

Process Dynamics and Control

5. Frequency-Response Analysis & Design

Kee-Youn Yoo

Department of Chemical & Biomolecular Engineering, SeoulTech

What are we up to?

Frequency-response analysis is the substitution $s = j\omega$: everything we need is the **magnitude** $|G(j\omega)|$ and the **phase** $\angle G(j\omega)$.

- Impose a **sinusoidal input** and measure the response
— dynamics can be identified even **without a model**
- **Bode** and **Nyquist** plots are two pictures of the same functions of ω
- Gives a **relative** stability measure (gain and phase margins) and handles **dead time exactly**

Running examples

- Ex 5.10–5.13: inverse-response process

$$\frac{3(-2s + 1)}{(10s + 1)(5s + 1)} \text{ under P and PI control}$$

- Ex 4.4 revisited: $\frac{0.8e^{-2s}}{5s + 1}$ — the dead-time case
Routh could not do exactly

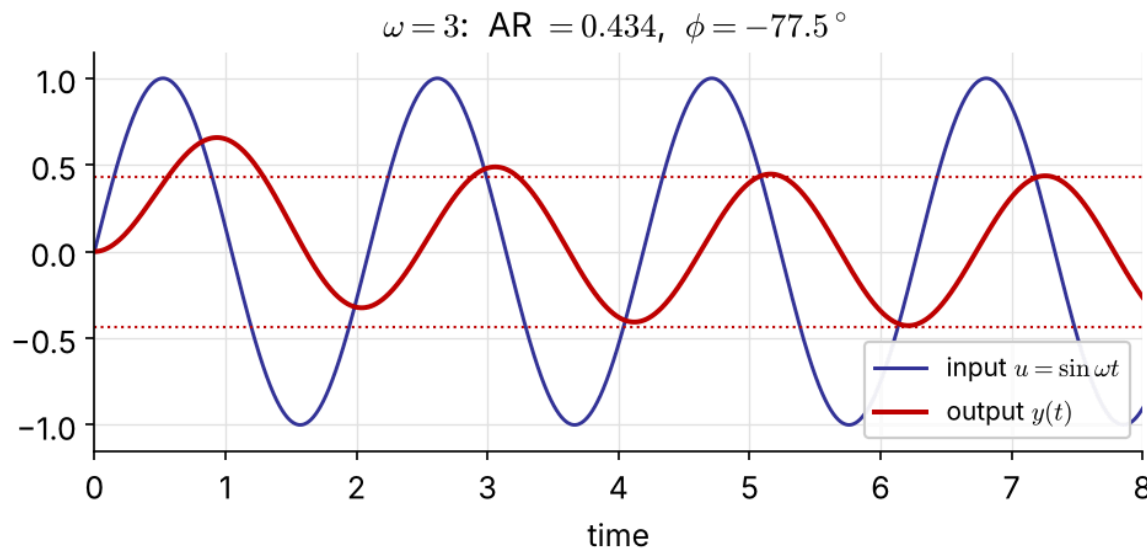
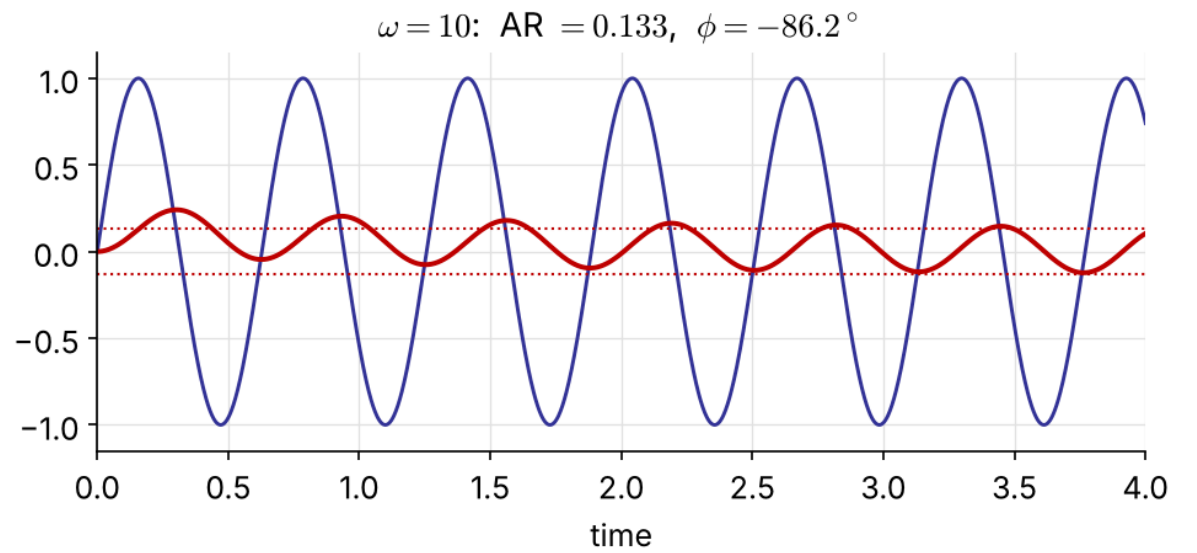
5.1 Magnitude and Phase Lag

A sine in, a sine out

First-order process $\tau_p \frac{dy}{dt} + y = K_p u$, $G(s) = \frac{K_p}{\tau_p s + 1}$, $u = A \sin \omega t$:

$$y(t) = \underbrace{\frac{AK_p\tau_p\omega}{\tau_p^2\omega^2 + 1} e^{-t/\tau_p}}_{\rightarrow 0} + \frac{AK_p}{\sqrt{\tau_p^2\omega^2 + 1}} \sin(\omega t + \phi), \quad \phi = -\tan^{-1}(\omega\tau_p)$$

$G(s) = \frac{2}{1.5s + 1}$: after the transient, the output is a sine of the same frequency



Same frequency; the **amplitude ratio** $AR = K_p / \sqrt{\tau_p^2\omega^2 + 1}$ falls and the **phase lag** grows as ω increases.

... and $G(j\omega)$ gives AR and ϕ directly

$$G(j\omega) = \frac{K_p}{j\omega\tau_p + 1} = \frac{K_p}{\tau_p^2\omega^2 + 1} - j \frac{K_p\omega\tau_p}{\tau_p^2\omega^2 + 1} = |G(j\omega)| e^{j\phi}$$

$$|G(j\omega)| = \frac{K_p}{\sqrt{\tau_p^2\omega^2 + 1}}, \quad \phi = \angle G(j\omega) = -\tan^{-1}(\omega\tau_p)$$

General result (any stable G , see Appendix for the derivation): at large times

$$\frac{y_\infty(t)}{A} = |G(j\omega)| \sin(\omega t + \angle G(j\omega))$$

No partial fractions needed — substitute $s = j\omega$ and take modulus and argument.

Products of transfer functions

For $G(s) = \frac{G_a(s)G_b(s) \cdots G_m(s)}{G_1(s)G_2(s) \cdots G_n(s)}:$

$$\log |G(j\omega)| = \sum_{\text{num}} \log |G_a(j\omega)| - \sum_{\text{den}} \log |G_i(j\omega)|, \quad \angle G(j\omega) = \sum_{\text{num}} \angle G_a(j\omega) - \sum_{\text{den}} \angle G_i(j\omega)$$

On log scales, magnitudes **add** and phases **add** → a Bode plot is the **sum of simple pieces**.

Example 5.1 $G(s) = \frac{\tau_a s + 1}{(\tau_1 s + 1)(\tau_2 s + 1)}$

$$|G(j\omega)| = \frac{\sqrt{1 + \omega^2 \tau_a^2}}{\sqrt{1 + \omega^2 \tau_1^2} \sqrt{1 + \omega^2 \tau_2^2}},$$

$$\phi = \tan^{-1}(\omega \tau_a) - \tan^{-1}(\omega \tau_1) - \tan^{-1}(\omega \tau_2)$$

(The original slide left this blank — solution added.)

5.2 Graphical Analysis Tools

Bode and Nyquist plots

Bode plot — two plots against $\log \omega$

- magnitude: log-log ($|G|$ or **dB** = $20 \log_{10} |G|$)
- phase: semi-log, in **degrees**
- frequency in **rad per unit time**

Nyquist (polar) plot — $G(j\omega)$ drawn in the **complex G -plane** with ω as the parameter

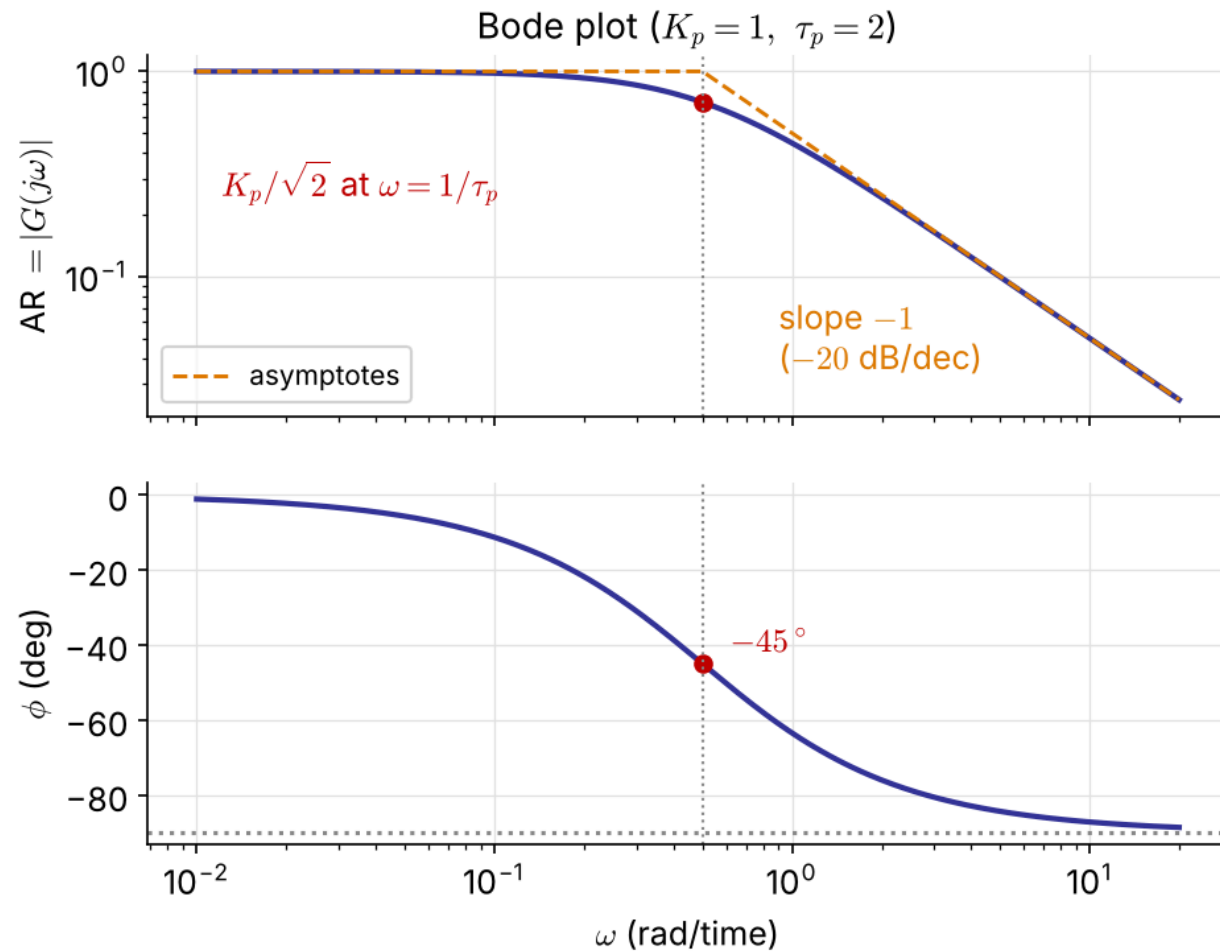
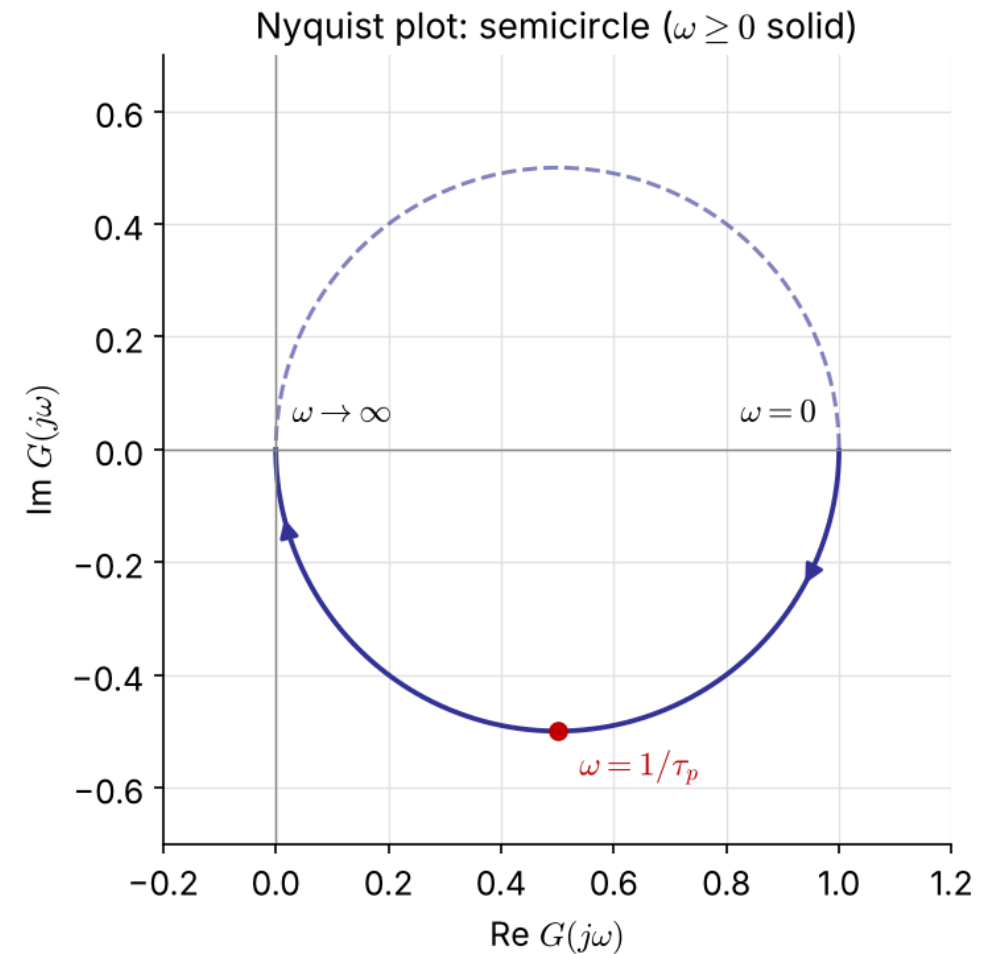
$$G(j\omega) = \text{Re } G + j \text{Im } G = |G|e^{j\phi}$$

