Process Dynamics and Control

4. Stability of Closed-Loop Systems

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What are we up to?

The foremost concern in control-system design is to keep the closed loop stable — and the stability limit itself is a design tool.

Three techniques for the same question

1. **Routh–Hurwitz** — the stability *region* in parameter space
2. **Direct substitution $s = j\omega$** — the *boundary*: ultimate gain K_{cu} and frequency ω_u
3. **Root locus** — *where* the closed-loop poles go as K_c varies → dynamics, design

Running examples (each solved by all three methods)

- Ex 4.2: P control of $\dfrac{1}{(s+1)(s+2)(s+3)}$
- Ex 4.3: PI control of $\dfrac{1}{(s+1)^2}$
- Ex 4.4: P control with dead time e^{-2s}

4.1 Definition of Stability

Closed-loop characteristic equation

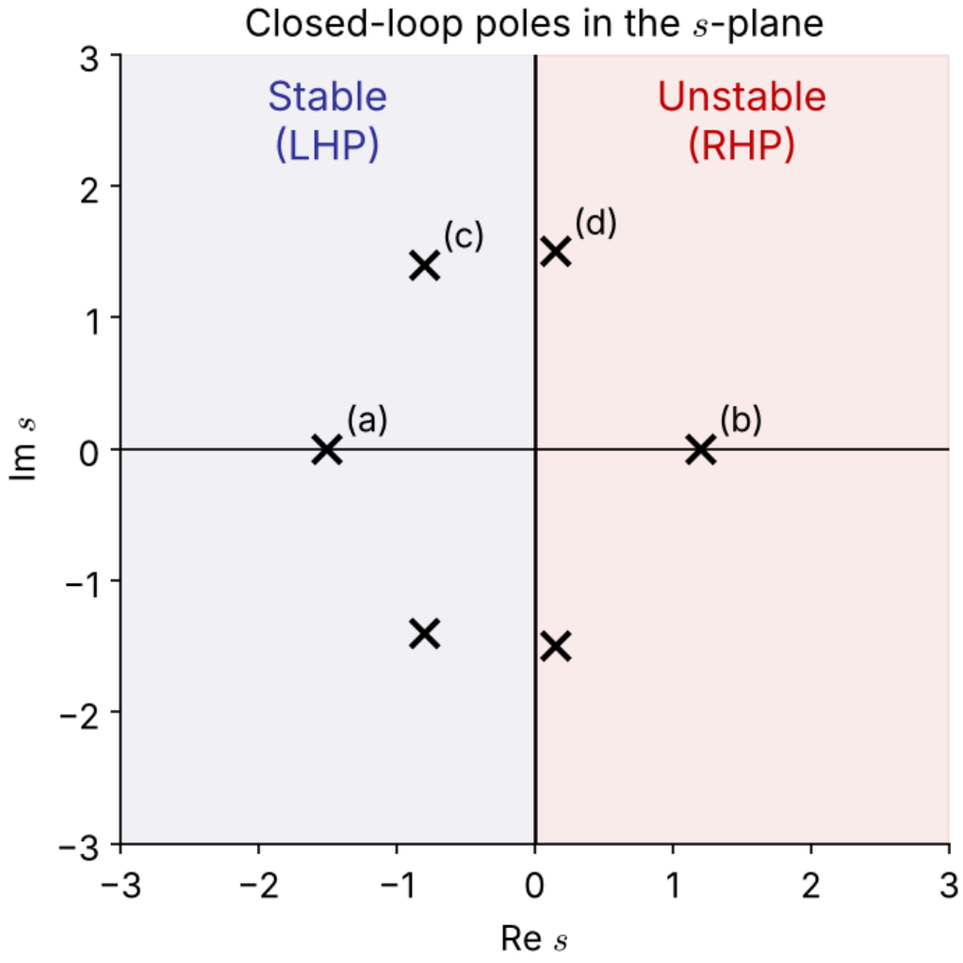
$$C = \left(\frac{K_m G_c G_a G_p}{1 + G_m G_c G_a G_p} \right) R + \left(\frac{G_L}{1 + G_m G_c G_a G_p} \right) L$$

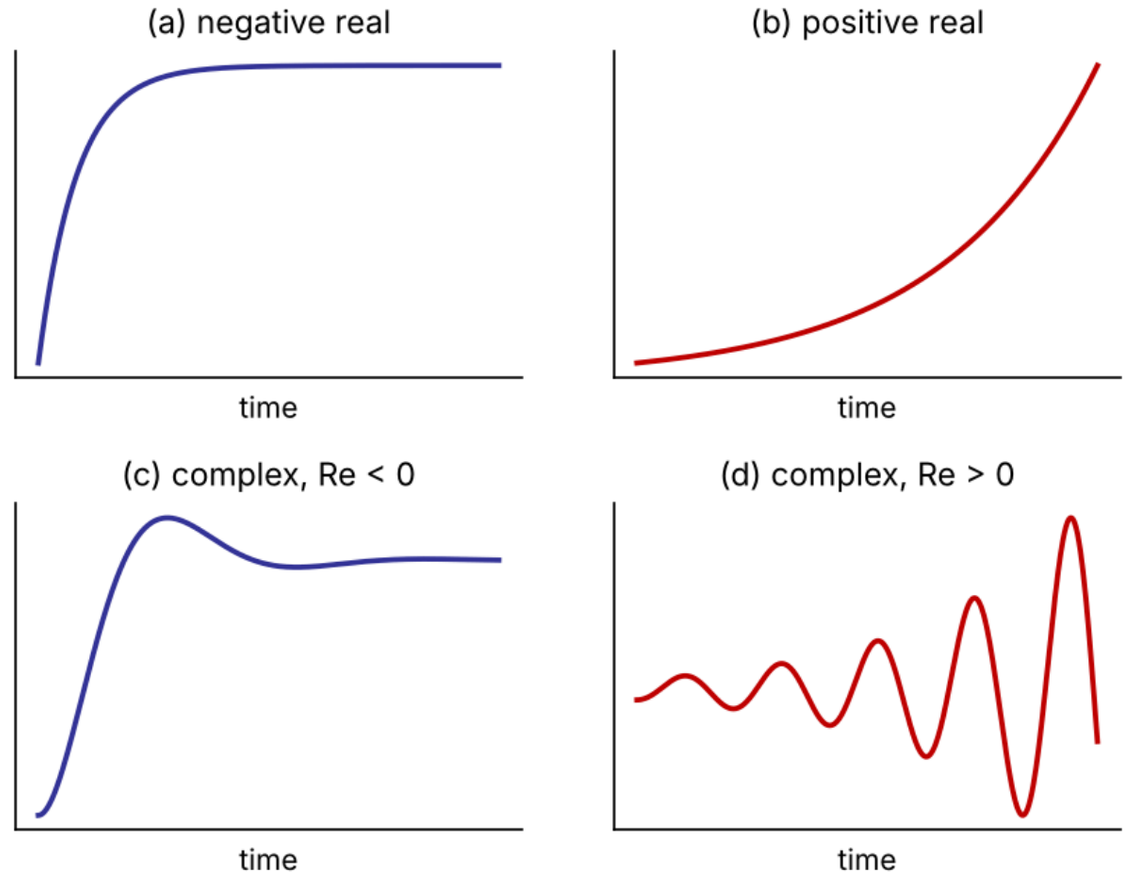
Both servo and regulator responses share the same denominator → **characteristic equation**

$$1 + G_m G_c G_a G_p = 0 \qquad (\text{often written } 1 + K_c G_0(s) = 0)$$

General stability criterion. The closed-loop system is stable **if and only if all roots of the characteristic equation lie in the open left-half plane** (negative real parts). A root on the imaginary axis → *marginally stable*; any root in the RHP → unstable.

Pole location ↔ time response





Real part → growth/decay rate; imaginary part → oscillation frequency. Stability depends on the **sign of the real part only**.

Example 4.1 Stabilizing an unstable process with P control

Unstable process $G_p = \dfrac{K}{s-a}$ $(a>0)$, unity feedback $G_m = G_a = 1, G_c = K_c$:

$$1 + K_c \frac{K}{s-a} = 0 \;\Rightarrow\; s - a + K_c K = 0 \;\Rightarrow\; s = a - K_c K$$

Closed loop is stable iff $K_c K > a$

- Feedback can **create** stability, not only lose it (e.g., exothermic CSTR operated at an unstable steady state).
- There is a **minimum** gain here; in Ex 4.2 we will meet a **maximum** gain.

4.2 Routh–Hurwitz Criterion

(1) Hurwitz test: a necessary condition

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0, \qquad a_n > 0$$

- If all roots are in the LHP, then **all coefficients are positive** (none missing).
- If **any coefficient is negative** → at least one root has a positive real part.
- If **any coefficient is zero** → not all roots are in the open LHP.

  (The roots may be on the imaginary axis *or* in the RHP — e.g. $s^3 + 1$ has roots $-1,\ \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$.)

All coefficients positive is **necessary but not sufficient** for $n \geq 3$ → we need the Routh array.

