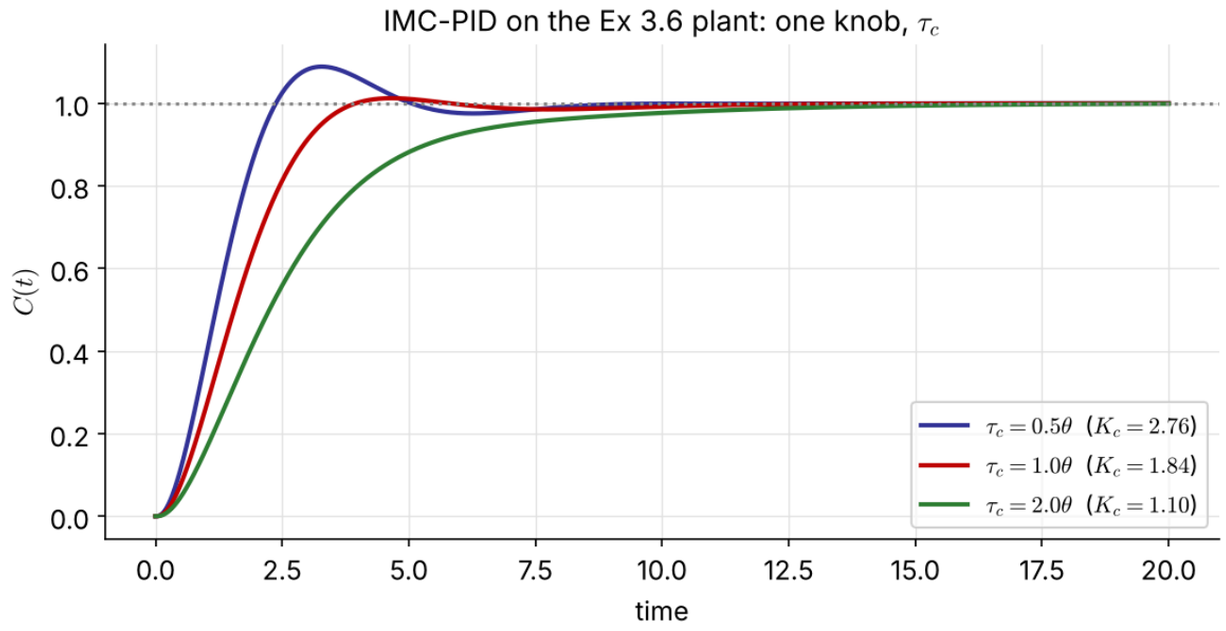
Ideal PID

$$K_c = \frac{1}{K_p} \frac{2\tau_p/\theta + 1}{2\tau_c/\theta + 1}$$

$$\tau_I = \tau_p + \frac{\theta}{2}, \qquad \tau_D = \frac{\tau_p}{2\tau_p/\theta + 1}$$

Checked symbolically in the notebook. Dropping the Padé factor in $\tilde{G}_{p-}$ gives the PI of Ex 3.3.

Choosing τ_c



τ_c is the **single knob** that trades speed for robustness:

- smaller τ_c: faster, more overshoot, less tolerant of model error
- larger τ_c: slower, more robust

Rule of thumb: $\tau_c \approx \theta$. For the IMC-PI on an exact FOPDT the loop becomes $e^{-\theta s}/(2\theta s)$:

$$\mathrm{GM} = \pi = 3.1, \qquad \mathrm{PM} = 61^\circ$$

Example 3.7 IMC for the Van de Vusse reactor

$\tilde{G}_{p+} = -0.3549s + 1$ (RHP zero),

$$\tilde{G}_{p-} = \frac{0.5858}{0.1858s^2 + 0.8627s + 1}$$

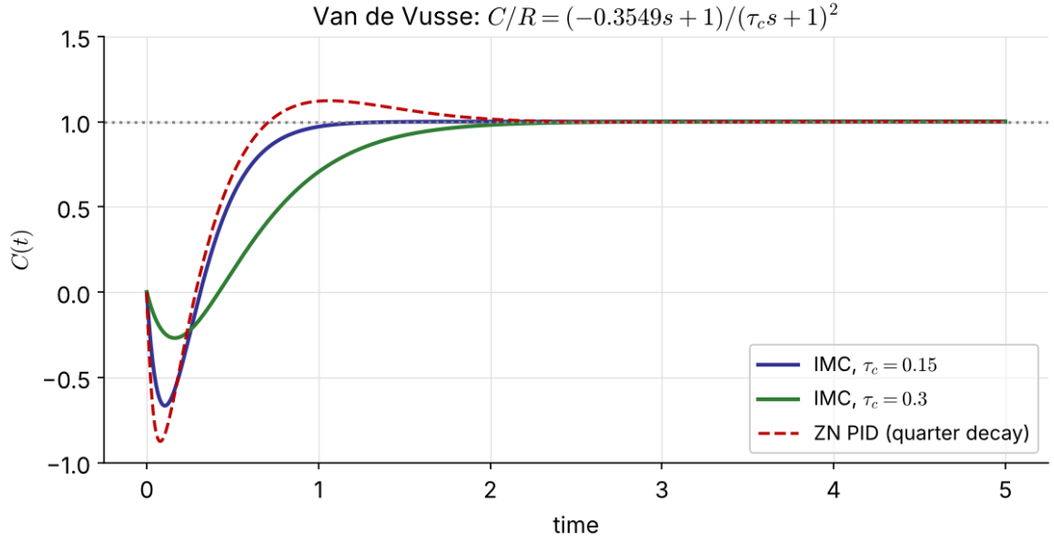
$\tilde{G}_{p-}$ has relative degree 2, so take $r = 2$:

$$\frac{C}{R} = \frac{-0.3549s + 1}{(\tau_c s + 1)^2}$$

(3.2-8) gives a **PID with a derivative filter**:

$$G_c = \frac{0.1858s^2 + 0.8627s + 1}{0.5858\, s\,(\tau_c^2 s + 2\tau_c + 0.3549)}$$

$$K_c = \frac{0.8627}{0.5858(2\tau_c + 0.3549)},\ \tau_I = 0.863,\ \tau_D = 0.215$$



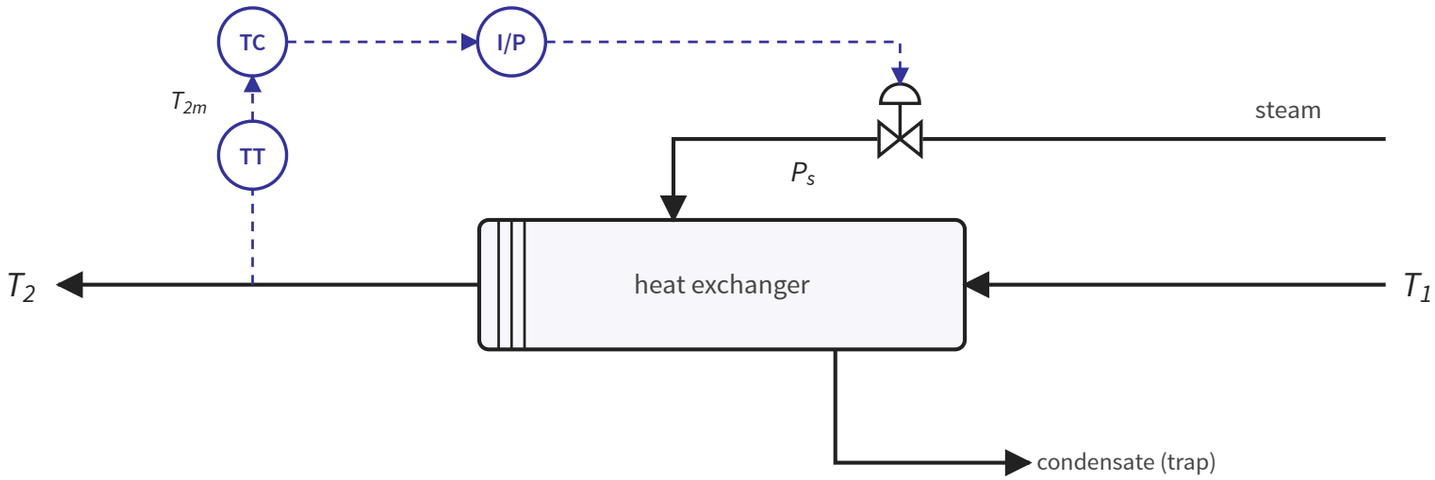
No overshoot, and τ_c sets the depth of the inverse response: -0.67 for $\tau_c = 0.15$ ($K_c = 2.25$), -0.27 for $\tau_c = 0.3$ ($K_c = 1.54$), against -0.88 for ZN PID.

Summary: which method when?

Situation	Method	Remarks
a step test is allowed	ZN / Cohen–Coon / ITAE from a FOPDT fit	fit by the two-point method
loop cannot be opened; plant tolerates cycling	ZN ultimate cycle	K_{cu}, T_u; Ch 4–5 compute them from a model
a model is available	**IMC** (or direct synthesis)	one physical knob τ_c; handles RHP zeros and dead time
robustness matters most	ITAE set point, IMC with $\tau_c \geq \theta$	check GM and PM (Ch 5)
load rejection matters most	ITAE load, Cohen–Coon	expect more overshoot on set-point changes

Whatever the rule: verify by simulation, then field-tune.

Problem #1 (homework)



A process stream is heated in a shell-and-tube exchanger, and T_2 is controlled with the steam valve. In an open-loop test the steam pressure P_s was stepped from 18 to 20 psig. Valve $K_v = 0.9$ psi/psi, I/P transducer $K_{IP} = 0.75$ psi/mA.

Determine PID settings by (a) IMC (choose a reasonable τ_c), (b) ITAE (set point), (c) ITAE (disturbance).

t (min)	T_{2m} (mA)
0	12.0
1	12.0
2	12.5
3	13.1
4	14.0
5	14.8
6	15.4
7	16.1
8	16.4
9	16.8
10	16.9
11	17.0
12	16.9

Resources

- **Slides (interactive HTML):** seoultechpse.github.io/ProCon/ch3/ch3_tuning.html
- **PDF:** slides · handout with note space
- **Python notebook:** Open in Colab — every calculation and figure of this chapter
- **Tuning relations:** `tuning.py` (ZN, Cohen–Coon, ITAE, IMC, FOPDT fitting)