Same information; more compact, but ω is not shown explicitly.

AR	dB
0.01	-40
0.1	-20
0.5	-6
1	0
2	+6
10	+20
100	+40

A factor 10 is 20 dB; a factor 2 is ≈ 6 dB.

Example 5.2 First-order lag $K_p/(\tau_p s + 1)$



- $\omega \rightarrow 0: |G| = K_p, \phi = 0; \omega \rightarrow \infty: |G| \rightarrow K_p/\tau_p\omega, \phi \rightarrow -90^\circ$
- Corner frequency $\omega = 1/\tau_p: |G| = K_p/\sqrt{2}, \phi = -45^\circ$

- Bandwidth ω_b : where $|G|$ drops to 0.707 of its low-frequency value ($= 1/\tau_p$ here)
- A low-pass filter: small $\tau \Rightarrow$ wide bandwidth $\Rightarrow$ fast system

Example 5.3 Second-order $K/(\tau^2 s^2 + 2\zeta\tau s + 1)$

$$G(j\omega) = \frac{K [(1 - \tau^2\omega^2) - j2\zeta\tau\omega]}{(1 - \tau^2\omega^2)^2 + (2\zeta\tau\omega)^2}$$

$$|G| = \frac{K}{\sqrt{(1 - \tau^2\omega^2)^2 + (2\zeta\tau\omega)^2}}$$

$$\phi = \tan^{-1} \frac{-2\zeta\tau\omega}{1 - \tau^2\omega^2} \quad (0 \rightarrow -180^\circ)$$

- $\omega \rightarrow 0$: $|G| = K$, $\phi = 0$
- $\omega = 1/\tau$: $|G| = K/2\zeta$, $\phi = -90^\circ$
- $\omega \rightarrow \infty$: $|G| \rightarrow K/\tau^2\omega^2$ (slope -2),
 $\phi \rightarrow -180^\circ$

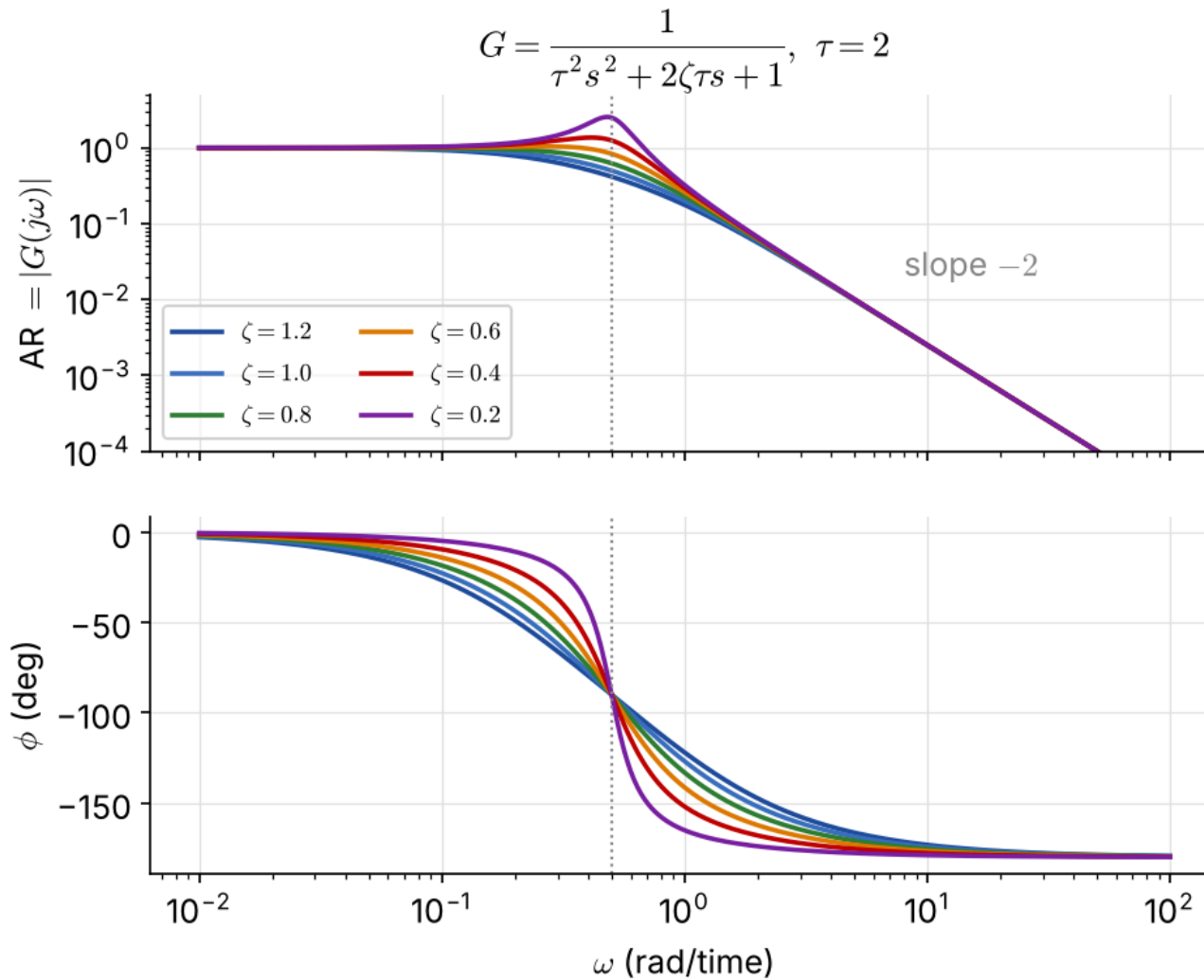
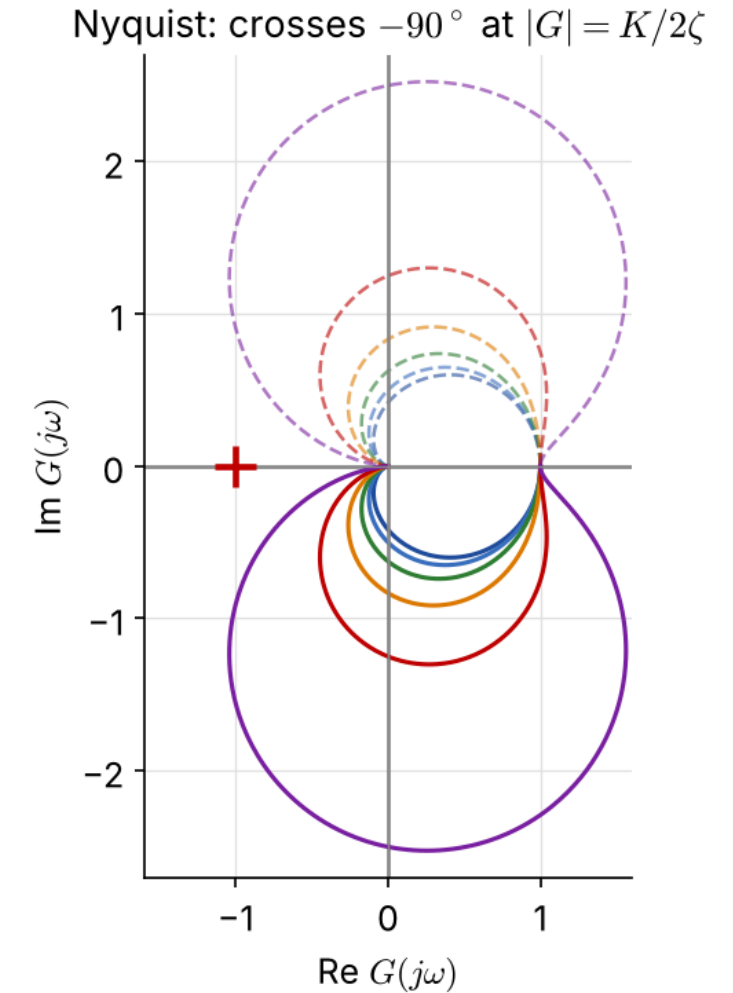
Resonance for $0 < \zeta < 1/\sqrt{2}$

$$\omega_r = \frac{\sqrt{1 - 2\zeta^2}}{\tau}, \quad M_{pw} = \frac{K}{2\zeta\sqrt{1 - \zeta^2}}$$

Smaller $\zeta \rightarrow$ higher peak $\rightarrow$ more overshoot in time (derivation: Appendix).

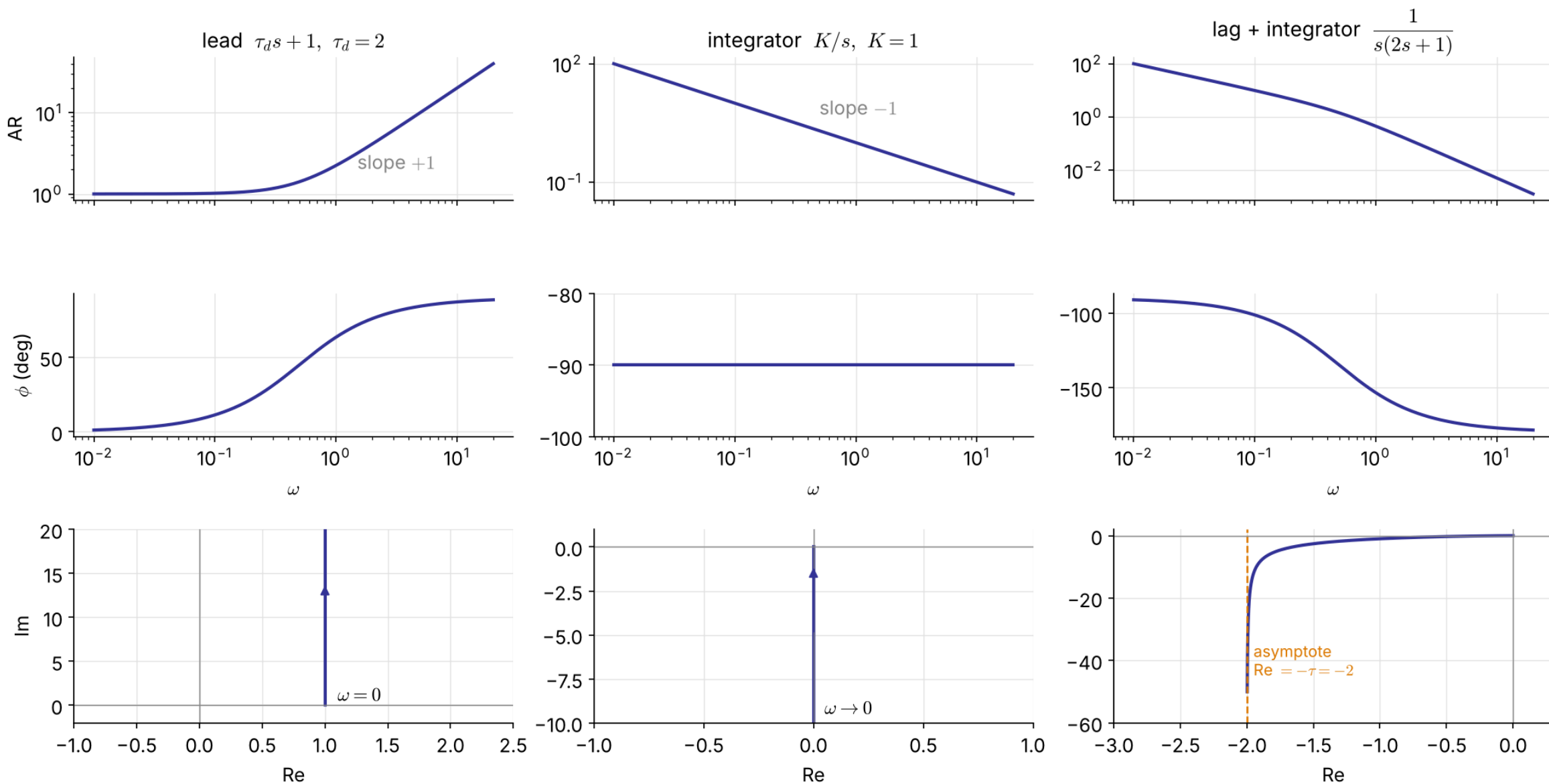
The original said the curve rises above K for $\zeta < 1/2$; that is the condition for $|G(j/\tau)| > K$. The resonant peak itself exists for $\zeta < 0.707$.