(2) Routh array: necessary and sufficient

row			
s^n	a_n	a_{n-2}	a_{n-4}
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}
s^{n-2}	b_1	b_2	$\cdots$
s^{n-3}	c_1	c_2	$\cdots$
$\vdots$			
s^0	$*$		

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}, \qquad b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}, \ \cdots$$

Routh criterion. All roots in the LHP $\Leftrightarrow$ every element of the **first column** is positive.

The **number of sign changes** in the first column = number of **RHP roots**.

Results worth memorizing

- $n = 2$: all $a_i > 0$ (that's all)
- $n = 3$: all $a_i > 0$ and $a_2 a_1 > a_3 a_0$

General 4th-order formulas → Appendix

(3) Special cases

Case 1 — zero in the first column

Replace the 0 by a small $\varepsilon > 0$, continue, then let $\varepsilon \to 0^{+}$.

$$P(s) = s^4 + s^3 + 2s^2 + 2s + 3$$

s^4	1	2	3
s^3	1	2	
s^2	$0 \to \varepsilon$	3	
s^1	$2 - 3/\varepsilon \to -\infty$		
s^0	3		

Two sign changes → **2 RHP roots** ($0.41 \pm 1.29j$)

Case 2 — a whole row of zeros

The roots contain pairs symmetric about the origin (typically $\pm j\omega$).

1. Form the **auxiliary polynomial** $A(s)$ from the row *above*.
2. Replace the zero row by dA/ds and continue.
3. Roots of $A(s) = 0$ are those symmetric roots.

In stability problems a zero row appears exactly **at the ultimate gain**, and $A(j\omega) = 0$ gives ω_u — see Ex 4.2 at $K_c = 60$.

Example 4.2 P control of a third-order process

$$1 + K_c \frac{1}{(s+3)(s+2)(s+1)} = 0 \;\Rightarrow\; s^3 + 6s^2 + 11s + (6 + K_c) = 0$$

s^3	1	11
s^2	6	$6 + K_c$
s^1	$\frac{66 - (6 + K_c)}{6} = 10 - \frac{K_c}{6}$	0
s^0	$6 + K_c$	

First column positive:

- $10 - K_c/6 > 0 \Rightarrow K_c < 60$
- $6 + K_c > 0 \Rightarrow K_c > -6$

$-6 < K_c < 60$ (with a positive controller gain: $0 < K_c < 60$)

Same as the 3rd-order rule $a_2 a_1 > a_3 a_0$:
$6 \cdot 11 > 6 + K_c$.

Example 4.2 at $K_c = 60$: a row of zeros

s^3	1	11
s^2	6	66
s^1	**0**	

Auxiliary polynomial from the s^2 row:

$$A(s) = 6s^2 + 66 = 0 \;\Rightarrow\; s = \pm j\sqrt{11}$$

At the stability limit the closed loop has a pair of **pure imaginary poles** $s = \pm j\omega_u$, $\omega_u = \sqrt{11}$ — the system oscillates with constant amplitude.

→ This is exactly what **direct substitution** ($s = j\omega$) looks for (§4.3).

Example 4.3 PI control of a critically damped process

$G_p = \dfrac{1}{s^2+2s+1}$, $G_c = K_c\left(1+\dfrac{1}{\tau_I s}\right)$:

$$1 + K_c\frac{\tau_I s + 1}{\tau_I s(s^2+2s+1)} = 0 \;\Rightarrow\; \tau_I s^3 + 2\tau_I s^2 + \tau_I(1+K_c)s + K_c = 0$$

s^3	τ_I	$\tau_I(1+K_c)$
s^2	$2\tau_I$	K_c
s^1	$\tau_I(1+K_c) - \dfrac{K_c}{2}$	
s^0	K_c	

$$K_c > 0,\ \tau_I > 0, \quad \tau_I > \frac{K_c}{2(1+K_c)}$$

- Since $\dfrac{K_c}{2(1+K_c)} < \dfrac{1}{2}$, **any** $\tau_I \geq 0.5$ is stable for all $K_c > 0$.
- Too much integral action (small τ_I) is destabilizing.

Routh array in code

Python

```python
import sympy as sp
from routh import routh_array          # routh.py (handles epsilon & zero-row cases)

s, Kc, tI = sp.symbols('s K_c tau_I', positive=True)

R = routh_array((s+3)*(s+2)*(s+1) + Kc, s)                # Ex 4.2
sp.solve([r[0] > 0 for r in R], Kc)                       # -> Kc < 60

R = routh_array(tI*s*(s**2 + 2*s + 1) + Kc*(tI*s + 1), s)  # Ex 4.3
sp.simplify(R[2][0])                                      # -> tau_I*(Kc+1) - Kc/2

routh_array((s+3)*(s+2)*(s+1) + 60, s)                    # zero row -> A(s) = 6(s^2+11)
```

MATLAB

```matlab
syms s Kc tI positive
p = sym2poly_sym(expand((s+3)*(s+2)*(s+1) + Kc), s);   % [1 6 11 Kc+6]
R = routh_array(p);   R(:,1)                          % first column

function c = sym2poly_sym(P, s), c = coeffs(P, s, 'All'); end

function R = routh_array(p)            % basic version (no special cases)
  n = numel(p)-1;  cols = floor(n/2)+1;  R = sym(zeros(n+1, cols));
  R(1,1:numel(p(1:2:end))) = p(1:2:end);
  R(2,1:numel(p(2:2:end))) = p(2:2:end);
  for i = 3:n+1
    for j = 1:cols-1
      R(i,j) = simplify((R(i-1,1)*R(i-2,j+1) - R(i-2,1)*R(i-1,j+1))/R(i-1,1));
    end
  end
end
```

4.3 Direct-Substitution Analysis

Finding the stability boundary with $s = j\omega$

At the moment a system becomes unstable, a pair of closed-loop poles sits **on the imaginary axis**. Substitute $s = j\omega$ and set **real and imaginary parts to zero**:

- the gain at this limit → **ultimate (critical) gain** K_{cu}
- the frequency → **ultimate (crossover) frequency** ω_u, ultimate period $P_u = 2\pi/\omega_u$

Example 4.2A $(j\omega)^3 + 6(j\omega)^2 + 11(j\omega) + 6 + K_c = 0$

$$\text{Im: } \omega(11 - \omega^2) = 0 \Rightarrow \omega_u = \sqrt{11} = 3.32, \qquad \text{Re: } 6 + K_c - 6\omega^2 = 0 \Rightarrow K_{cu} = 60$$

Example 4.3A With $s = j\omega$ in $\tau_I s^3 + 2\tau_I s^2 + \tau_I(1 + K_c)s + K_c = 0$: Im gives $\omega_u^2 = 1 + K_c$, Re gives $K_c = 2\tau_I\omega_u^2 \Rightarrow \tau_I = \dfrac{K_c}{2(1 + K_c)}$ — the same boundary as Routh. Direct substitution finds the **boundary**; Routh gives the **region**.

Example 4.4 First-order process with dead time

$$1 + K_c \frac{0.8e^{-2s}}{5s+1} = 0$$

e^{-2s} is not a polynomial → Routh cannot be applied directly. With $s = j\omega$:

$$0.8K_c\, e^{-2j\omega} = -(1 + 5j\omega)$$

Magnitude: $0.8K_c = \sqrt{1+25\omega^2}$

Phase: $-2\omega - \tan^{-1}(5\omega) = -\pi$

Solve the phase equation numerically:

$\omega_u = 0.895\ \text{rad/time}$

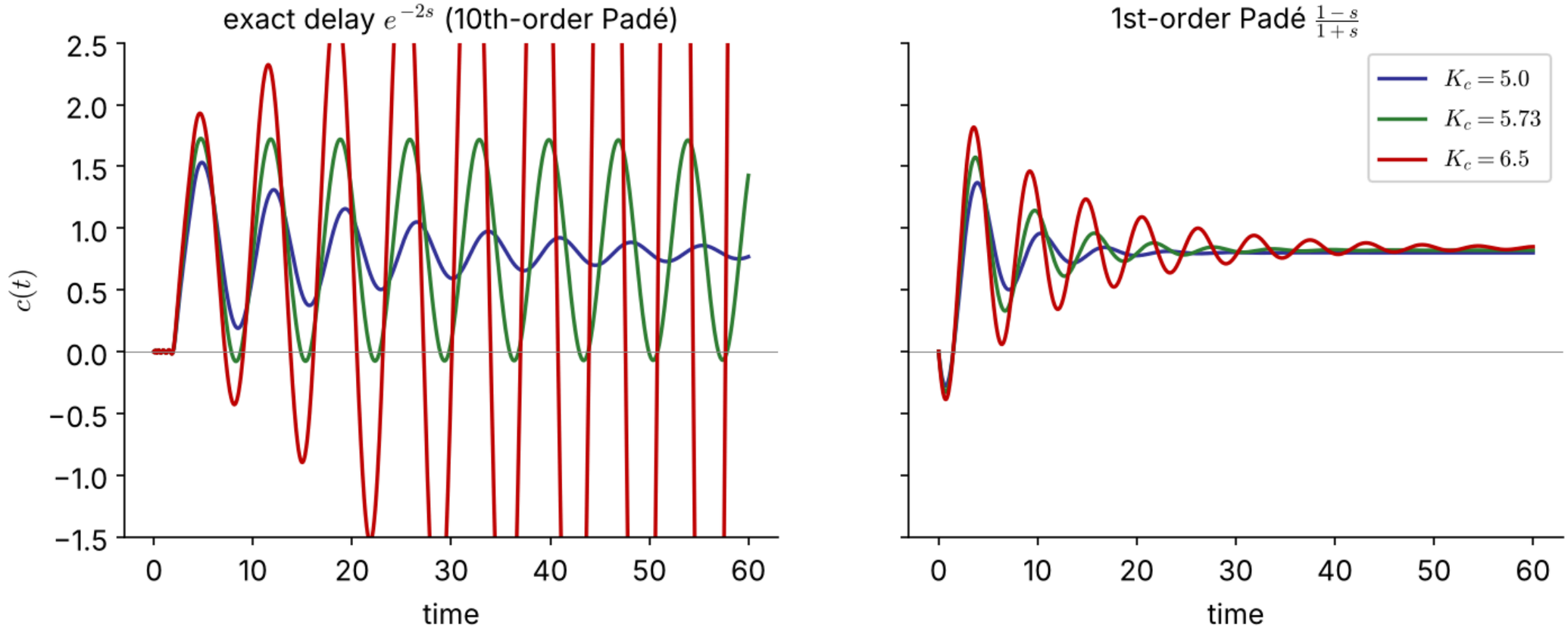
$$K_{cu} = \frac{\sqrt{1+25\omega_u^2}}{0.8} = 5.73$$

Stable for $0 < K_c < 5.73$.

Dead time: exact vs. Padé approximation

First-order Padé: $e^{-\theta s} \approx \dfrac{1-\theta s/2}{1+\theta s/2} = \dfrac{1-s}{1+s} \Rightarrow 5s^2 + (6-0.8K_c)s + (1+0.8K_c) = 0 \Rightarrow$

$K_{cu} = 7.5, \omega_u = 1.18$



The 1st-order Padé model **overestimates** the ultimate gain by 31 % (7.5 vs 5.73): at $K_c = 6.5$ it predicts a stable loop that actually oscillates with growing amplitude. Use approximations with care — they err on the **unsafe** side here.

Dead time in code

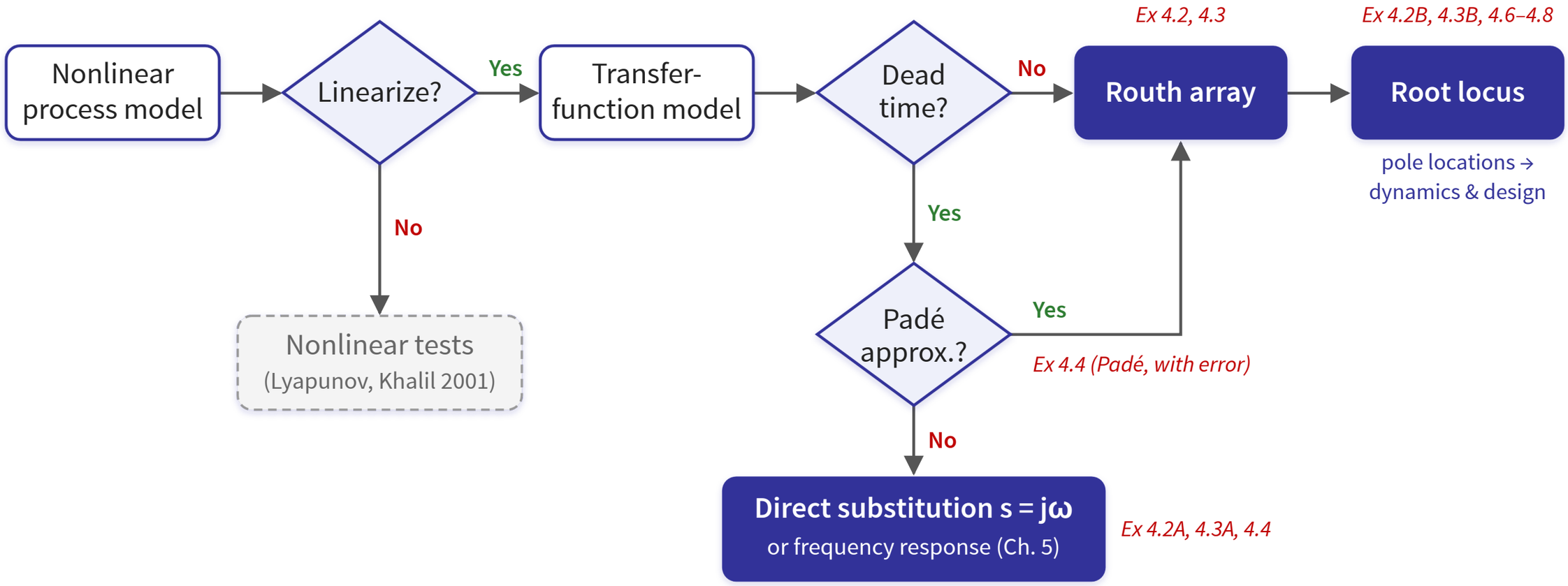
Python

```python
import numpy as np, control as ct
from scipy.optimize import brentq

# exact: phase condition -2w - atan(5w) = -pi
wu  = brentq(lambda w: 2*w + np.arctan(5*w) - np.pi, 0.1, 2.0)
Kcu = np.sqrt(1 + 25*wu**2)/0.8                    # 0.895, 5.73

# Pade approximations of e^{-2s}
G = ct.tf(0.8, [5, 1])
for order in (1, 10):
    D = ct.tf(*ct.pade(2, order))
    gm, pm, wcg, wcp = ct.margin(G*D)              # gain margin = Kcu
    print(order, gm, wcg)                          # 1: 7.50, 1.18   10: 5.73, 0.895
```

MATLAB

```matlab
wu  = fzero(@(w) 2*w + atan(5*w) - pi, [0.1 2]);
Kcu = sqrt(1 + 25*wu^2)/0.8                % 5.73

G = tf(0.8, [5 1]);
[n1,d1] = pade(2,1);   [Gm1,~,Wcg1] = margin(G*tf(n1,d1))     % 7.50, 1.18
Gd = tf(0.8, [5 1], 'InputDelay', 2);
[Gm, ~, Wcg] = margin(Gd)                                     % exact: 5.73, 0.895
```

Which technique? (decision chart)



From K_{cu}, P_u to controller tuning

The ultimate gain and period are the basis of **closed-loop (Ziegler–Nichols) tuning** — the starting point for the PID term project.

Controller	K_c	τ_I	τ_D
P	$0.5K_{cu}$	—	—
PI	$0.45K_{cu}$	$P_u/1.2$	—
PID	$0.6K_{cu}$	$P_u/2$	$P_u/8$

	K_{cu}	P_u	ZN-PI (K_c, τ_I)
Ex 4.2	60	1.89	27, 1.58
Ex 4.4	5.73	7.02	2.58, 5.85

- In the plant, K_{cu} and P_u are found **experimentally** (gain ramp-up, or a relay test) — no model needed.
- ZN settings give a quarter-decay response: a *starting point*, usually detuned afterwards (tuning rules: Ch 3).

4.4 Root-Locus Analysis

What is a root locus?

Root locus: the paths traced by the closed-loop poles as a parameter (usually K_c) varies from 0 to ∞.

$$1 + K_c G_0(s) = 0, \qquad G_0 = \frac{Q(s)}{P(s)} = \frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

$K_c G_0$ is the **open-loop** transfer function ($G_0 = G_m G_a G_p$ for P control; other controllers contribute to G_0).

$$(s - p_1)\cdots(s - p_n) + K_c(s - z_1)\cdots(s - z_m) = 0$$

- n loci (order of the characteristic polynomial)
- $K_c = 0$: loci **start at the open-loop poles**; $K_c \to \infty$: they **end at the open-loop zeros**
- Proper systems $n \geq m$: the remaining $(n - m)$ loci go to infinity along asymptotes

With first- or second-order processes, no time delay and no RHP zeros, the closed loop is stable for all $K_c > 0$ (Ex 4.6 a–b).

Angle and magnitude conditions

A point s is on the locus iff $K_c G_0(s) = -1$ for some $K_c > 0$:

Angle condition (shapes the locus, independent of K_c)

$$\sum_j \angle(s - z_j) - \sum_i \angle(s - p_i) = (2k+1)\pi$$

Magnitude condition (gives K_c at that point)

$$K_c = \frac{\prod_i |s - p_i|}{\prod_j |s - z_j|}$$

Check on Ex 4.2 at $s = j\sqrt{11}$:

- angles: $\tan^{-1}\sqrt{11} + \tan^{-1}\frac{\sqrt{11}}{2} + \tan^{-1}\frac{\sqrt{11}}{3} = 73.2^\circ + 58.9^\circ + 47.9^\circ = 180^\circ$ ✓
- gain: $K_c = |j\sqrt{11}+1|\,|j\sqrt{11}+2|\,|j\sqrt{11}+3| = \sqrt{12}\sqrt{15}\sqrt{20} = 60$ ✓

Every sketching rule on the next slides follows from these two conditions — and the magnitude condition replaces graphical `rlocfind` with an exact number.