Example 5.3: Bode and Nyquist for $\zeta = 1.2 \dots 0.2$



Low damping: a sharp **resonant peak** in $|G|$ and a fast phase drop at $\omega = 1/\tau$; the Nyquist curve bulges far beyond the radius- K circle.

Examples 5.4, 5.7, 5.8: lead, integrator, lag + integrator



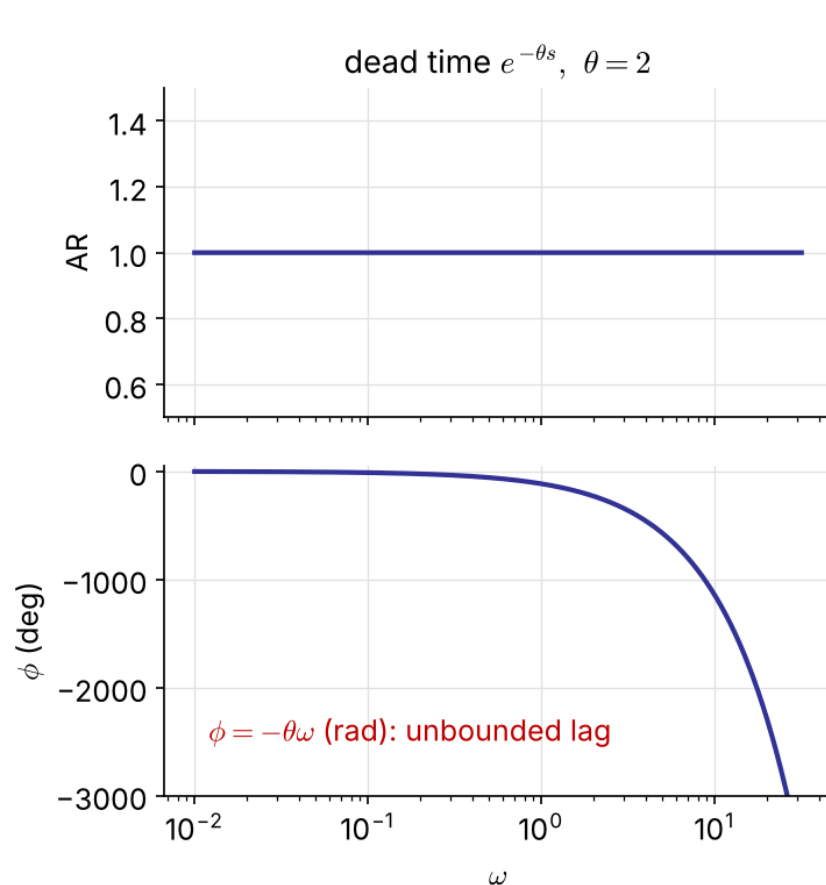
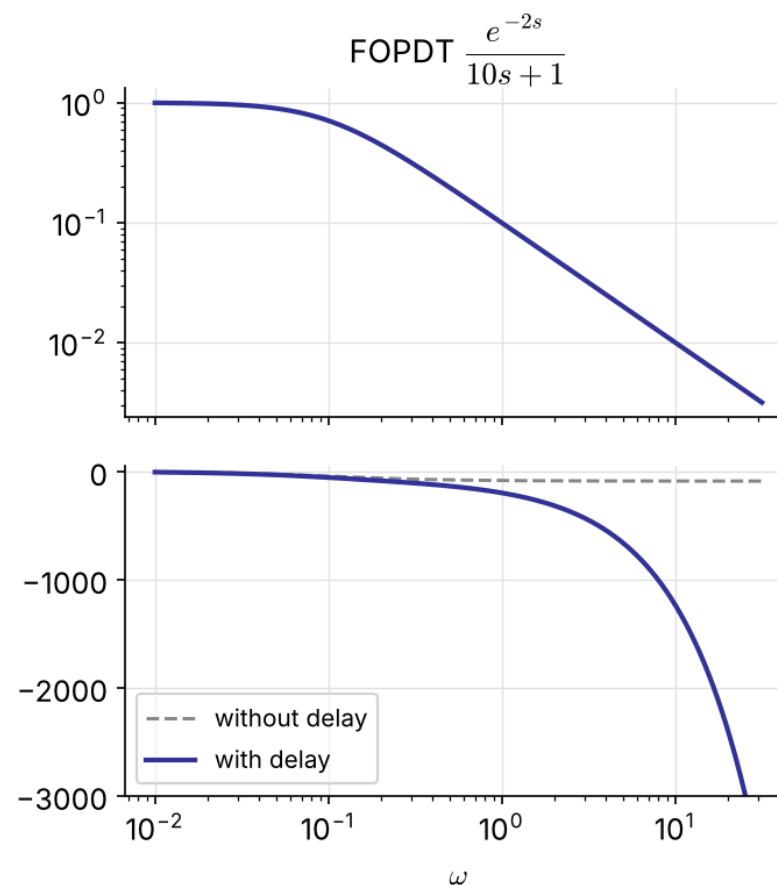
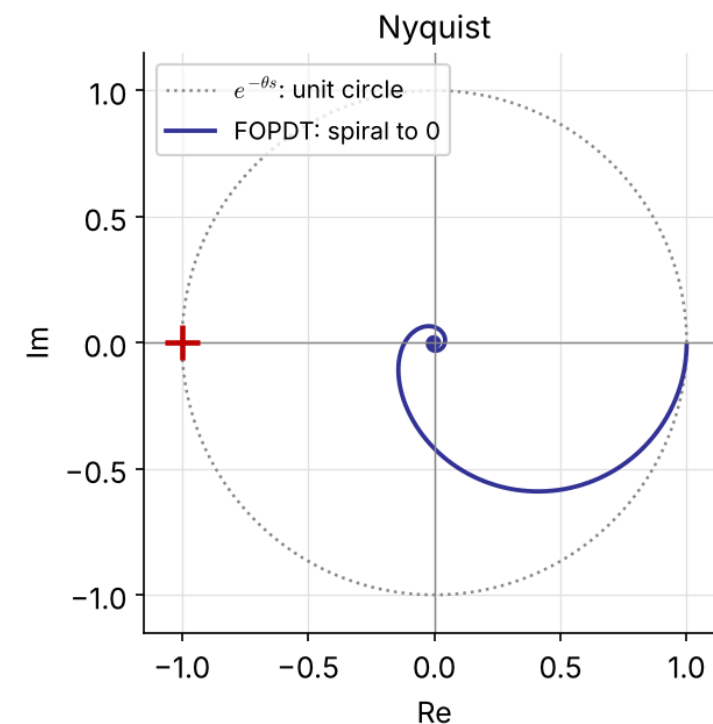
$$|G| = \sqrt{1 + \omega^2 \tau_d^2}, \quad \phi = +\tan^{-1} \omega \tau_d$$

(phase lead)

$$|G| = K/\omega, \quad \phi = -90^\circ \text{ for all } \omega$$

ϕ from -90° to -180° ; Nyquist asymptote $\text{Re} = -\tau$

Examples 5.5, 5.6: dead time



$|e^{-j\omega\theta}| = 1$, $\angle e^{-j\omega\theta} = -\omega\theta$ (rad): the phase lag grows without bound. FOPDT: $|G| = \frac{K_p}{\sqrt{1 + \omega^2\tau_p^2}}$, $\phi = -\tan^{-1}(\omega\tau_p) - \omega\theta$.

Nonminimum-phase systems contain dead time or RHP zeros (Ex 5.10); a **minimum-phase** system has neither, and no RHP poles.

Building blocks at a glance

Factor	AR = $ G(j\omega) $	$\phi = \angle G(j\omega)$	high- ω slope	ϕ range
K	K	0	0	0
$\frac{1}{\tau s + 1}$	$\frac{1}{\sqrt{1 + \omega^2 \tau^2}}$	$-\tan^{-1} \omega \tau$	-1	$0 \rightarrow -90^\circ$
$\tau s + 1$ (LHP zero)	$\sqrt{1 + \omega^2 \tau^2}$	$+\tan^{-1} \omega \tau$	+1	$0 \rightarrow +90^\circ$
$-\tau s + 1$ (RHP zero)	$\sqrt{1 + \omega^2 \tau^2}$	$-\tan^{-1} \omega \tau$	+1	$0 \rightarrow -90^\circ$
$\frac{1}{s}$	$\frac{1}{\omega}$	-90°	-1	—
s	ω	$+90^\circ$	+1	—
$\frac{1}{\tau^2 s^2 + 2\zeta \tau s + 1}$	see Ex 5.3		-2	$0 \rightarrow -180^\circ$
$e^{-\theta s}$	1	$-\omega \theta$	0	unbounded

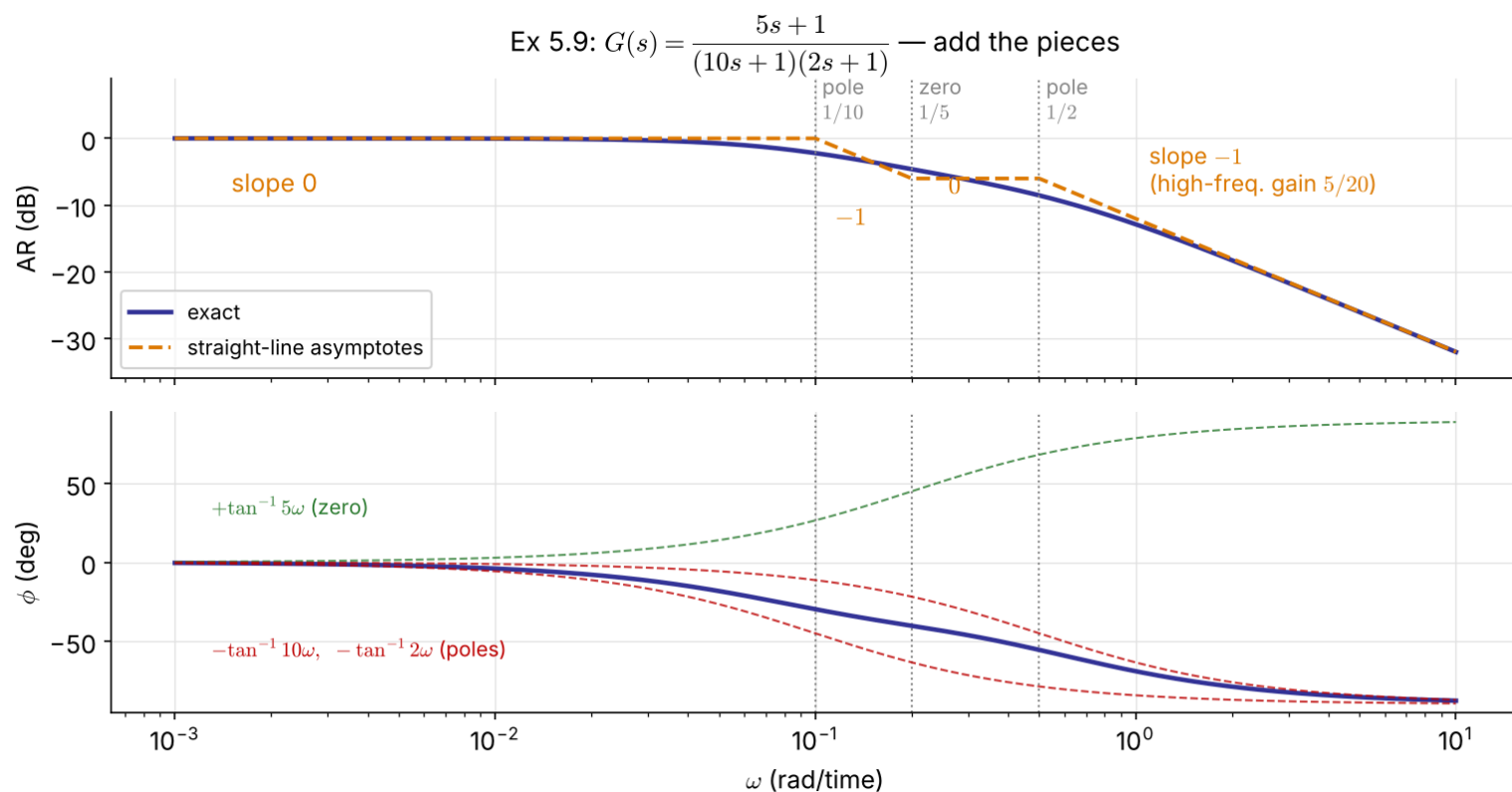
Slopes are in decades of AR per decade of ω ($\times 20$ for dB/dec). The RHP zero has the **magnitude** of an LHP zero but the **phase** of a pole — the origin of its trouble.

Sketching a Bode plot by asymptotes

1. Write G in **time-constant form** $K \frac{\prod(\tau_a s + 1)}{s^q \prod(\tau_i s + 1)} e^{-\theta s}$ and list the **corner frequencies** $1/\tau$.
2. **Low frequency:** $|G| \approx K/\omega^q$ — a line of slope $-q$ ($q = 0$: horizontal at K).
3. Moving right, at each corner the slope changes by **-1 for a pole, +1 for a zero** (-2 for an underdamped pair).
4. **Phase:** each pole contributes $0 \rightarrow -90^\circ$ (-45° at its corner), each zero $0 \rightarrow +90^\circ$; dead time adds $-\omega\theta$.

For real poles and zeros the asymptotes are within 3 dB of the exact curve (largest error at a corner; underdamped pairs can deviate more near resonance) — good enough to read crossover frequencies and margins by hand.

Example 5.9 $(5s + 1)/[(10s + 1)(2s + 1)]$ by asymptotes



Corners: 0.1 (pole), 0.2 (zero), 0.5 (pole)

ω range	slope
< 0.1	0
0.1-0.2	-1
0.2-0.5	0
> 0.5	-1

High frequency: $|G| \rightarrow \frac{5}{20\omega}$,
 $\phi \rightarrow -90^\circ$

Phase:
 $\tan^{-1} 5\omega - \tan^{-1} 10\omega - \tan^{-1} 2\omega$
 never reaches -90° at finite ω

Bode and Nyquist in code

Python

```

1 import numpy as np, control as ct, matplotlib.pyplot as plt
2
3 G = ct.tf(1, [2, 1]) # Ex 5.2
4 ct.bode_plot(G, dB=False, deg=True) # AR (log) and phase (deg)
5 ct.nyquist_plot(G)
6
7 # dead time: evaluate G(jw) e^{-jw*theta} exactly (no Pade)
8 w = np.logspace(-3, 2, 2000)
9 g = ct.tf(1, [10, 1])(1j*w) * np.exp(-2j*w) # Ex 5.6, theta = 2
10 mag, phase = np.abs(g), np.degrees(np.unwrap(np.angle(g)))
11 fig, (a1, a2) = plt.subplots(2, 1, sharex=True)
12 a1.loglog(w, mag); a2.semilogx(w, phase)

```

MATLAB

```

1 G = tf(1, [2 1]); % Ex 5.2
2 figure(1), bode(G)
3 figure(2), nyquist(G)
4
5 G6 = tf(1, [10 1], 'InputDelay', 2); % Ex 5.6: exact dead time
6 figure(3), bode(G6)
7 figure(4), nyquist(G6)
8
9 K = 1; tau = 2; % Ex 5.3: family in zeta
10 for zeta = [1.2 1.0 0.8 0.6 0.4 0.2]
11     bode(tf(K, [tau^2 2*zeta*tau 1])), hold on
12 end

```

5.3 Stability Analysis

Two frequency-domain tests

Bode stability criterion

G_{OL} strictly proper, **no poles on or right of the imaginary axis** (a single pole at the origin is allowed), and **one** phase crossover ω_c ($\angle G_{OL} = -180^\circ$) and one gain crossover ω_g ($|G_{OL}| = 1$):

$$\text{closed loop stable} \iff |G_{OL}(j\omega_c)| < 1$$

Applies with dead time — unlike Routh.

Nyquist stability criterion

G_{OL} proper, no unstable pole-zero cancellations.

- N = **clockwise** encirclements of $(-1, 0)$ by the Nyquist plot ($\omega : -\infty \rightarrow \infty$)
- P = open-loop poles in the RHP
- Z = closed-loop poles in the RHP

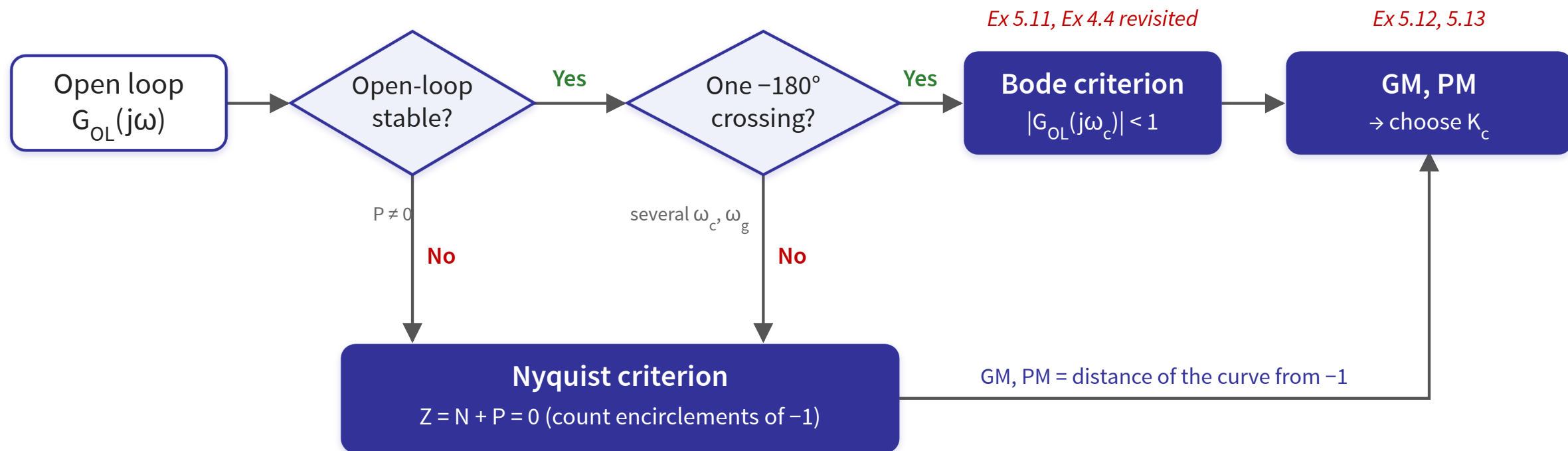
$$Z = N + P \quad \text{stable} \iff Z = 0$$

(the original had $Z = N - P$; with clockwise N the sign is +)

Why -1 ? The characteristic equation $1 + G_{OL} = 0$ means $G_{OL} = -1$, i.e. $|G_{OL}| = 1$ and $\phi_{OL} = -180^\circ$ — the **critical point**.

Reading the Nyquist criterion

- Most process loops are **open-loop stable**: $P = 0 \Rightarrow Z = N$ – unstable iff the curve **encircles** -1 .
- A **counterclockwise** encirclement counts $N = -1$: each one can stabilize one open-loop unstable pole.
- Nyquist works where Bode does not: **open-loop unstable** processes, **several** crossover frequencies.



Ex 5.11, Ex 4.4 revisited

Ex 5.12, 5.13

Ex 5.14: open-loop unstable process

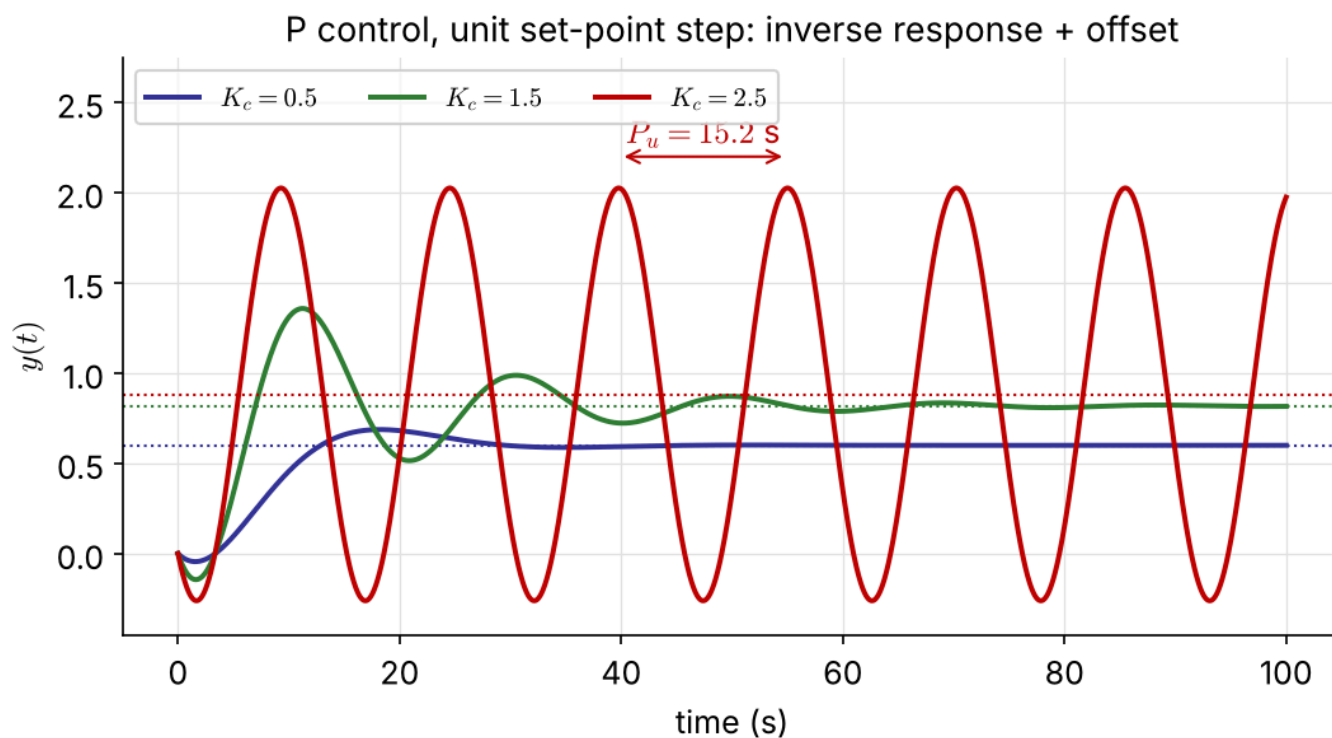
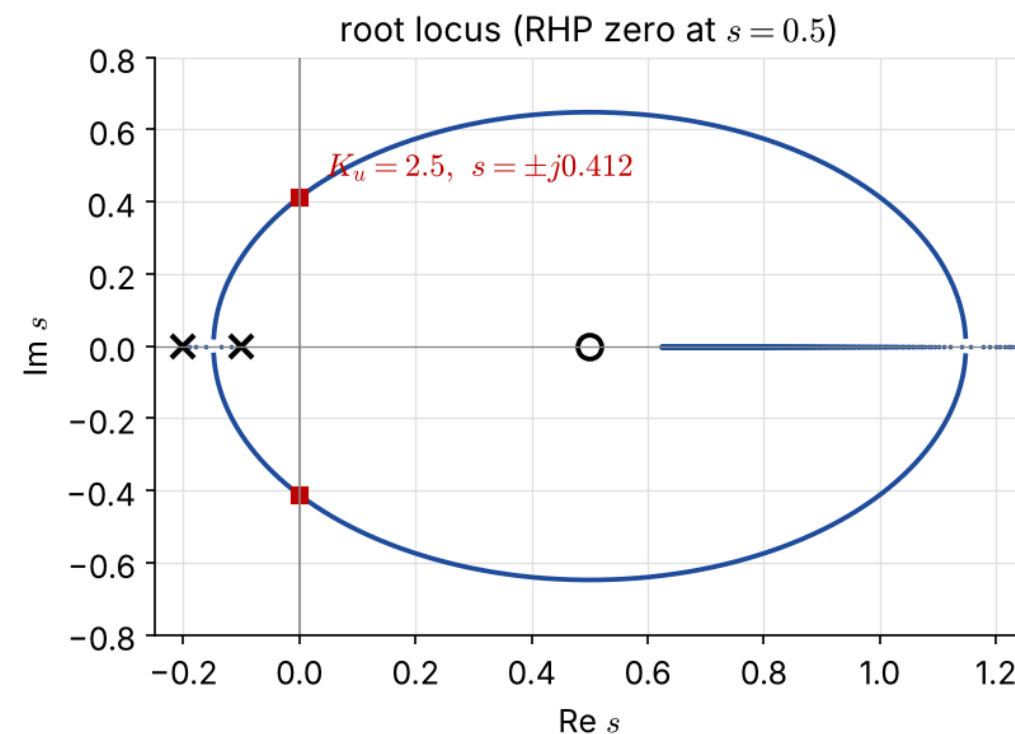
(a single pole at the origin is allowed in “open-loop stable”)