Sketching rules (1)–(5)

1. Mark open-loop poles × and zeros ○. The plot is **symmetric** about the real axis.
2. **Number of loci** = number of open-loop poles n.
3. Loci **start at poles** ($K_c = 0$) and **end at zeros** or at **infinity**. A q-th order pole (zero) emits (absorbs) q loci.
4. **Real axis:** a point is on the locus iff the number of real poles + zeros **to its right** is **odd** (count multiplicities).
5. **Imaginary-axis crossings:** by Routh (zero row) or by $s = j\omega$.

Sketching rules (6)–(8)

6. **Asymptotes** of the $(n-m)$ loci going to infinity:

$$\gamma = \frac{\sum p_i - \sum z_j}{n-m}, \qquad \theta_k = \frac{(2k+1)\pi}{n-m},\ k = 0, \dots, n-m-1$$

7. **Departure / arrival angles** at complex poles / zeros (q = multiplicity):

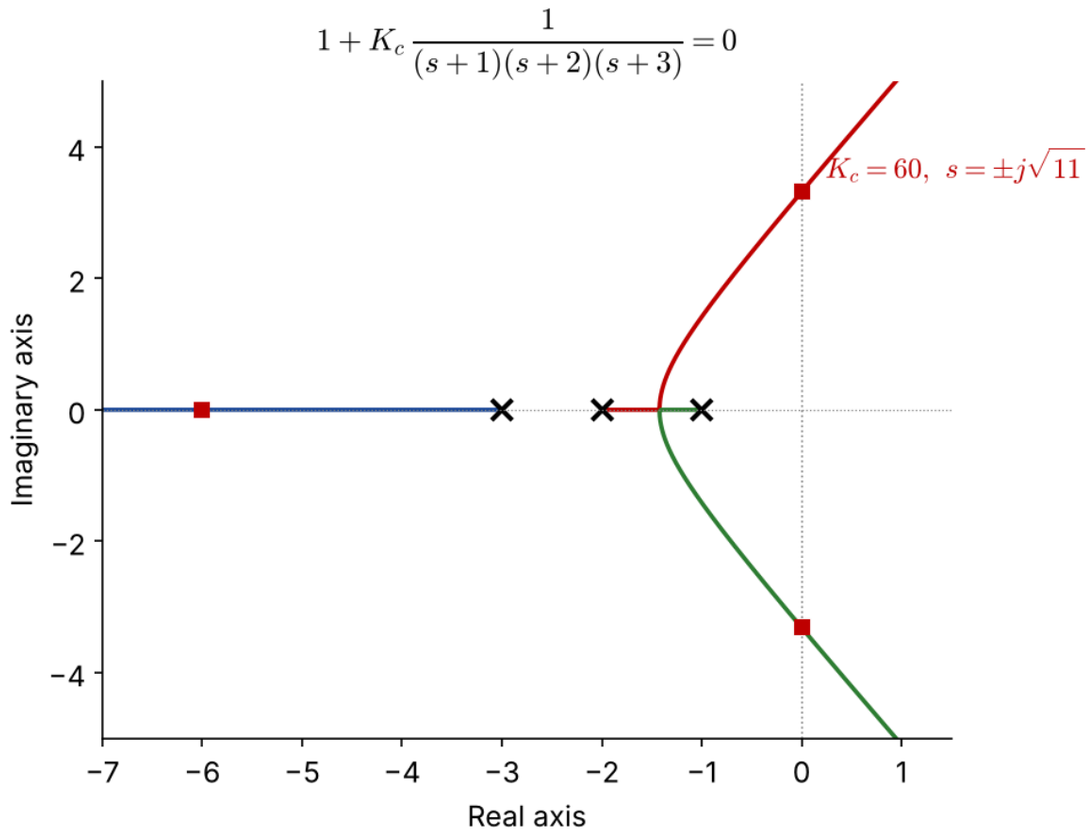
$$q\phi_{\text{dep}} = \sum \psi_j - \sum_{\text{other}} \phi_i - (2k+1)\pi, \qquad q\psi_{\text{arr}} = \sum \phi_i - \sum_{\text{other}} \psi_j + (2k+1)\pi$$

8. **Break-away / break-in points** (multiple roots) on the real axis:

$$Q(s)\frac{dP}{ds} - P(s)\frac{dQ}{ds} = 0$$

Two branches meet at 180° and leave at ±90°; three meet at 120° and leave rotated by 60°.

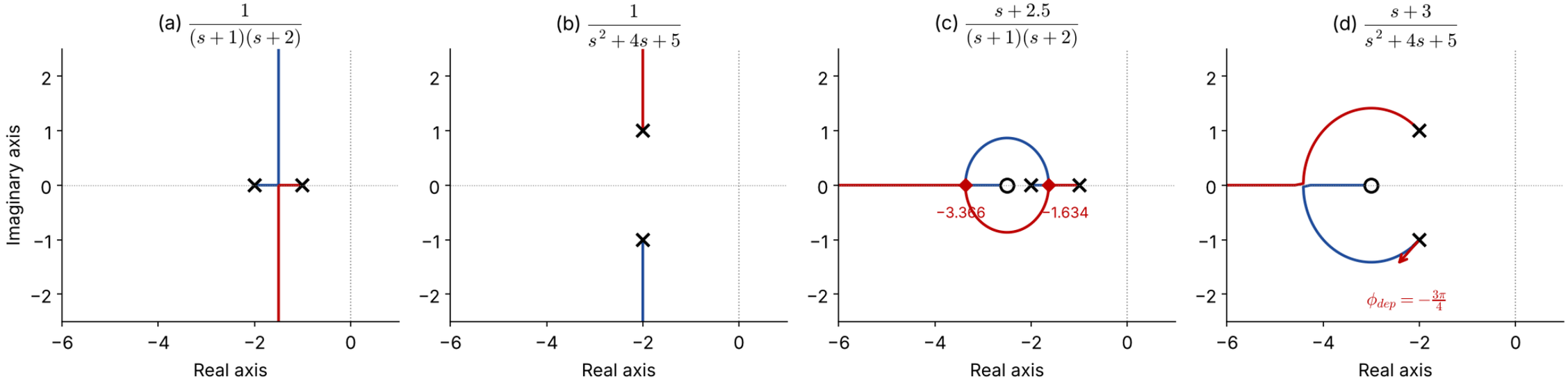
Example 4.2B Root locus of Ex 4.2



- Poles $-1, -2, -3$; no zeros; $n - m = 3$
- Real axis: $(-2, -1)$ and $(-\infty, -3)$
- Asymptotes: $\gamma = \dfrac{-6}{3} = -2,\ \theta = \pm 60°, 180°$
- Break-away: $\dfrac{dP}{ds} = 3s^2 + 12s + 11 = 0 \Rightarrow s = -1.42$ (the other root -2.58 is not on the locus)
- Crossing: $s = \pm j\sqrt{11}$ at $K_{cu} = 60$ (Routh / direct substitution)

Three methods, one answer: $K_{cu} = 60,\ \omega_u = \sqrt{11}$

Example 4.6 Representative root-locus shapes



a. real poles: break-away at -1.5, vertical asymptotes; **always stable**

b. complex poles: vertical lines from $-2 \pm j$; **always stable**, more oscillatory

c. zero at -2.5: loci leave the axis and **return** — a circle around the zero

d. complex poles + zero: departure angle $\mp 3\pi/4$, break-in at -4.41

Example 4.6 (c), (d): the calculations

(c) $G_0 = \dfrac{s+2.5}{(s+1)(s+2)}$, $P = s^2+3s+2$, $Q = s+2.5$

$$Q\frac{dP}{ds} - P\frac{dQ}{ds} = (s+2.5)(2s+3) - (s^2+3s+2) = s^2+5s+5.5 = 0$$

$s = -1.634$ (break-away), $s = -3.366$ (break-in)