Examples 5.10, 5.11 Inverse-response process, P control

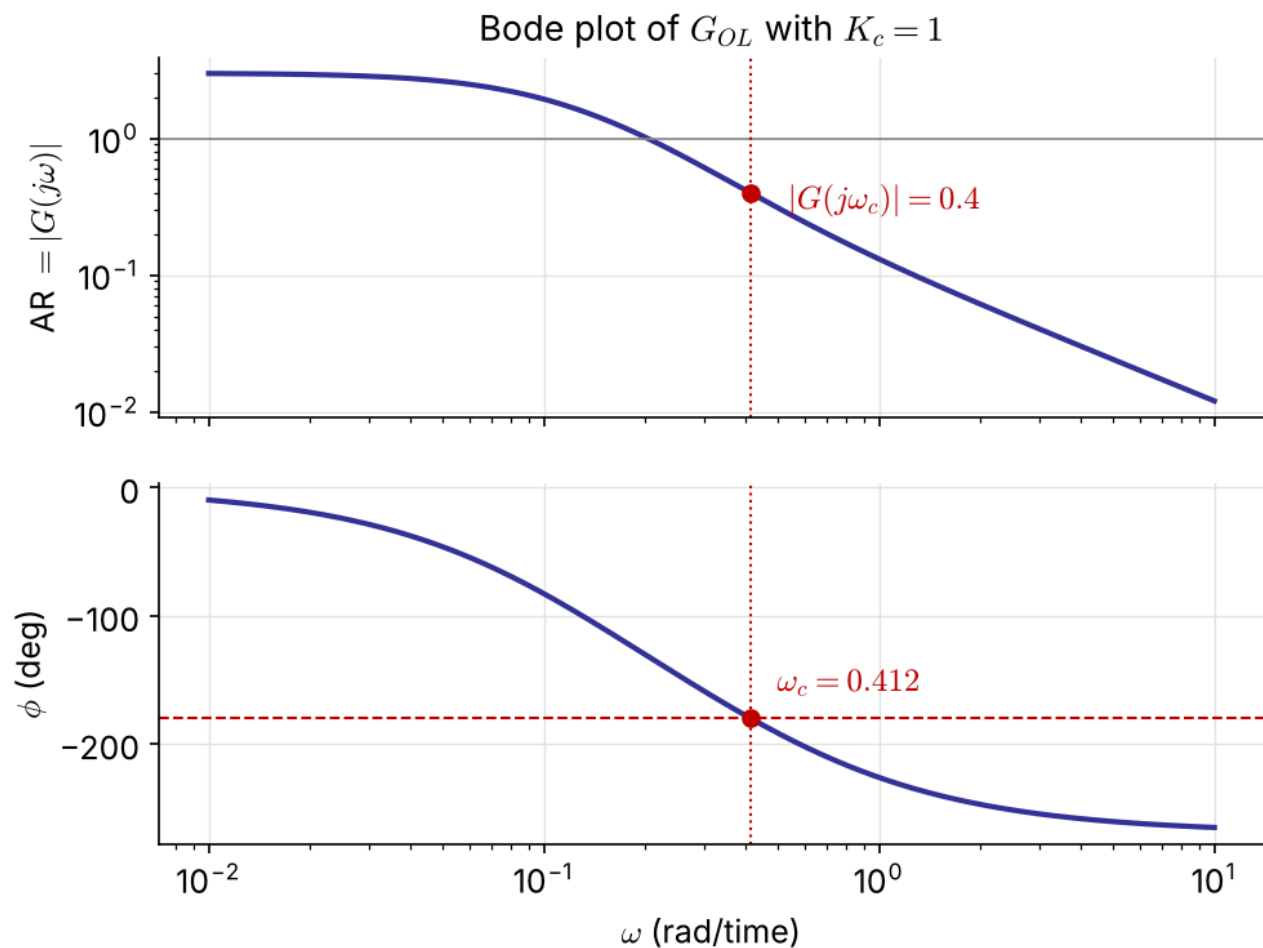
$G = \frac{3(-2s + 1)}{(10s + 1)(5s + 1)}$ has a **RHP zero** at $s = 0.5 \rightarrow$ high gain is fundamentally limited.

$$(10s + 1)(5s + 1) + 3K_c(-2s + 1) = 0 \iff 50s^2 + (15 - 6K_c)s + (1 + 3K_c) = 0$$

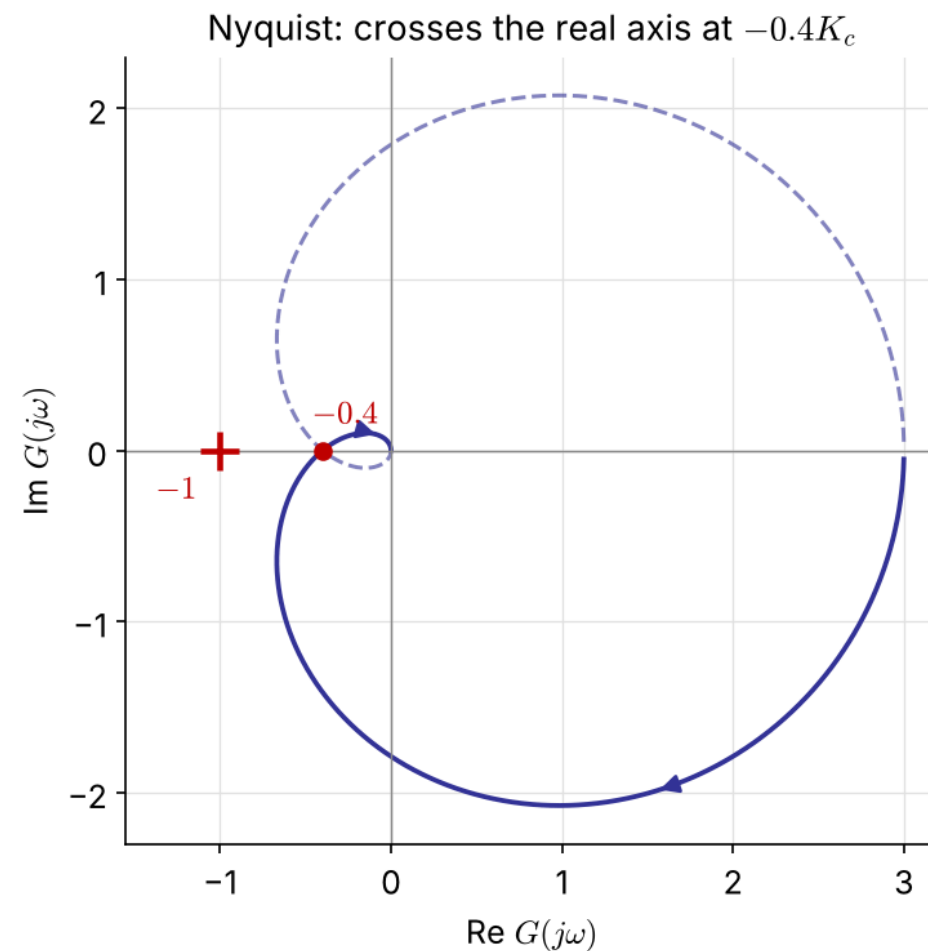
Routh: $15 - 6K_c > 0$, $1 + 3K_c > 0 \Rightarrow -\frac{1}{3} < K_c < 2.5 \rightarrow$ **ultimate gain $K_u = 2.5$** ; at K_u :
 $50s^2 + 8.5 = 0$, $s = \pm j0.412$



Example 5.11 by frequency response

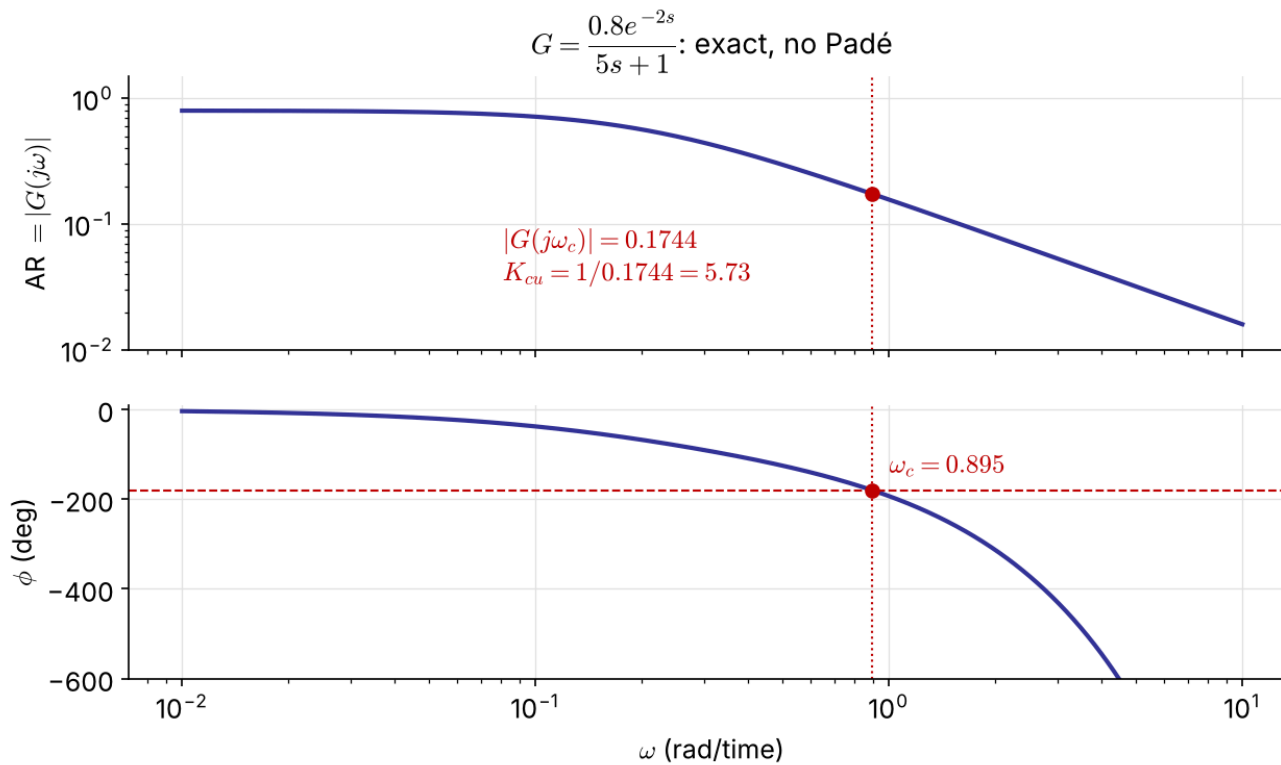


With $K_c = 1$: phase crossover $\omega_c = 0.412$, $|G(j\omega_c)| = 0.4$
 $|G_{OL}(j\omega_c)| = 0.4K_c < 1 \iff K_c < 2.5$ – the same as Routh



Nyquist: the real-axis crossing at $-0.4K_c$ reaches -1
 when $K_c = 2.5$; $P_u = 2\pi/\omega_c = 15.2$ s

Ex 4.4 revisited: dead time handled exactly



$$G_{OL} = K_c \frac{0.8e^{-2s}}{5s+1}$$

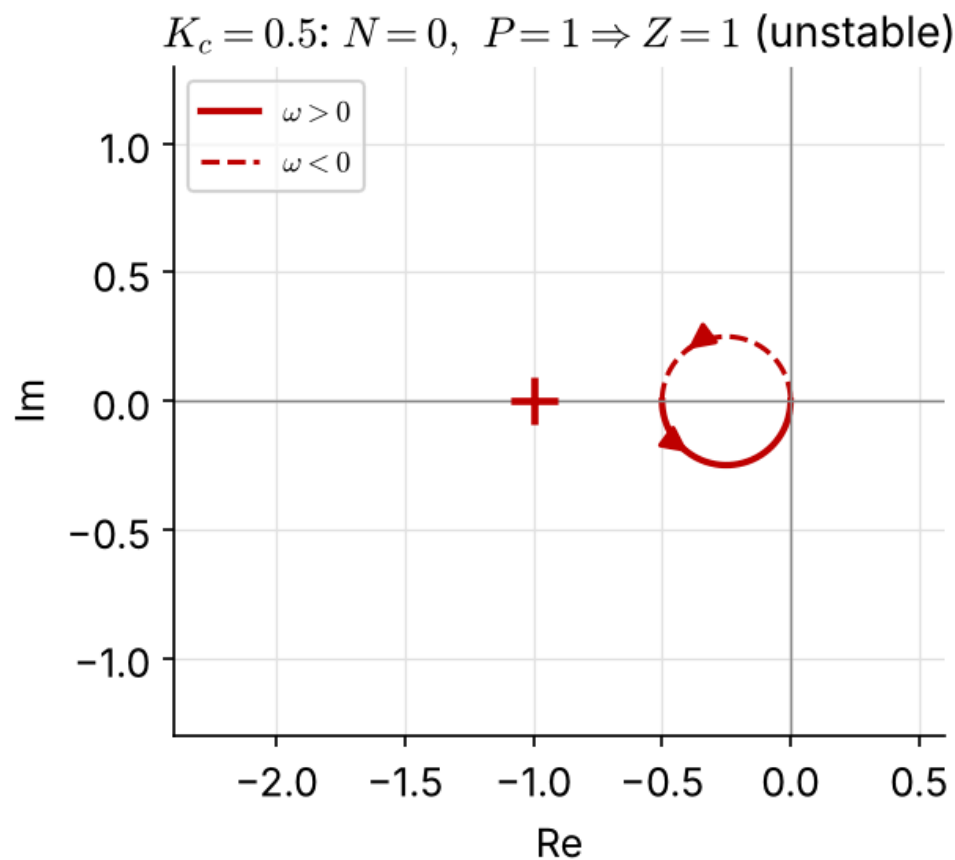
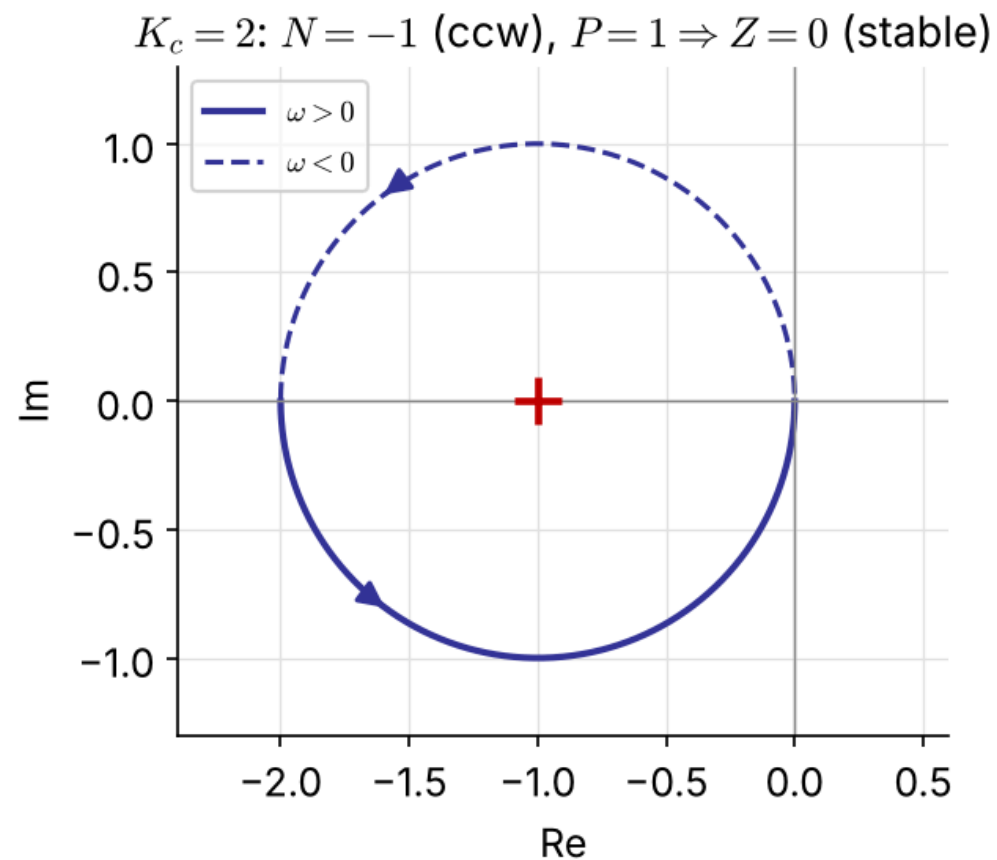
Phase crossover: $-\tan^{-1}(5\omega_c) - 2\omega_c = -\pi$
 $\Rightarrow \omega_c = 0.895$

$$K_{cu} = \frac{1}{|G(j\omega_c)|} = \frac{\sqrt{1+25\omega_c^2}}{0.8} = 5.73$$

The **same** answer as direct substitution in Ch 4 — read off one Bode plot, with no Padé approximation (1st-order Padé gave 7.5).

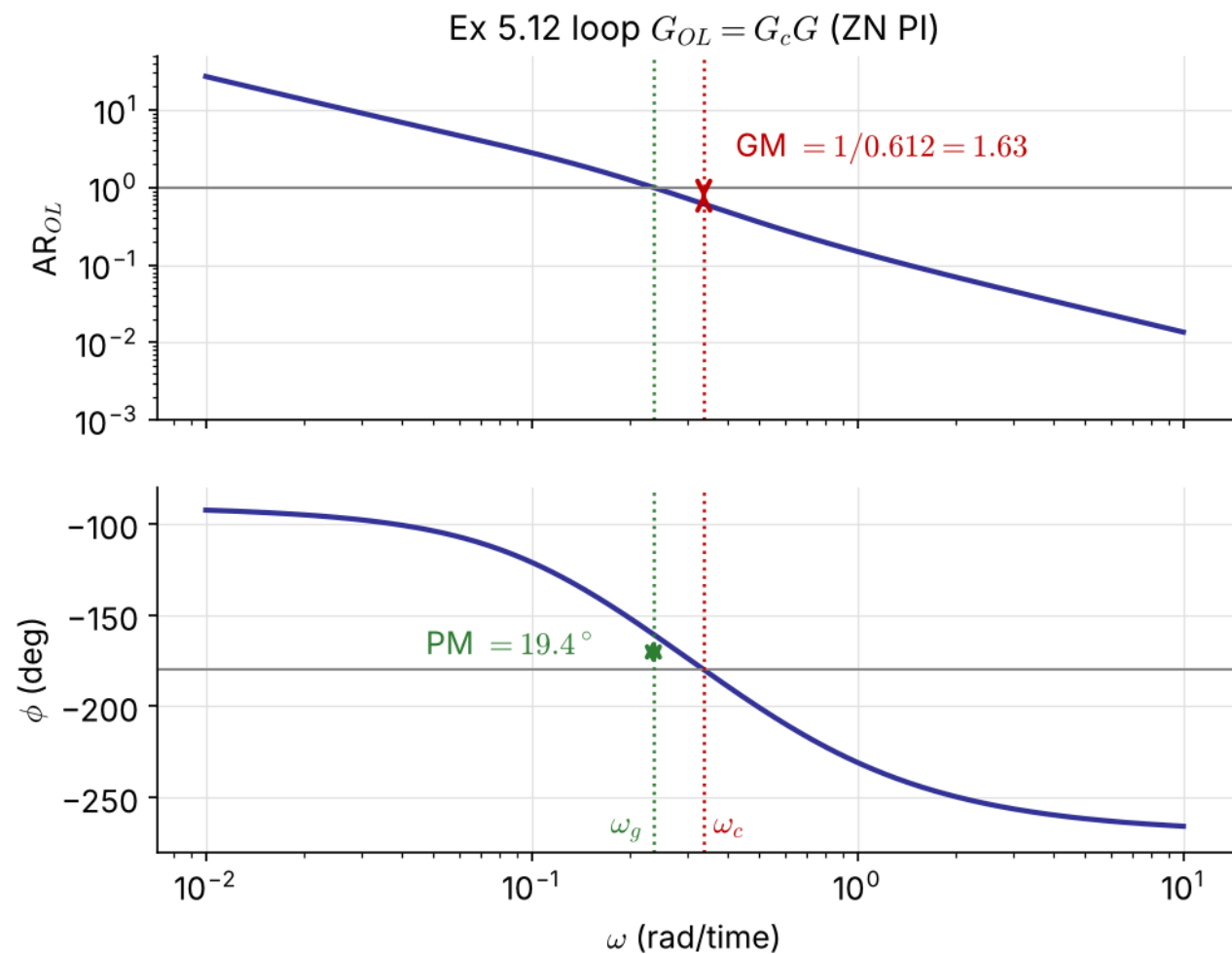
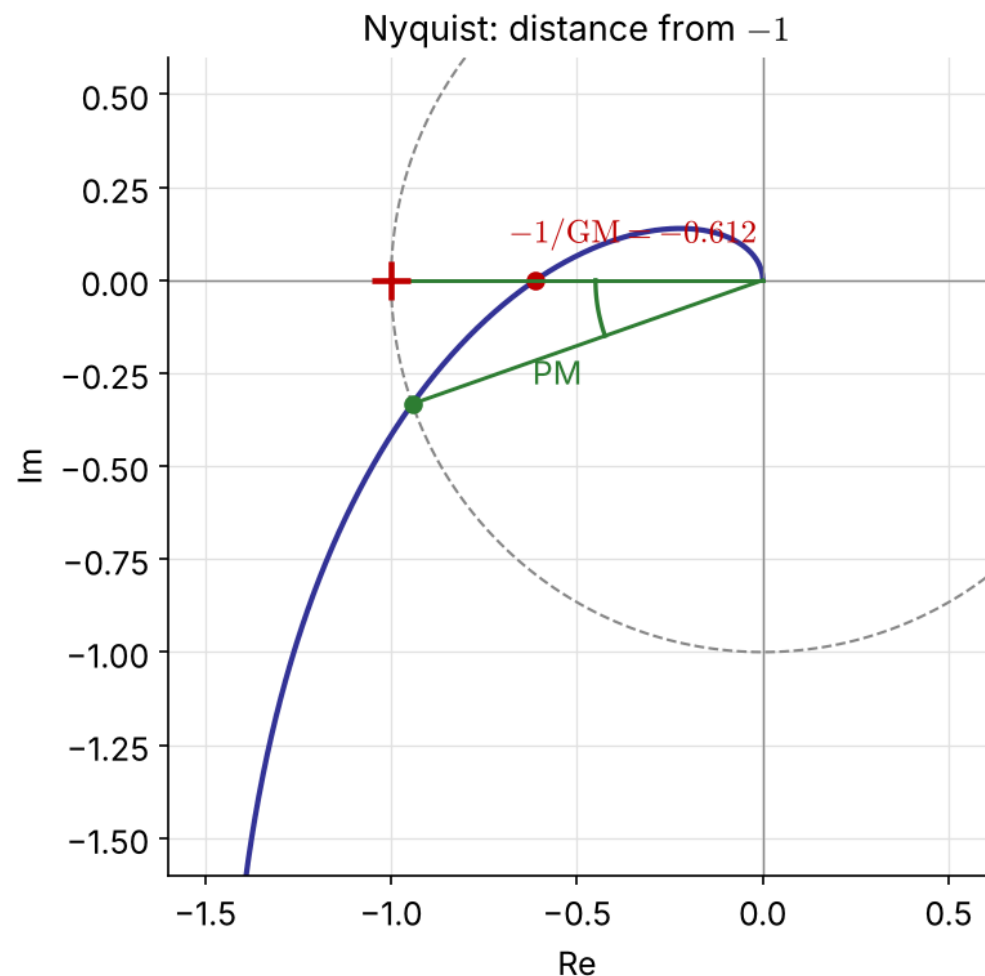
Example 5.14 Nyquist for an open-loop unstable process

$$G_{OL} = \frac{K_c}{s-1} \text{ (Ex 4.1 with } K = a = 1 \text{)}$$



$K_c/(s-1)$ is a circle of diameter K_c through 0 and $-K_c$, traversed **counterclockwise**. $P = 1$, so stability needs $N = -1$: the circle must enclose $-1 \Rightarrow K_c > 1$ — exactly Ex 4.1 ($K_c K > a$). The Bode criterion cannot be used here ($P \neq 0$).

Gain and phase margins



Gain margin (at ω_c , where $\angle G_{OL} = -180^\circ$)

$$\text{GM} = \frac{1}{|G_{OL}(j\omega_c)|}$$

Phase margin (at ω_g , where $|G_{OL}| = 1$)

$$\text{PM} = 180^\circ + \angle G_{OL}(j\omega_g)$$

What the margins protect against

GM — factor by which the loop gain can grow before instability → safeguard against **gain (model) error**

Typical spec: **GM** > 2 (6 dB)

$GM = 1/|G_{OL}(j\omega_c)|$ is the distance from -1 along the real axis

PM — extra phase lag tolerated at ω_g → safeguard against **dead-time error**

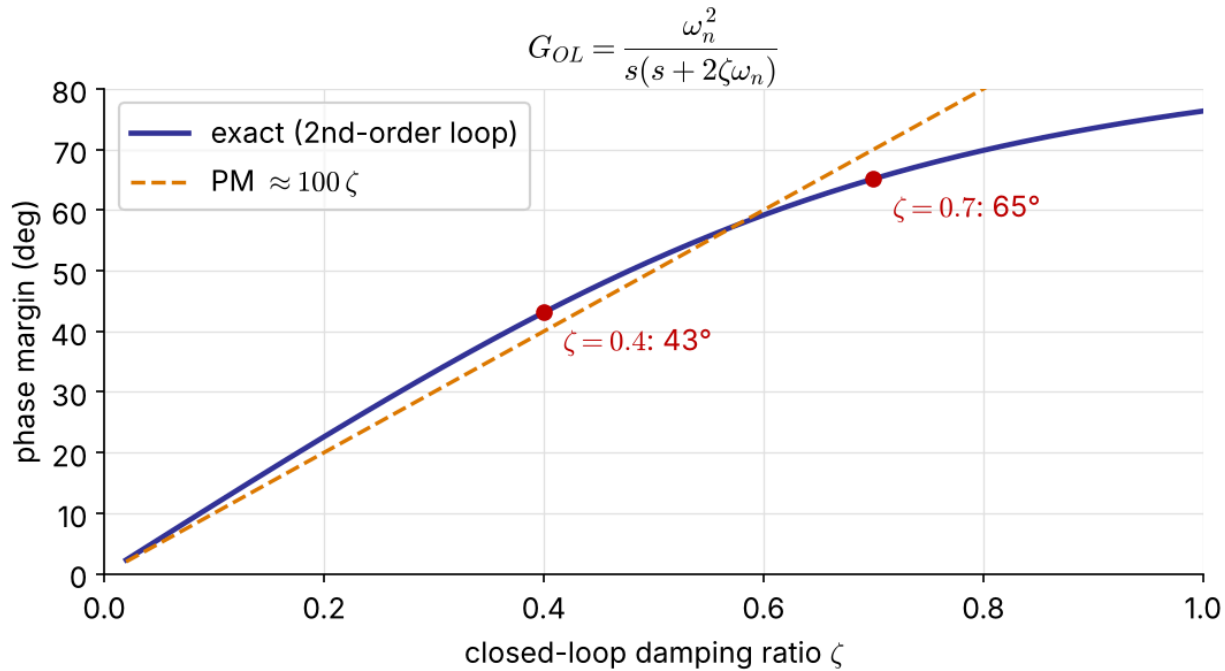
Typical spec: **PM** > 30° (often 45°)

Extra delay that destabilizes:

$$\theta_{\max} = \frac{\text{PM (rad)}}{\omega_g}$$

- If several crossings: take the one closest to -1 (largest $|G_{OL}|$) for GM, the smallest PM. Low-order minimum-phase plants may have $GM = \infty$.
- Lowering ω_g (smaller bandwidth, slower loop) tolerates larger delay errors. Ex 5.12: $\theta_{\max} = 0.339/0.236 = 1.4$ s.