(d) $G_0 = \dfrac{s+3}{s^2+4s+5}$, poles $-2 \pm j$, zero -3

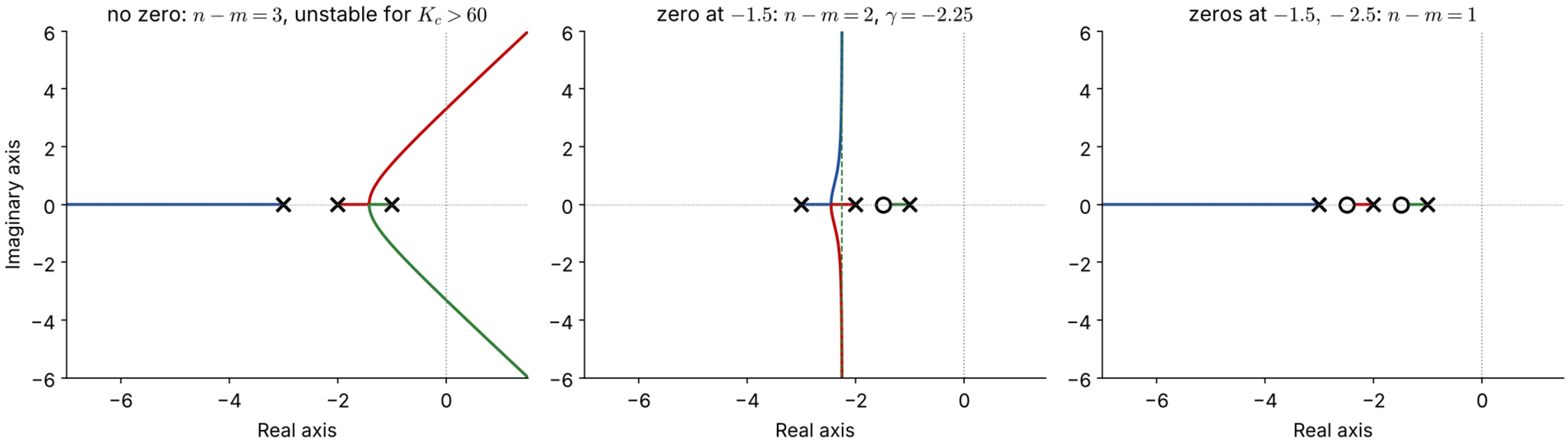
Departure from the lower pole $p_2 = -2-j$:

- from zero -3: $\psi = \angle(1-j) = -\pi/4$
- from pole $p_1 = -2+j$: $\phi_1 = \angle(-2j) = -\pi/2$

$$\phi_{\text{dep}} = -\tfrac{\pi}{4} - \left(-\tfrac{\pi}{2}\right) - \pi = -\tfrac{3\pi}{4}$$

(upper pole: $+3\pi/4$ by symmetry)

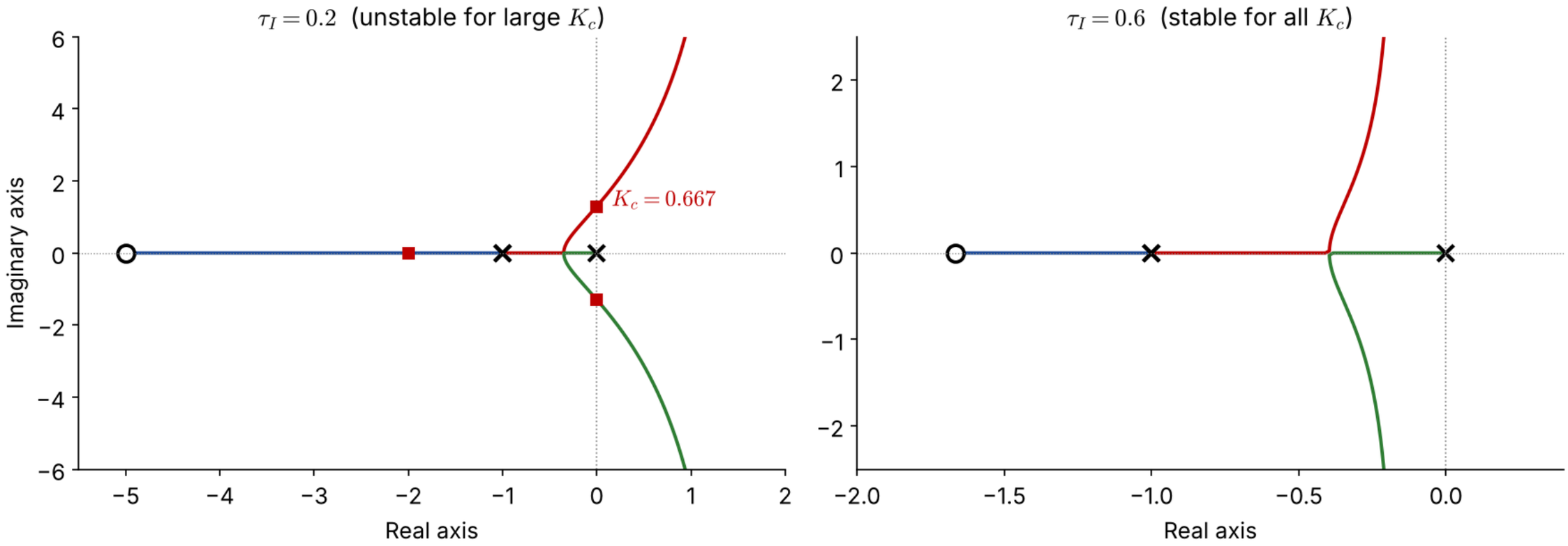
Effect of adding a **zero**



- One LHP zero reduces $n-m$ from 3 to 2: asymptotes become **vertical** at $\gamma = \dfrac{-6-(-z)}{2} < 0$ → loci never cross into the RHP
- Holds for any zero $z \in \{0.5, 1.5, 2.5, 3.5\}$: $\gamma = -2.75 \ldots -1.25$

An LHP zero **pulls the loci to the left** (stabilizing). A PD controller $K_c(1+\tau_D s)$ adds exactly such a zero → Problem #1.

Effect of adding a **pole**: PI control (**Example 4.3B**)



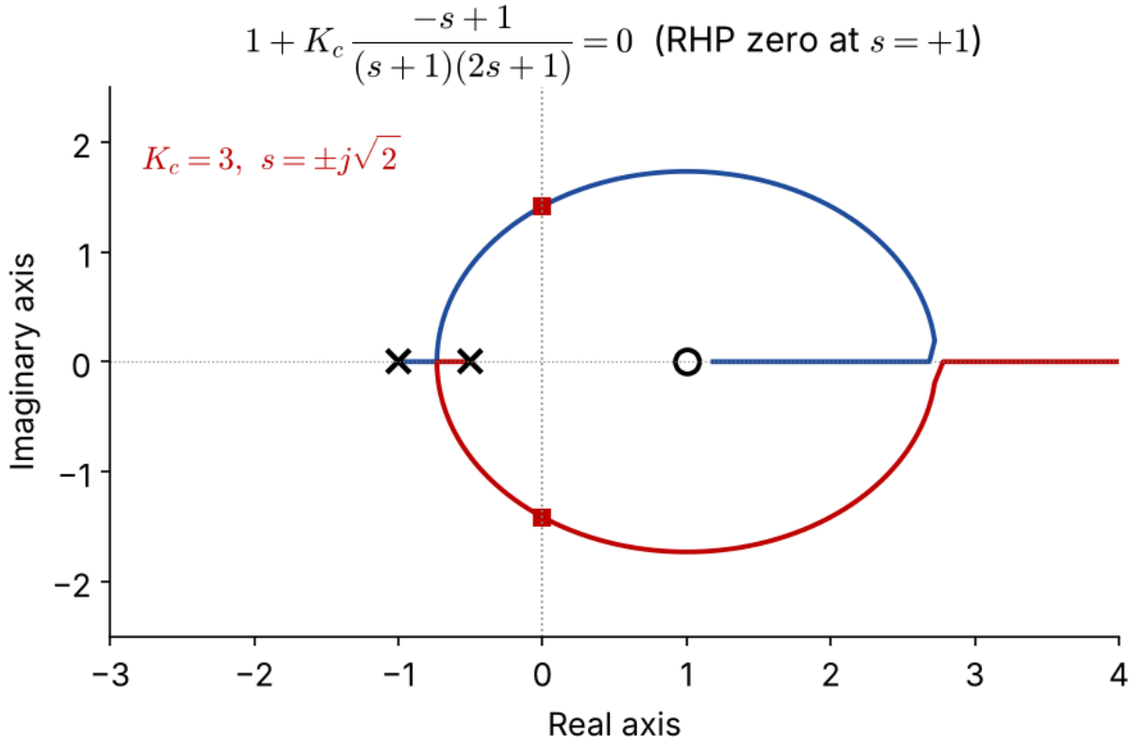
PI adds a pole at the **origin** and a zero at $-1/\tau_I$; $n - m = 2$,

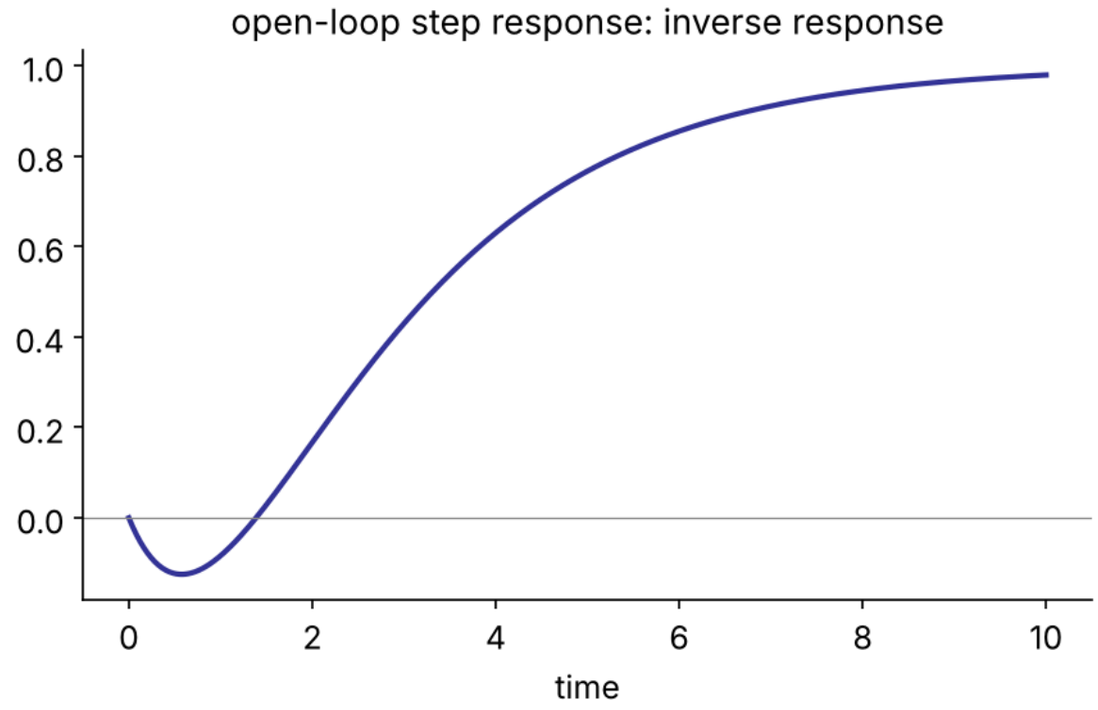
$$\gamma = \frac{-2 + 1/\tau_I}{2}$$

- $\tau_I = 0.2$: $\gamma = +1.5$ → crosses at $K_c = 0.667$ (Routh: $\tau_I = \frac{K_c}{2(1+K_c)}$ ✓)
- $\tau_I = 0.6$: $\gamma = -0.17$ → stable for all K_c

Adding poles pushes loci **to the right** (destabilizing).
Keep the PI zero far enough left ($\tau_I > 0.5$ here).

RHP zero: inverse response





$1 + K_c \frac{-s+1}{(s+1)(2s+1)} = 0 \Rightarrow 2s^2 + (3 - K_c)s + (1 + K_c) = 0$ → stable only for $0 < K_c < 3$.

A locus **ends at the RHP zero** $s = +1$, so high gain is **always** unstable — no controller tuning can remove this limit. Typical of inverse-response processes (e.g. drum-level "swell", some reactors).

4.5 Root-Locus Design

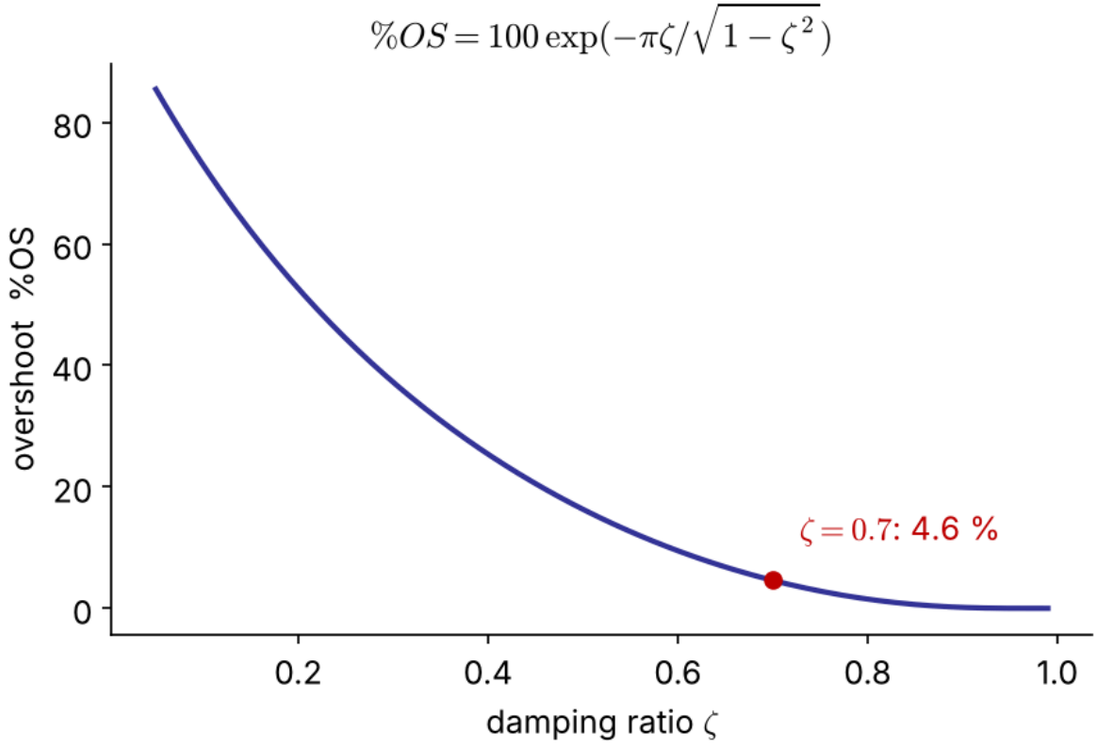
Dominant poles and performance specs

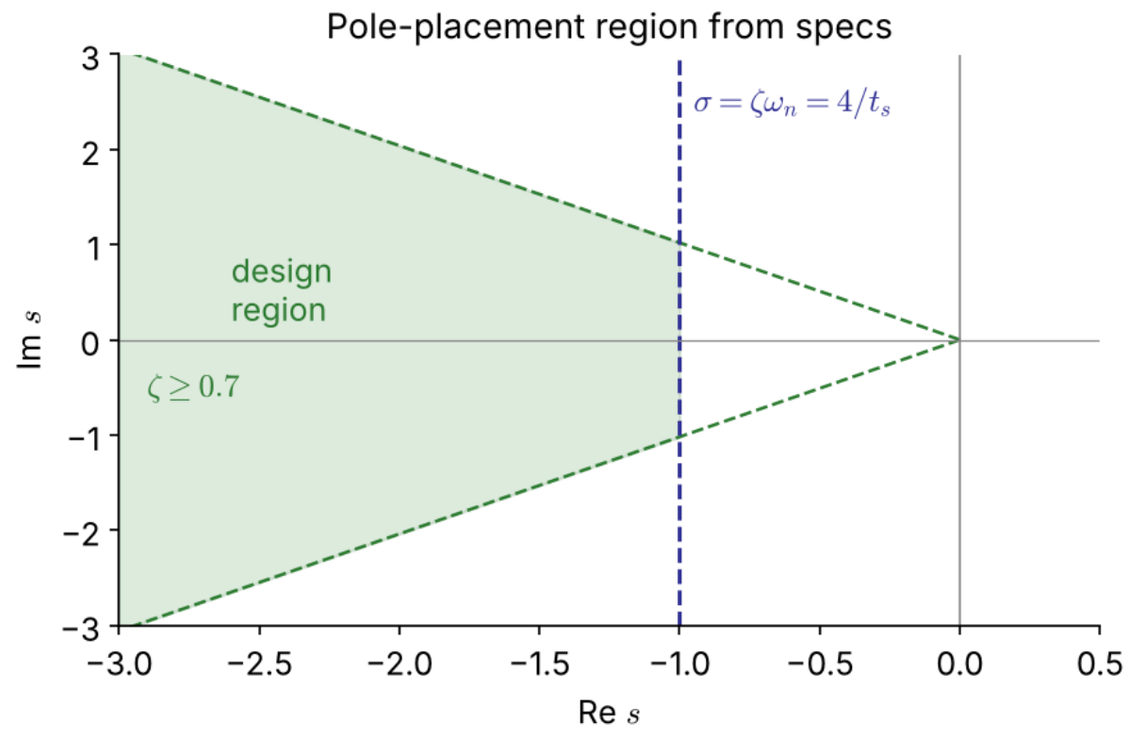
Approximate the closed loop by its **dominant underdamped pair**

$$s = -\frac{\zeta}{\tau} \pm j\frac{\sqrt{1-\zeta^2}}{\tau}, \quad |s| = \frac{1}{\tau} = \omega_n, \quad \zeta = \cos\psi$$

$$\%OS = 100\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right), \qquad t_s(2\%) \approx \frac{4}{\zeta\omega_n}$$

ζ spec → a **ray** at angle ψ; t_s spec → a **vertical line** $\mathrm{Re}\, s = -4/t_s$





Example 4.7 Gain for $\zeta = 0.7$ (P control)