Phase margin and damping



For a loop that behaves like $\frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$:

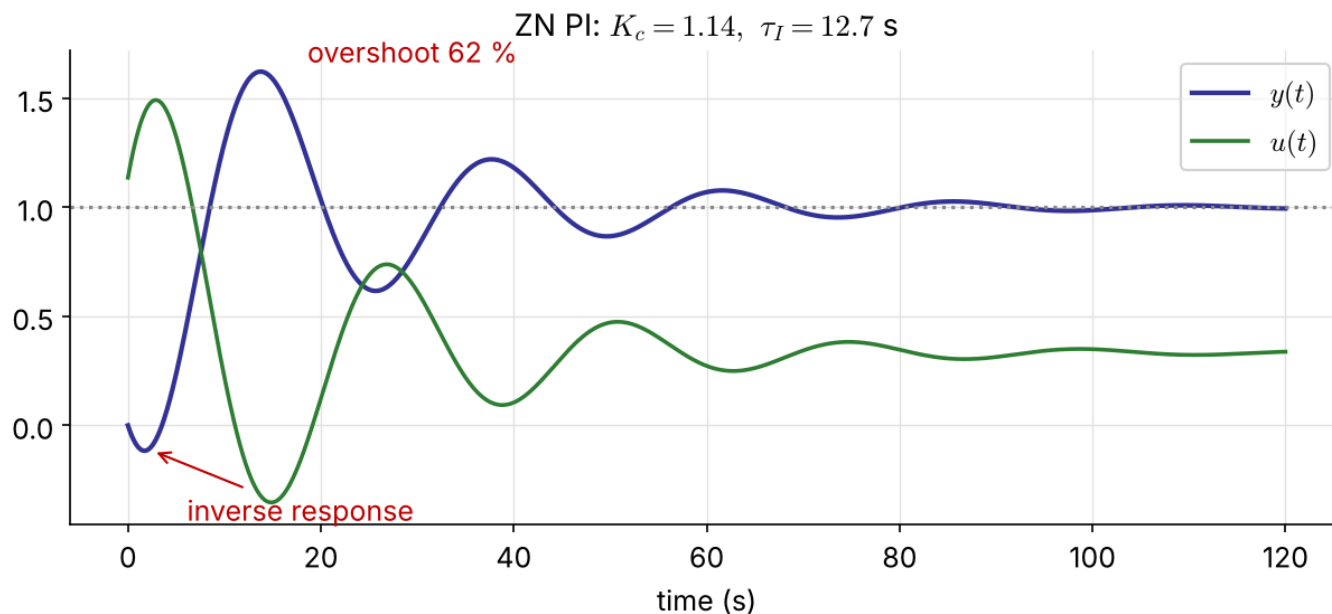
$$PM = \tan^{-1} \frac{2\zeta}{\sqrt{\sqrt{1 + 4\zeta^4} - 2\zeta^2}} \approx 100 \zeta \text{ (deg)}$$

PM 45° ↔ $\zeta \approx 0.4$ (overshoot ≈ 25 %)

PM 65° ↔ $\zeta \approx 0.7$ (overshoot ≈ 5 %)

A small PM means an oscillatory closed loop — the bridge to the ζ -based root-locus design of Ch 4.

Example 5.12 Ziegler–Nichols PI for the inverse-response process



$G_c = K_c \left(1 + \frac{1}{\tau_I s} \right)$ removes the P-control offset.

ZN settings from Ex 5.11 ($K_u = 2.5$, $P_u = 15.2$ s):

$$K_c = \frac{K_u}{2.2} = 1.14, \quad \tau_I = \frac{P_u}{1.2} = 12.7 \text{ s}$$

$$GM = 1.63, PM = 19.4^\circ$$

— **too small** ($GM > 2$, $PM > 30^\circ$)

→ 62 % overshoot, slow-decaying oscillation

ZN tunings are deliberately “aggressive” (quarter-decay); use them only as a starting point.

Margins in code

Python

```

1 import numpy as np, control as ct
2
3 G = ct.tf([-6, 3], np.polymul([10, 1], [5, 1]))
4 gm, pm, wc, wg = ct.margin(G)           # P control with Kc = 1
5 Ku, Pu = gm, 2*np.pi/wc                # 2.5, 15.2 s
6
7 Kc, tI = Ku/2.2, Pu/1.2                  # Ziegler-Nichols PI
8 L = Kc*ct.tf([tI, 1], [tI, 0])*G
9 gm, pm, wc, wg = ct.margin(L)           # GM 1.63, PM 19.4 deg
10 theta_max = np.radians(pm)/wg          # 1.43 s of extra dead time
11 ct.bode_plot(L, display_margins=True)

```

MATLAB

```

1 G = tf([-6 3], conv([10 1],[5 1]));
2 [Ku,~,wc] = margin(G);  Pu = 2*pi/wc;    % 2.5, 15.2 s
3
4 Kc = Ku/2.2;  tI = Pu/1.2;                % Ziegler-Nichols PI
5 L = Kc*tf([tI 1],[tI 0])*G;
6 [GM,PM,wc,wg] = margin(L)                % 1.63, 19.4 deg
7 theta_max = PM*pi/180/wg                  % 1.43 s
8 margin(L)                                % Bode plot with margins marked

```

5.4 Controller Design

Choosing K_c without trial and error

Write the loop as $1 + G_{OL} = 1 + K_c G^* = 0$, where G^* contains **everything except** K_c (process, valve, sensor, and the τ_I, τ_D terms of the controller).

From a GM spec

$\angle K_c = 0$, so G^* and G_{OL} share ω_c :

$$K_{cu} = \frac{1}{|G^*(j\omega_c)|}, \quad K_c = \frac{K_{cu}}{\text{GM}}$$

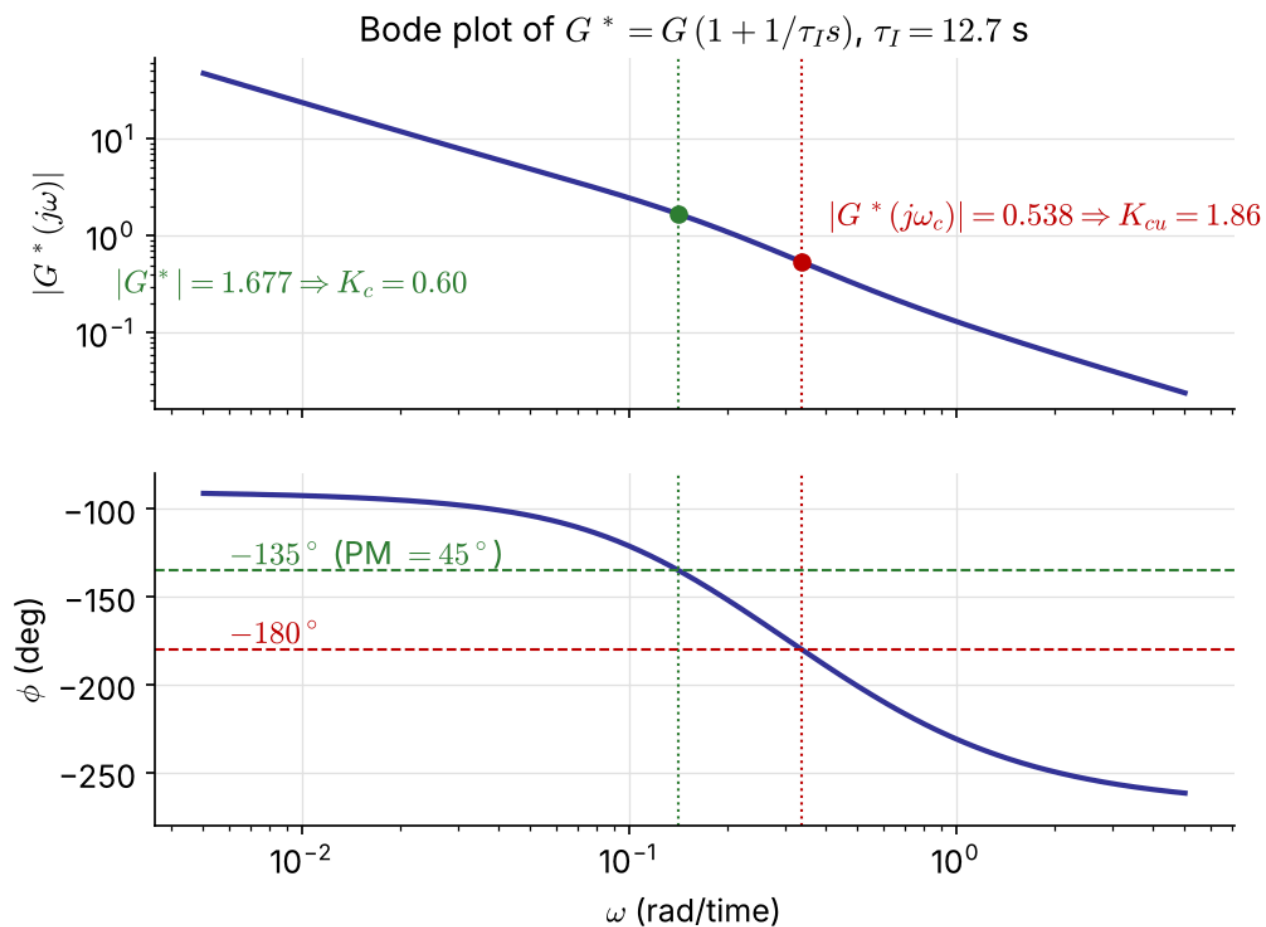
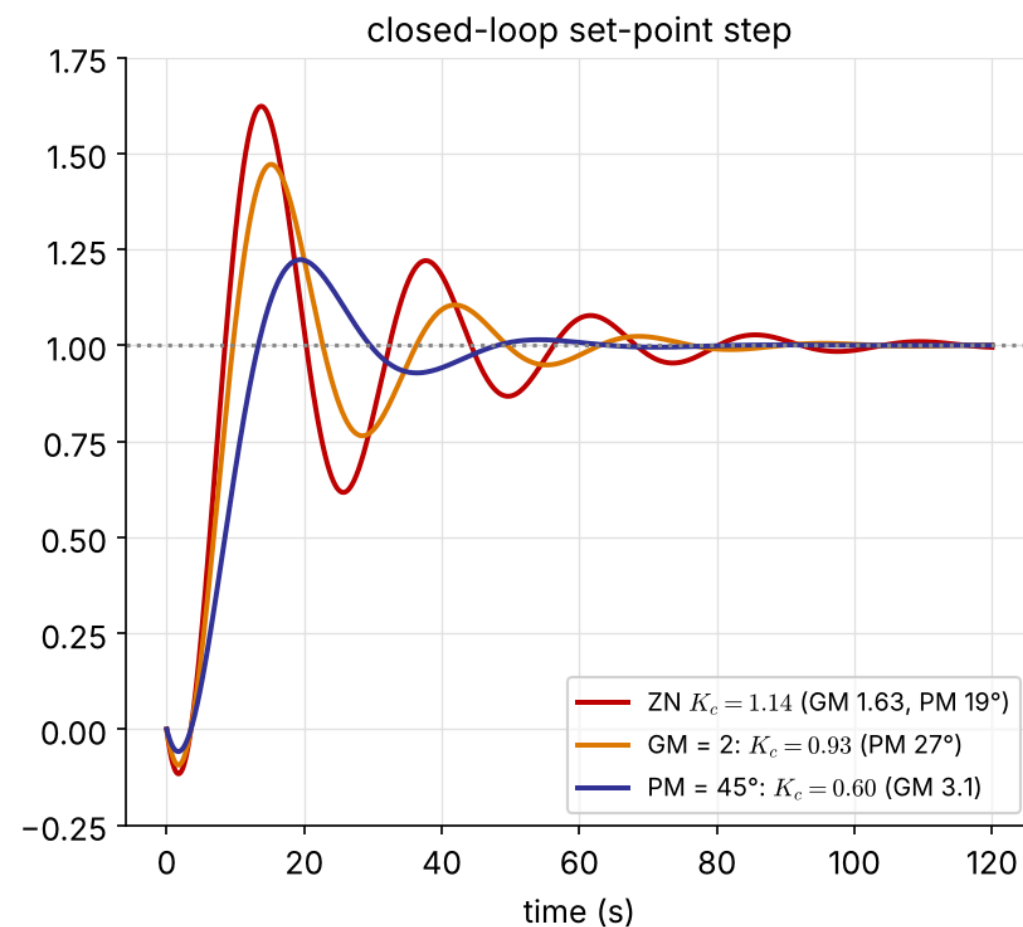
From a PM spec

Find ω_g where $\angle G^* = -180^\circ + \text{PM}$
(e.g. -135° for 45°):

$$K_c = \frac{1}{|G^*(j\omega_g)|}$$

Compute both and take the **smaller** K_c , which satisfies both specs. One Bode plot of G^* does it all.