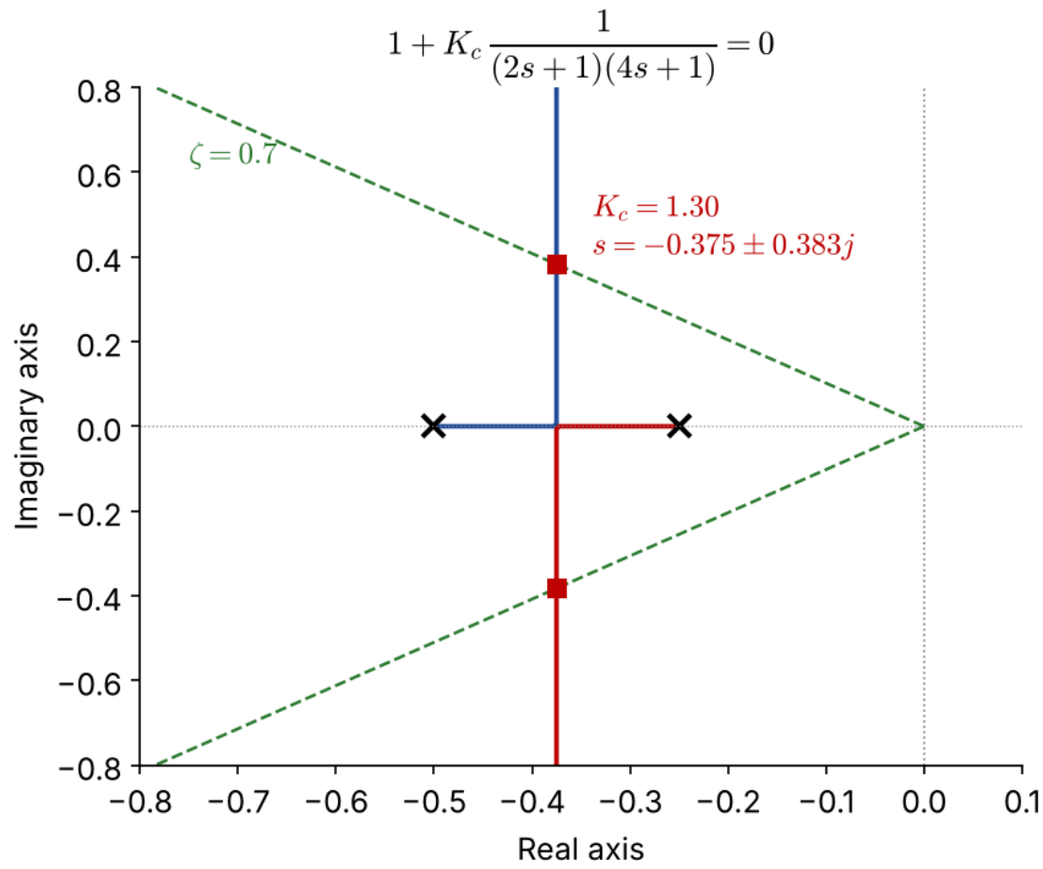
$$G_p = \frac{1}{(2s+1)(4s+1)} \quad (K_p = 1,\ \tau_1 = 2,\ \tau_2 = 4)\text{: } 8s^2 + 6s + (1 + K_c) = 0$$

$$\omega_n^2 = \frac{1+K_c}{8}, \quad 2\zeta\omega_n = \frac{6}{8} \ \Rightarrow\ \zeta = \frac{3}{\sqrt{8(1+K_c)}}$$

$$\zeta = 0.7 \ \Rightarrow\ K_c = \frac{127}{98} = 1.30$$

poles $-0.375 \pm 0.383j$, $\%OS = 4.6\%$, $t_s \approx 11$

- Graphical `rlocfind` gave 1.23 — use the **magnitude condition** for an exact value.
- Offset: $\dfrac{1}{1+K_c} = 0.44$ (!) → P control alone is not enough → Ex 4.8 (PI)



Example 4.7 in code

Python

```python
import numpy as np, control as ct, matplotlib.pyplot as plt

G = ct.tf(1, np.polymul([2, 1], [4, 1]))
ct.root_locus_plot(G, grid=True)                # s-plane grid incl. zeta lines
zeta = 0.7
# analytic: zeta = 3/sqrt(8(1+Kc))
Kc = 9/(8*zeta**2) - 1                          # 1.296
print(Kc, ct.feedback(Kc*G, 1).poles())         # -0.375 ± 0.383j
print(ct.step_info(ct.feedback(Kc*G, 1)))       # overshoot 4.6 %
```

MATLAB

```matlab
G = tf(1, conv([2 1],[4 1]));
rlocus(G); axis([-0.8 0 -1 1]); sgrid(0.7, 0.2:0.2:1)   % wn > 0
zeta = 0.7;
Kc = 9/(8*zeta^2) - 1                % exact: 1.296  (rlocfind click: ~1.23)
pole(feedback(Kc*G,1))
stepinfo(feedback(Kc*G,1))
```

Example 4.8 PI design for $\zeta = 0.7$

First-order process $K_p = 0.5, \tau_p = 2$ min, $G_a = G_m = 1$:

$$1 + K_c \frac{0.5(\tau_I s + 1)}{\tau_I s(2s + 1)} = 0 \;\Rightarrow\; 2\tau_I s^2 + \tau_I(1 + 0.5K_c)s + 0.5K_c = 0$$

(a) $\tau_I = 3$ min: zero at $-1/3$ lies between the poles 0 and $-1/2$

$$\zeta = \frac{3(1 + 0.5K_c)}{2\sqrt{3K_c}} \;\geq\; 1.22 \text{ for all } K_c$$

→ always **overdamped**; $\zeta = 0.7$ is **not attainable**

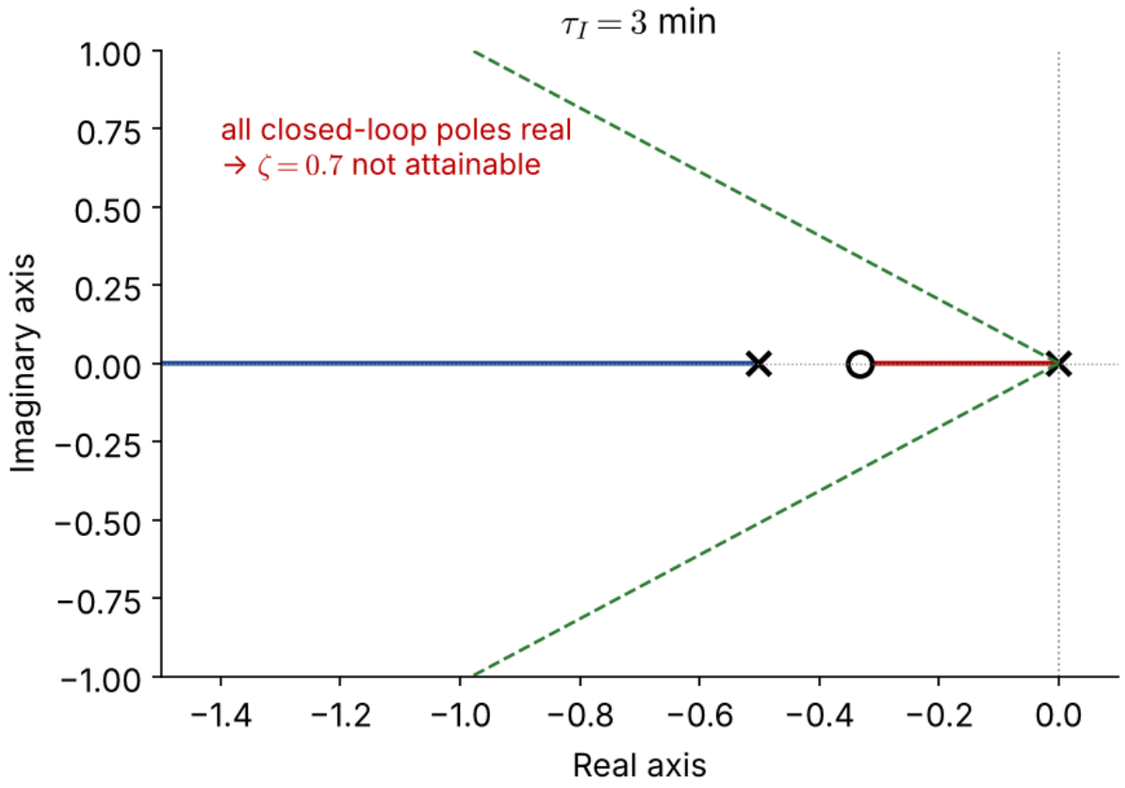
(b) $\tau_I = 2/3$ min: zero at -1.5 — loci form a circle

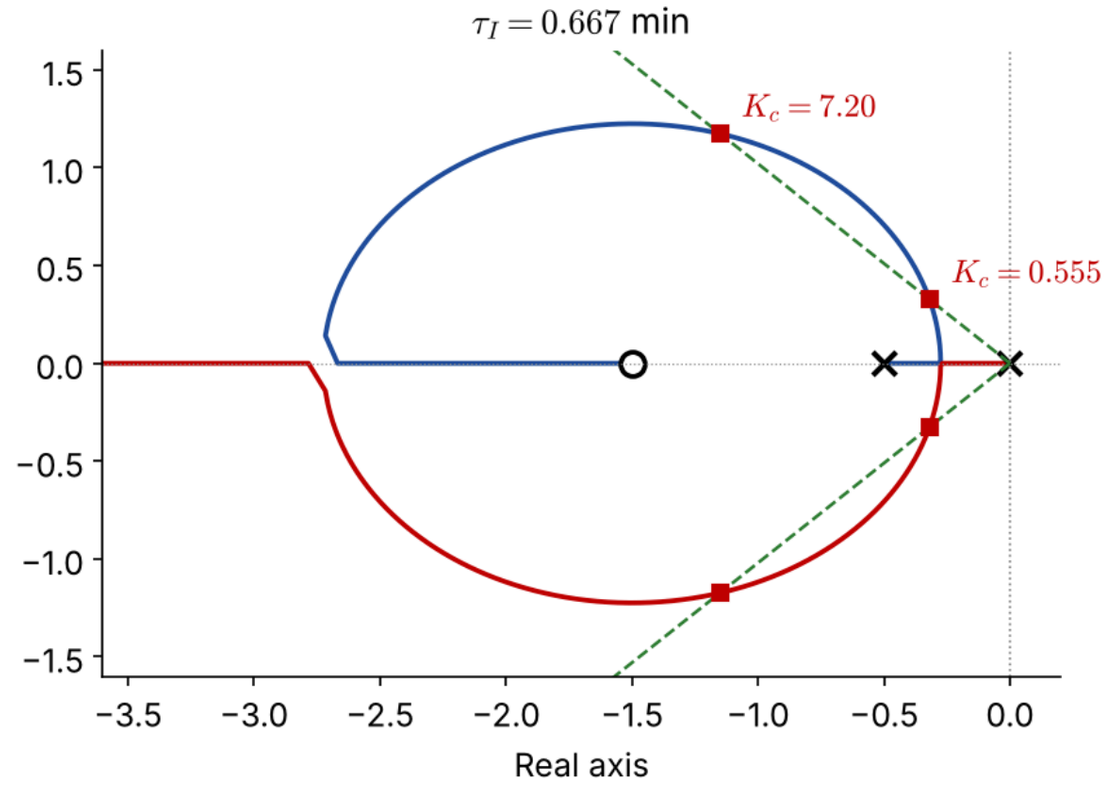
$$\zeta = 0.7 \Rightarrow 0.0625K_c^2 - 0.485K_c + 0.25 = 0$$

$K_c = 0.555$: poles $-0.32 \pm 0.33j$, $t_s \approx 12$ min

$K_c = 7.20$: poles $-1.15 \pm 1.17j$, $t_s \approx 3$ min

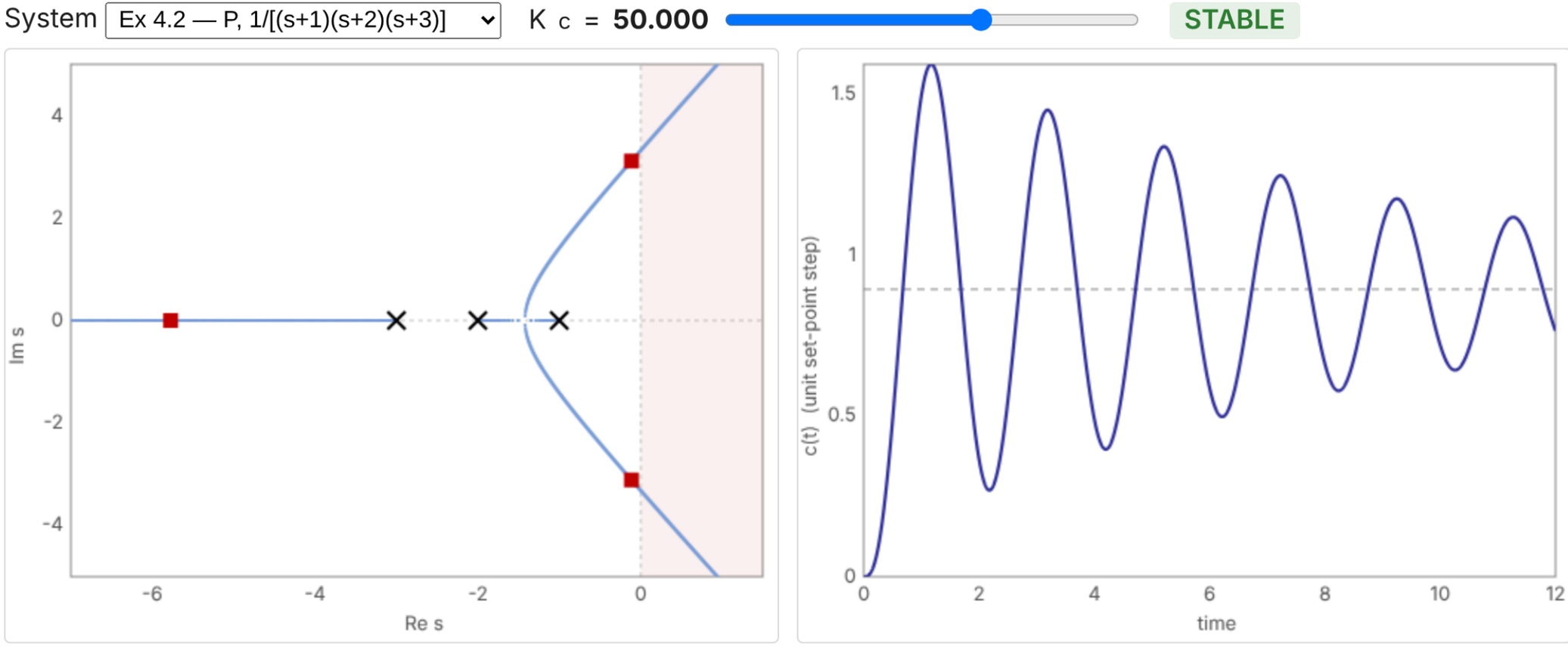
Example 4.8: root loci and a caveat





- Choose $K_c = 7.20$ for the **faster** response.
- **Caveat:** the PI zero at -1.5 is close to the dominant poles → actual overshoot is **13 %**, not 4.6 %. The second-order approximation must be checked by simulation.
- Alternative: $\tau_I = \tau_p = 2$ cancels the process pole → $1 + \dfrac{0.5K_c}{2s} = 0$, a first-order closed loop with $\tau_{cl} = 4/K_c$.

Interactive: move K_c, watch poles and response



poles: -5.774, -0.113+3.112j, -0.113−3.112j dominant ζ = 0.036

Interactive version: seoultechpse.github.io/ProCon/ch4/ch4_stability.html — move K_c and switch between Ex 4.2, 4.3, 4.8 and the RHP-zero case.

Final remarks on root-locus plots

1. Root locus **cannot handle dead time easily** — frequency-response analysis handles it exactly (Ch. 5).
2. Root locus gives **absolute** stability only; there is no general “safety margin” — frequency response defines **relative** stability (gain/phase margins).
3. In return, root locus shows the **dynamic response** (damping, speed) directly — something frequency response does not do easily.

Toolbox summary

Task	MATLAB	Python (`control`, `scipy`, `sympy`)
transfer function	`tf(num,den)`	`ct.tf(num,den)`
root locus	`rlocus(G)`, `sgrid`	`ct.root_locus_plot(G, grid=True)`
closed loop / poles	`feedback`, `pole`	`ct.feedback`, `.poles()`
Padé / margins	`pade`, `margin`	`ct.pade`, `ct.margin`
Routh array	(custom function)	`routh.py` (sympy)
step response	`step`, `stepinfo`	`ct.step_response`, `ct.step_info`

Problem #1 (homework)

$$G_p = \frac{2(s+1)}{s(s+2)(s^2+1)}, \qquad 1 + G_c G_p = 0$$

a. Show with an analytical technique of your choice that the closed loop **cannot** be stabilized by a proportional controller.

b. With a PI or a PD controller (positive gain), write the characteristic equations. *Without further analysis*, which is more likely to stabilize the system?

c. Use root-locus plots to illustrate your answer.

Hint for (b): recall what an added zero and an added pole at the origin do to $n - m$ and to the asymptote centroid.

Resources

- **Slides (interactive HTML):** seoultechpse.github.io/ProCon/ch4/ch4_stability.html
- **PDF:** slides · handout with note space
- **Python notebook:** Open in Colab — every calculation and figure of this chapter
- **Routh-array helper:** `routh.py`

Appendix

Routh array: 4th-order polynomial

$$p(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$b_1 = \dfrac{a_3 a_2 - a_4 a_1}{a_3}$	$b_2 = a_0$	
s^1	$c_1 = \dfrac{b_1 a_1 - a_3 a_0}{b_1}$	0	
s^0	$d_1 = a_0$		

Stable $\Leftrightarrow$ all $a_i > 0,\ b_1 > 0,\ c_1 > 0$, i.e. $a_3 a_2 > a_4 a_1$ and $a_3 a_2 a_1 > a_4 a_1^2 + a_3^2 a_0$.