Example 5.13 Retuning Ex 5.12 for $GM \geq 2$ and $PM \geq 45^\circ$



Keep $\tau_I = 12.7$ s: $G^* = G(1 + 1/\tau_I s)$, $\omega_c = 0.338$, $K_{cu} = 1.86$

- GM = 2 $\rightarrow K_c = 0.93$ (but PM only 27°)
- PM = $45^\circ \rightarrow \omega_g = 0.142$, $K_c = 0.60$ (GM 3.1)

$K_c = 0.60$ meets **both** specs: overshoot 62 % $\rightarrow$ 22 %, settles in 45 s instead of 89 s

Example 5.13 in code

Python

```

1  tI = Pu/1.2
2  Gs = ct.tf([tI, 1], [tI, 0])*G          # G* = everything except Kc
3
4  Kcu = ct.margin(Gs)[0]                  # 1.86
5  Kc_GM = Kcu/2                          # GM = 2    -> 0.93
6
7  w = np.logspace(-3, 1, 100000)
8  g = Gs(1j*w)
9  phase = np.degrees(np.unwrap(np.angle(g)))
10 i = np.argmin(abs(phase - (-180 + 45)))  # phase = -135 deg
11 Kc_PM = 1/abs(g[i])                    # PM = 45 deg -> 0.60
12
13 Kc = min(Kc_GM, Kc_PM)
14 print(ct.margin(Kc*Gs), ct.step_info(ct.feedback(Kc*Gs, 1)))

```

MATLAB

```

1  tI = Pu/1.2;
2  Gs = tf([tI 1],[tI 0])*G;              % G* = everything except Kc
3
4  Kcu = margin(Gs);    Kc_GM = Kcu/2;      % 1.86 -> 0.93
5
6  w = logspace(-3, 1, 1e5);
7  [m, p] = bode(Gs, w);    m = squeeze(m);    p = squeeze(p);
8  [~, i] = min(abs(p + 135));              % phase = -135 deg
9  Kc_PM = 1/m(i)                          % 0.60
10
11 Kc = min(Kc_GM, Kc_PM);
12 margin(Kc*Gs), stepinfo(feedback(Kc*Gs, 1))

```

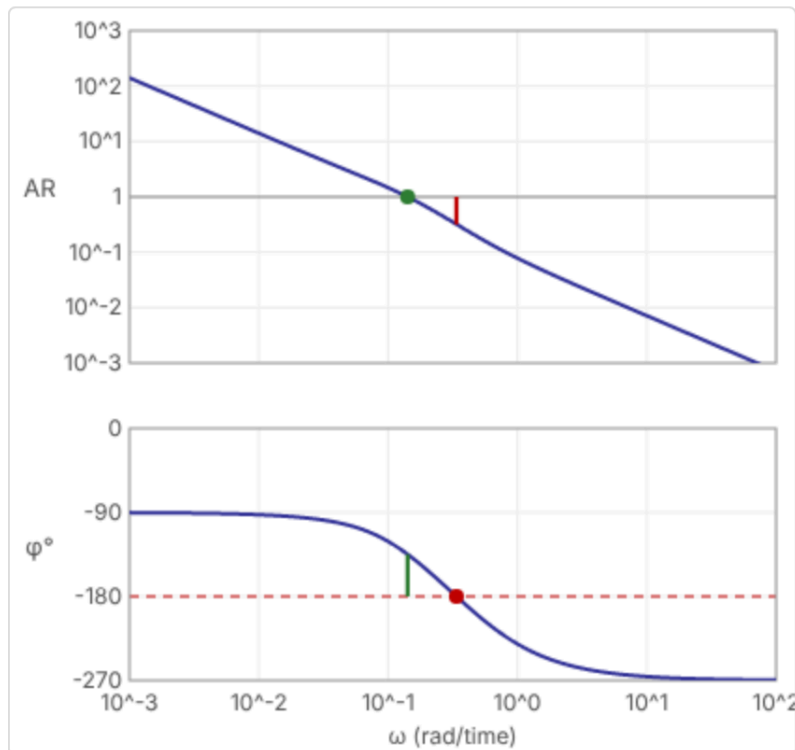
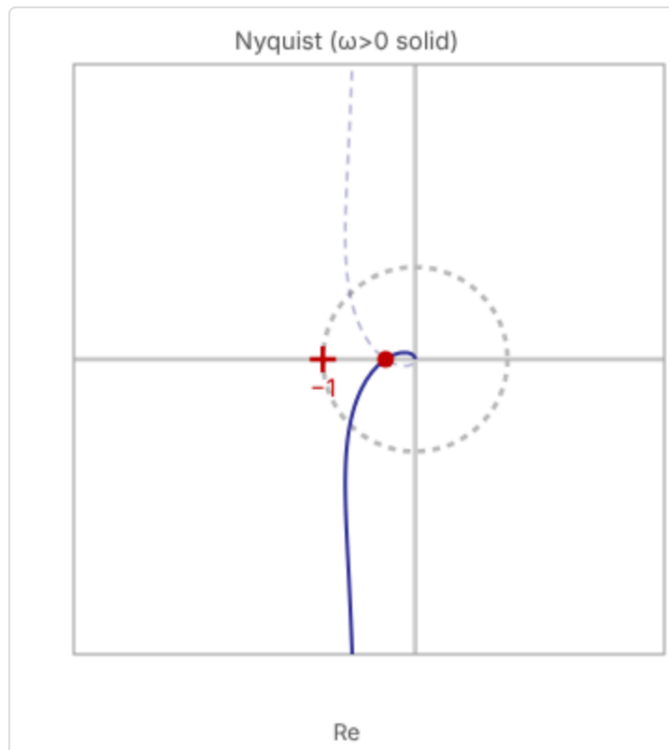
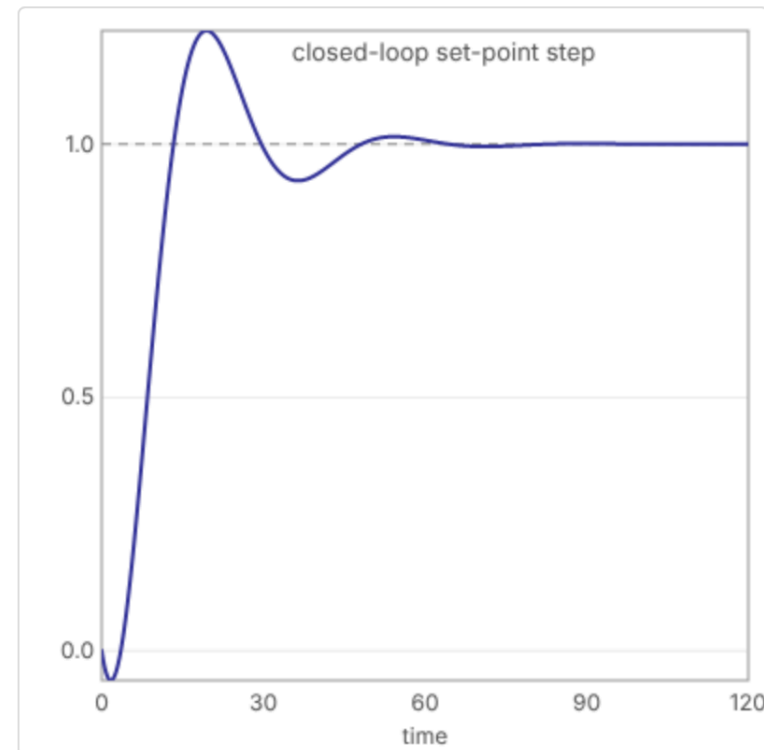
Interactive: move K_c , watch the margins

System Ex 5.12 — PI ($\tau_I=12.7$), same plant ▼

$K_c = 0.595$

STABLE

GM = 3.122 ($\omega_c = 0.338$) PM = 45.1° ($\omega_g = 0.142$) $K_{cu} = K_c \cdot GM = 1.858$



Interactive version: seoultechpse.github.io/ProCon/ch5/ch5_frequency.html — move K_c for Ex 5.11, 5.12, Ex 4.4 and Ex 4.2.

A final word: frequency response or root locus?

They **complement** each other; a design decision is often made only after both.

Root locus (Ch 4)

- shows the **transient response** and how it moves with K_c
- needs a fairly accurate model
- does not handle dead time well

Frequency response (Ch 5)

- **relative** stability: GM, PM; dead time **exactly**
- Bode plot can come from **experiments** (sine or pulse tests)
— no model needed, or a check of one
- several crossovers: use Nyquist, not Bode alone

Task	MATLAB	Python (control)
Bode / Nyquist	<code>bode(G), nyquist(G)</code>	<code>ct.bode_plot(G), ct.nyquist_plot(G)</code>
dead time	<code>tf(..., 'InputDelay', θ)</code>	<code>G(1j*w)*np.exp(-1j*w*θ)</code>
margins	<code>[GM, PM, wc, wg] = margin(L)</code>	<code>gm, pm, wc, wg = ct.margin(L)</code>

Problem #1 (homework)

$$G_p = \frac{2e^{-s}}{10s + 1}, \quad G_c = K_c \left(1 + \frac{1}{\tau_I s} \right), \quad \tau_I = 10$$

- Show that $\tau_I = 10$ makes G_{OL} a pure integrator with dead time. Sketch its Bode plot by hand.
- Find the phase crossover ω_c and the ultimate gain K_{cu} analytically.
- Find K_c for $\text{PM} = 45^\circ$. What is the resulting GM? Comment on the result.
- Check (b)–(c) with `ct.margin` and simulate the set-point response.

Hint: the phase of $e^{-j\omega}/j\omega$ is $-\pi/2 - \omega$.

Resources

- Slides (interactive HTML): seoultechpse.github.io/ProCon/ch5/ch5_frequency.html
- PDF: [slides](#) · [handout with note space](#)
- Python notebook: [Open in Colab](#) — every calculation and figure of this chapter

Appendix

General frequency response: derivation

Stable $G = Q/P$, input $A \sin \omega t$: $Y(s) = \frac{Q(s)}{P(s)} \frac{A\omega}{(s + j\omega)(s - j\omega)}$

All roots of P have negative real parts, so their terms decay. At large times only the input poles remain:

$$Y_{\infty}(s) = \frac{a}{s + j\omega} + \frac{a^*}{s - j\omega}, \quad a = (s + j\omega)Y(s)|_{s=-j\omega} = \frac{AG(-j\omega)}{-2j}$$

With $G(\pm j\omega) = |G(j\omega)|e^{\pm j\phi}$:

$$\frac{y_{\infty}(t)}{A} = |G(j\omega)| \left(\frac{e^{-j(\omega t + \phi)}}{-2j} + \frac{e^{j(\omega t + \phi)}}{2j} \right) = |G(j\omega)| \sin(\omega t + \phi), \quad \phi = \angle G(j\omega)$$

Resonance of the second-order system

$$\frac{d}{d\omega} \left[(1 - \tau^2 \omega^2)^2 + (2\zeta \tau \omega)^2 \right] = 0 \Rightarrow \tau^2 \omega_r^2 = 1 - 2\zeta^2$$

$$M_{p\omega} = |G(j\omega_r)| = \frac{K}{\sqrt{(2\zeta^2)^2 + 4\zeta^2(1 - 2\zeta^2)}} = \frac{K}{2\zeta\sqrt{1 - \zeta^2}}$$

A real ω_r requires $1 - 2\zeta^2 > 0$, i.e. $\zeta < 1/\sqrt{2}$.

ζ	0.2	0.4	0.6
$\tau\omega_r$	0.96	0.82	0.53
$M_{p\omega}/K$	2.55 (8.1 dB)	1.36 (2.7 dB)	1.04 (0.4 dB)